Problem 1. (17pts)

11 pts a) Find parametric equations for the line through the point $(4, 2, 6)$ that is perpendicular to the plane

$$3x - y + z = 7,$$

and find the points where the line intersects the coordinate planes.

The line has direction vector $\underbrace{\vec{u} = \langle 3, -1, 1 \rangle}_{2\text{ pts}}$. The parametric equations of the line are

$\underbrace{x = 4 + 3t}_{1\text{ pt}}$, $\underbrace{y = 2 - t}_{1\text{ pt}}$, $\underbrace{z = 6 + t}_{1\text{ pt}}$

Intersection with coordinate planes

* xy-plane : $z = 0 \Rightarrow t = -6$

2pts
$x = -14$, $y = 8$. The point is
$(-14, 8, 0)$

* xz-plane : $y = 0 \Rightarrow$
$t = 2$, $x = 10$, $z = 8$.
The point is $(10, 0, 8)$
2 pts

* yz-plane : $x = 0 \Rightarrow$
$t = -\frac{4}{3}$, $y = \frac{10}{3}$
$z = \frac{14}{3}$. The point
is $(0, \frac{10}{3}, \frac{14}{3})$.
2 pts

6 pts b) If θ is the angle between the planes

$$3(x - 1) - 2(y - 5) + 2(z + 1) = 0, \text{ and } 2x + 5(y - 1) + (z + 4) = 0,$$

find the value of $4\sec^2\theta$

$\underbrace{\vec{n}_1 = \langle 3, -2, 2 \rangle}_{1\text{ pt}}$, $\underbrace{\vec{n}_2 = \langle 2, 5, 1 \rangle}_{1\text{ pt}}$

$$\underbrace{\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}}_{2\text{pts}} = \underbrace{\frac{2}{\sqrt{510}}}_{1\text{ pt}}$$

$$\sec^2\theta = \frac{510}{4} \Rightarrow \underbrace{4\sec^2\theta = 510}_{1\text{ pt}}$$

Problem 2. (17pts)

6 pts a) Find an equation of the plane that contains the line

$$x = 10 + t, \qquad y = 9 - 5t, \qquad z = t$$

and is perpendicular to the plane $3x + 2y - 2z = 7$.

Let $\vec{u} = \langle 1, -5, 1 \rangle$ and $\vec{n} = \langle 3, 2, -2 \rangle$.

A normal vector to the plane is given by

$$\vec{u} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & 1 \\ 3 & 2 & -2 \end{vmatrix} = \langle 8, 5, 17 \rangle.$$

3 pts

$\underbrace{(10, 9, 0) \text{ is a point on that plane}}_{1\,pt}$. Hence, the equation of the plane is

$$8(x-10) + 5(y-9) + 17(z-0) = 0 \iff$$

$$\underbrace{8x + 5y + 17z = 125}_{2\,pts}$$

11 pts b) Identify and sketch the surface given by

$$-4x^2 + 8x + y^2 + 2y - \frac{4z^2}{9} = 7$$

The standard form of the equation is (after completing the squares)

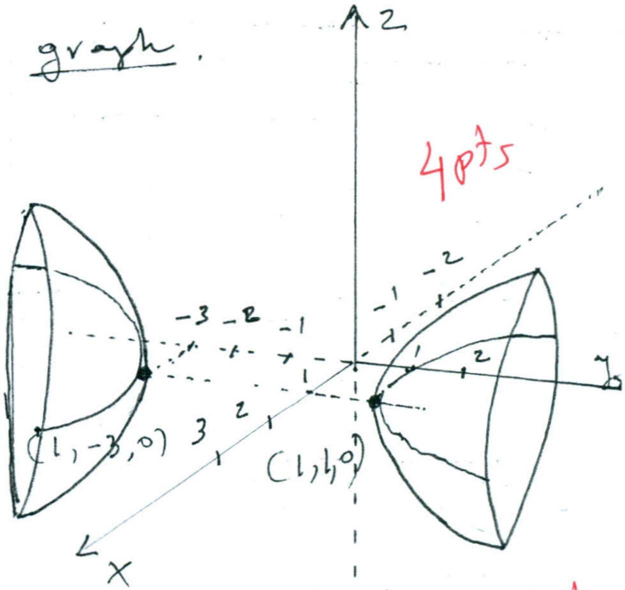
$$-\frac{(x-1)^2}{1^2} + \frac{(y+1)^2}{2^2} - \frac{z^2}{3^2} = 1$$

4 pts

This is a hyperboloid of two sheets. (Like $\frac{-x^2}{1^2} + \frac{y^2}{2^2} - \frac{z^2}{3^2} = 1$, except y is shifted by -1 and x is shifted by 1).

3 pts

graph.



4pts

Rk. * general shape = 2pts
* shifting = 1pt
* axe = 1pt

Problem 3. (16pts)

8pts a) Show that the function

$$\mathbf{f(x,y)} = \begin{cases} \frac{\sin(xy)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}.$$

is not continuous at $(\mathbf{0},\mathbf{0})$

$f(0,0) = 1$.

4pts * Along the x-axis $(y=0,\ x\neq 0)$, $f(x,y) = 0$.
So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis.

4pts * Along the line $y = x$ $(x\neq 0,\ y\neq 0)$, $f(x,y) = \frac{\sin x^2}{2x^2}$

$f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $y = x$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

f is not continuous at $(0,0)$.

Rk. * choice of a curve and value of limit along that curve is 4pts
* limit $\neq f(0,0)$ 4pts

8pts b) Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{3-\sqrt{x^2+y^2+9}}{x^2+y^2}$$

Set $x = r\cos\theta$, $y = r\sin\theta$. $\lim_{(x,y)\to(0,0)} \frac{3-\sqrt{x^2+y^2+9}}{x^2+y^2} =$

$$= \lim_{r\to 0} \frac{3-\sqrt{r^2+9}}{r^2}$$

$$= \lim_{r\to 0^+} \frac{9-(r^2+9)}{r^2(3+\sqrt{r^2+9})} \quad \text{4pts}$$

$$= \underbrace{\lim_{r\to 0} \frac{-1}{3+\sqrt{r^2+9}}}_{\text{3pts}} = \underbrace{-\frac{1}{6}}_{\text{1 pt}}.$$

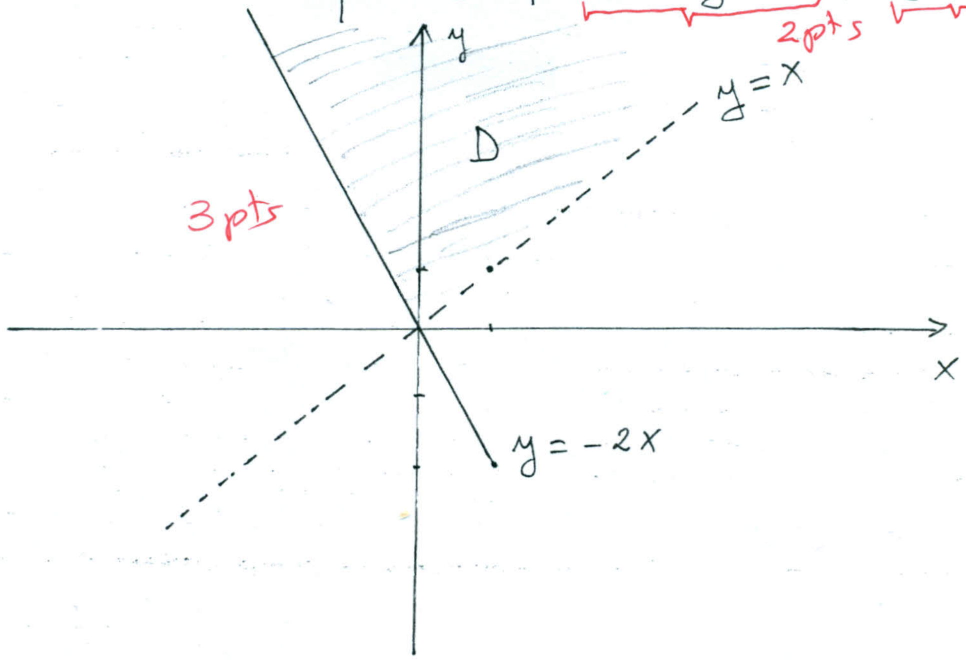
Rk. If the conjugate or L'Hospital rule is applied for a specific curve, give only 2pts.

Problem 4. (17pts)

7pts a) Find and sketch the domain of the function

$$f(x, y) = \sqrt{2x + y}\,\ln(y - x).$$

The domain $D = \{ (x, y) \mid \underbrace{2x + y \geq 0}_{2pts},\ \underbrace{y > x}_{2pts} \}$



10 pts b) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$yz = \ln(x + z).$$

$\frac{\partial z}{\partial x} = ?$

1pt $\longrightarrow$ $\frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}\left[\ln(x+z)\right] \Leftrightarrow$

2pts $\longrightarrow$ $y\frac{\partial z}{\partial x} = \frac{1 + \frac{\partial z}{\partial x}}{x+z} \Leftrightarrow$

$y(x+z)\frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x} \Leftrightarrow$

1pt $\longrightarrow$ $\frac{\partial z}{\partial x}\left[y(x+z) - 1\right] = 1 \Leftrightarrow$

1pt $\longrightarrow$ $\frac{\partial z}{\partial x} = \frac{1}{xy + yz - 1}$

$\frac{\partial z}{\partial y} = ?$

1pt $\frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}\left(\ln(x+z)\right) \Leftrightarrow$

$\underbrace{z + y\frac{\partial z}{\partial y} = \frac{\frac{\partial z}{\partial y}}{x+z}}_{2pts} \Leftrightarrow$

$z(x+z) + y(x+z)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \Leftrightarrow$

$\underbrace{\frac{\partial z}{\partial y}(1 - yx - yz) = zx + z^2}_{1pt} \Leftrightarrow$

$\underbrace{\frac{\partial z}{\partial y} = \frac{zx + z^2}{1 - yx - yz}}_{1pt}$

Problem 5. (16pts)

8pts a) Find an equation of the tangent plane to the surface

$$z = xye^{-xy^2}$$

at the point $(1,1,e^{-1})$.

$$\frac{\partial z}{\partial x} = y e^{-xy^2} + (xy)(-y^2) e^{-xy^2} = y e^{-xy^2} - x y^3 e^{-xy^2} \quad \text{2pts}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = e^{-1} - e^{-1} = 0 \quad \text{1pt}$$

$$\frac{\partial z}{\partial y} = x e^{-xy^2} + (xy)(-2xy) e^{-xy^2} = (x - 2x^2y^2) e^{-xy^2} \quad \text{2pts}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = (1-2) e^{-1} = -e^{-1} \quad \text{1pt}$$

An equation of the tangent line is thus given by

$$z = e^{-1} + 0(x-1) + (-e^{-1})(y-1) \iff$$

$$\text{1pt} \quad z = \frac{1}{e} - \frac{1}{e}(y-1) \iff ez + y = 2 \quad \text{1pt}$$

8pts b) Use the linearization L(x,y) of the function

$$f(x, y) = \sqrt{9 - x^2 - 4y^2},$$

at $(-1, 1)$ to find the best estimate for $f(-0.9, 1.1)$.

$$f_x = \frac{-2x}{2\sqrt{9 - x^2 - 4y^2}} = \frac{-x}{\sqrt{9 - x^2 - 4y^2}} \ \text{1pt}, \quad f_x(-1,1) = \frac{1}{2} \ \text{1pt}$$

$$f_y = \frac{-8y}{2\sqrt{9 - x^2 - 4y^2}} = \frac{-4y}{\sqrt{9 - x^2 - 4y^2}} \ \text{1pt}, \quad f_y(-1,1) = -2 \ \text{1pt}$$

$$f(-1,1) = 2 \ \text{1pt}, \quad L(x,y) = 2 + \frac{1}{2}(x+1) - 2(y-1) \ \text{2pts}$$

$(-0.9, 1.1)$ is close to $(-1,1)$ $\Rightarrow$

$$f(-0.9, 1.1) \approx L(-0.9, 1.1) = 2 + \frac{1}{2}(0.1) - 2(0.1)$$

$$= 1.85 \quad \text{1pt}$$

Rk. give 2 pts for the general formula of L(x,y).

Problem 6. (17pts)

8pts a) Find all second partial derivatives of

$$\mathbf{f(x,y) = xe^{-2y}}$$

$f_x = e^{-2y}$ 2pts , $f_y = -2x e^{-2y}$ 2pts

$f_{xx} = 0$ 1pt , $f_{yy} = 4x e^{-2y}$ 1pt

$f_{xy} = -2e^{-2y} = f_{yx}$ 2 pts

8pts b) If $\mathbf{z = y + \cos(x^2 - y^2)}$, show that

$$y\frac{\partial \mathbf{z}}{\partial \mathbf{x}} + x\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = \mathbf{x}$$

$\frac{\partial z}{\partial x} = -2x \sin(x^2 - y^2)$ 3pts , $\frac{\partial z}{\partial y} = 1 + 2y \sin(x^2 - y^2)$ 3pts

$y \frac{\partial z}{\partial x} = -2xy \sin(x^2 - y^2)$ 1pt , $x \frac{\partial z}{\partial y} = x + 2yx \sin(x^2 - y^2)$ 1pt

$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = \cancel{-2xy \sin(x^2 - y^2)} + x + \cancel{2xy \sin(x^2 - y^2)}$

$= x$. 1pt