

6 pts 1. (a) Eliminate the parameter to find a cartesian (rectangular) equation of the curve whose parametric equations are $x = 2\sin^2 t$ and $y = 4\sec t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

$$y = \frac{4}{\cos t} \quad \text{or}$$

$$\cos t = \sqrt{1 - \sin^2 t} \Rightarrow$$

$$y = \frac{4}{\sqrt{1 - \sin^2 t}} \quad 2 \text{ pts}$$

$$\sin^2 t = \frac{x}{2} \Rightarrow$$

$$y = \frac{4}{\sqrt{1 - \frac{x}{2}}} \quad , 3 \text{ pts}$$

$$0 \leq x < 2 \quad 1 \text{ pt}$$

$$y \cos t = 4$$

$$y^2 \cos^2 t = 16 \Rightarrow 1 \text{ pt}$$

$$y^2 (1 - \sin^2 t) = 16 \Rightarrow 1 \text{ pt}$$

$$y^2 \left(1 - \frac{x}{2}\right) = 16 \Rightarrow 2 \text{ pts}$$

$$x = 2 - \frac{32}{y^2} \quad ,$$

1 pt

with $y \geq 4$

1 pt

8 pts (b) Find an equation of the tangent line to the curve in part (a) at $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \sec t \tan t}{4 \cos t \sin t} = \frac{\sec t \tan t}{\cos t \sin t} \quad 2 \text{ pts}$$

at $t = \frac{\pi}{4}$, $x = 1$ and $y = 4\sqrt{2}$ 2 pts

$$\left. \frac{dy}{dx} \right|_{t = \frac{\pi}{4}} = \frac{2/\sqrt{2}}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad 2 \text{ pts}$$

The equation of the tangent line is:

$$y = 2\sqrt{2}(x-1) + 4\sqrt{2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 2 \text{ pts}$$

$$= 2\sqrt{2}x + 2\sqrt{2}$$

8 pts 2. (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve given by

$$x = t + t^2 \text{ and } y = t - t^2.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-2t}{1+2t} \quad 2 \text{ pts}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{dy}{dx} \right]}{\frac{dx}{dt}} = \frac{-2(1+2t) - 2(1-2t)}{(1+2t)^3} \quad 4 \text{ pts}$$

$$= \frac{-4}{(1+2t)^3} \quad 2 \text{ pts}$$

6 pts (b) Find the values of t for which the curve given in part (a) is concave down.

The graph is concave down if

$$\frac{d^2y}{dx^2} < 0 \Leftrightarrow \frac{-4}{(1+2t)^3} < 0 \Leftrightarrow$$

$\underbrace{\frac{d^2y}{dx^2}}_{3 \text{ pts}}$
 $\underbrace{(1+2t)^3}_{2 \text{ pts}}$

$$1+2t > 0 \Leftrightarrow t > -\frac{1}{2} \quad 1 \text{ pt}$$

- 10 pts 3. (a) Find the area of the surface obtained by rotating about the x -axis the curve given by

$$x = \sqrt{7} \cos^3 \theta \text{ and } y = \sqrt{7} \sin^3 \theta \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$$A = 2\pi \int_0^{\frac{\pi}{2}} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \quad 2 \text{ pts}$$

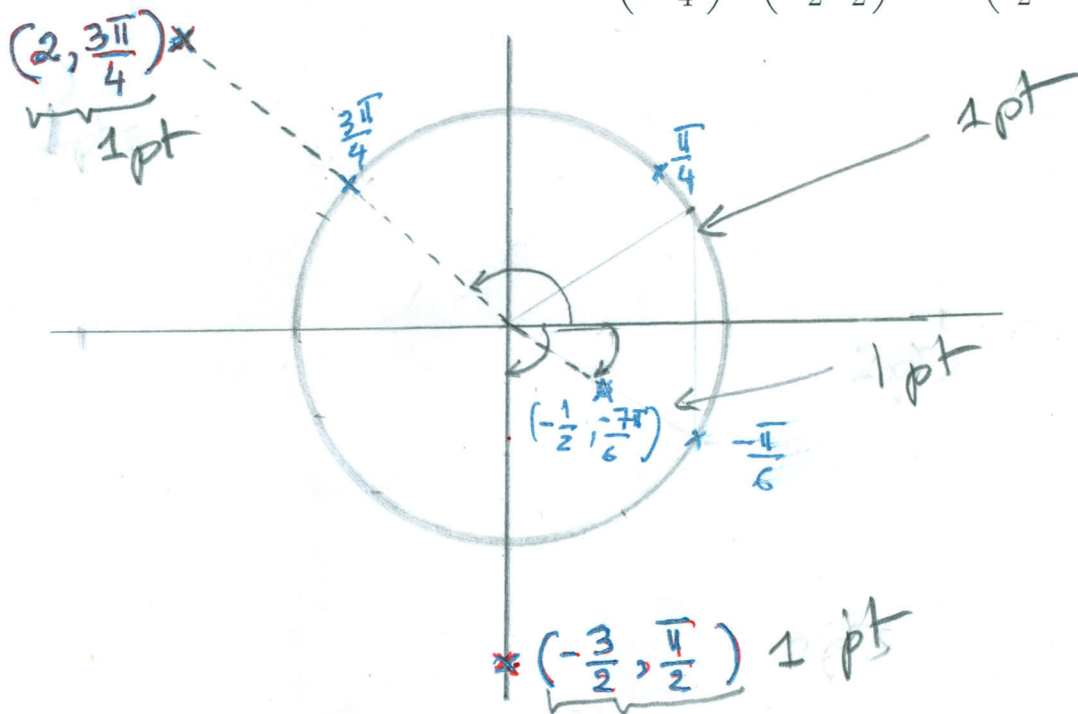
$$= 2\pi \int_0^{\frac{\pi}{2}} \sqrt{7} \sin^3 \theta \sqrt{(3\sqrt{7})^2 \cos^4 \theta \sin^2 \theta + (3\sqrt{7})^2 \sin^4 \theta \cos^2 \theta} d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 21 \sin^3 \theta \sqrt{\cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)} d\theta$$

$$= 42\pi \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos \theta d\theta \quad 6 \text{ pts}$$

$$= 42\pi \left(\frac{\sin^5 \theta}{5} \Big|_0^{\frac{\pi}{2}} \right) = \frac{42\pi}{5} \quad 2 \text{ pts}$$

- 4 pts (b) On the same polar coordinate system, draw the unit circle centered at the origin and plot the points whose polar coordinates are $\left(2, \frac{3\pi}{4}\right)$, $\left(-\frac{3}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{-1}{2}, -\frac{7\pi}{6}\right)$.



8 pts

4. (a) Sketch on the same polar coordinate system the curves

$$r = \sqrt{3} \sin \theta \quad \text{and} \quad r = 1 + \cos \theta.$$

and determine polar coordinates of the intersecting point(s) if any.

$r = \sqrt{3} \sin \theta$ is the circle centered at $(0, \frac{\sqrt{3}}{2})$ with radius $\frac{\sqrt{3}}{2}$.

Intersection points

$$1 \text{ pt } \left\{ \begin{aligned} \sqrt{3} \sin \theta &= 1 + \cos \theta \Rightarrow 3 \sin^2 \theta = 1 + 2 \cos \theta + \cos^2 \theta \\ \Leftrightarrow 3(1 - \cos^2 \theta) &= 1 + 2 \cos \theta + \cos^2 \theta \Leftrightarrow \end{aligned} \right.$$

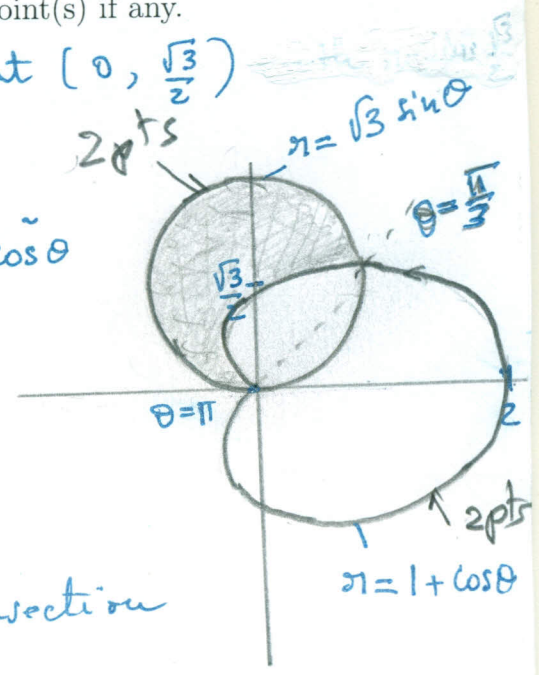
$$4 \cos^2 \theta + 2 \cos \theta - 2 = 0 \Rightarrow$$

$$\cos \theta = -1 \quad \text{or} \quad \cos \theta = \frac{1}{2} \Rightarrow$$

$$\theta = \pi \quad \text{or} \quad \theta = \frac{\pi}{3}. \quad 1 \text{ pt}$$

The polar coordinates of the intersection points are:

$$\underbrace{(0, \pi)}_{1 \text{ pt}} \quad \text{and} \quad \underbrace{\left(\frac{3}{2}, \frac{\pi}{3}\right)}_{1 \text{ pt}}$$



8 pts

(b) Find the area of the region that lies inside the curve $r = \sqrt{3} \sin \theta$ and outside the curve $r = 1 + \cos \theta$.

$$\begin{aligned}
 A &= \pi \left(\frac{\sqrt{3}}{2}\right)^2 - \text{Area inside both curves} \quad 2 \text{ pts} \\
 &= \frac{3\pi}{4} - \left[\underbrace{\frac{1}{2} \int_0^{\frac{\pi}{3}} 3 \sin^2 \theta \, d\theta}_{2 \text{ pts}} + \underbrace{\frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (1 + \cos \theta)^2 \, d\theta}_{2 \text{ pts}} \right] \\
 &= \frac{3\pi}{4} - \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16} + \frac{\pi}{2} - \frac{9\sqrt{3}}{16} \right) \quad 1 \text{ pt} \\
 &= \frac{3\pi}{4} - \frac{3\pi}{4} + \frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} \quad 1 \text{ pt}
 \end{aligned}$$

5. (a) Find an equation of the sphere whose diameter has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

8 pts
center: $\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3, 2, 7)$ 2 pts

Radius: $R = \sqrt{(2-3)^2 + (1-2)^2 + (4-7)^2}$
 $= \sqrt{1+1+9} = \sqrt{11}$ 2 pts

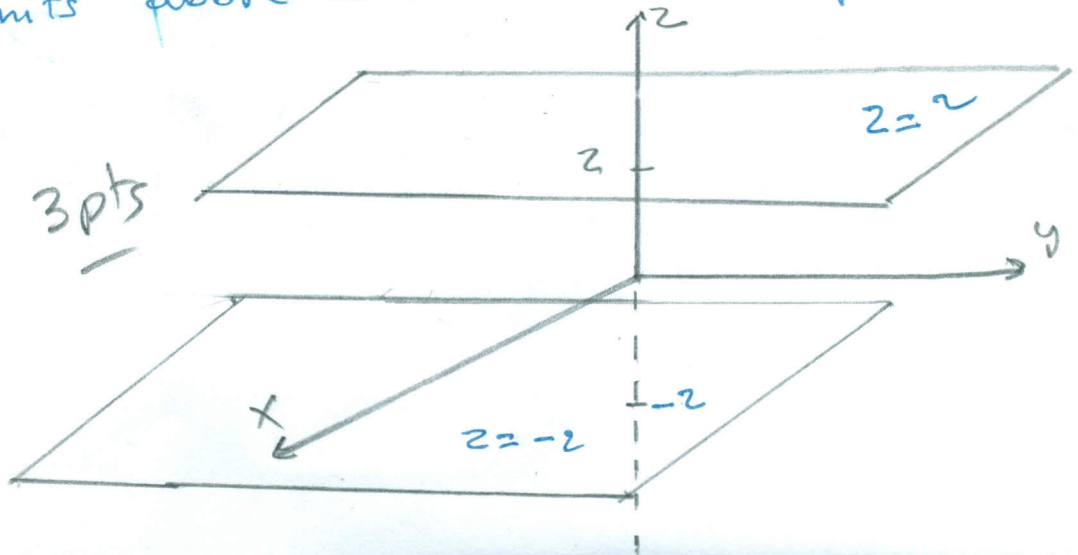
The equation of the sphere is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11 \quad 4 \text{ pts}$$

6 pts

- (b) Describe in words and sketch the region of the three-dimensional space represented by $z^2 = 4$.

3 pts } $z^2 = 4$ means $z = 2$ or $z = -2$. The region is the union of the two planes $z = 2$ and $z = -2$. The planes are both parallel to the xy -planes and are two units above and below it respectively.



6. (a) Given the points $A(1, 0, -1)$, $B(2, 1, 0)$, find the coordinates of the point C such that \vec{AC} and \vec{AB} have opposite directions and $|\vec{AC}| = 3|\vec{AB}|$.

7 pts

We have $\vec{AC} = -3\vec{AB}$, so if C has coordinates (x, y, z) then

$$\langle x-1, y, z+1 \rangle = -3\langle 1, 1, 1 \rangle \quad 2 \text{ pts}$$

$$\text{i.e. } \left. \begin{array}{l} x-1 = -3 \\ y = -3 \\ z+1 = -3 \end{array} \right\} \Rightarrow \begin{array}{l} x = -2 \quad 1 \text{ pt} \\ y = -3 \quad 1 \text{ pt} \\ z = -4 \quad 1 \text{ pt} \end{array}$$

The point C is given by $C(-2, -3, -4)$.

7 pts

- (b) Consider the vectors

$$\vec{u} = \langle \sin x, 2, 1 \rangle \text{ and } \vec{v} = \langle 1, \cos x, \sin x \rangle, \quad 0 \leq x \leq 2\pi.$$

Find the values of x for which the two vectors \vec{u} and \vec{v} are orthogonal.

$$\begin{aligned} 2 \text{ pts } \left\{ \begin{array}{l} \vec{u} \text{ and } \vec{v} \text{ are orthogonal if and only if} \\ \vec{u} \cdot \vec{v} = 0 \Leftrightarrow \sin x + 2\cos x + \sin x = 0 \Leftrightarrow \end{array} \right. & \quad 3 \text{ pts} \\ \sin x = -\cos x \Leftrightarrow \tan x = -1 \Rightarrow & \quad x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \\ & \quad \underbrace{\hspace{1cm}}_{1 \text{ pt}} \quad \underbrace{\hspace{1cm}}_{1 \text{ pt}} \\ x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4} & \end{aligned}$$

7. (a) Find the volume of the parallelepiped with adjacent edges DA , DB , and DC , where $A(3, 0, 1)$, $B(1, 2, 0)$, $C(2, 1, 0)$, $D(0, -1, 0)$.

7 pts

$$\vec{DA} = \langle 3, 1, 1 \rangle \quad , \quad \vec{DB} = \langle 1, 3, 0 \rangle \quad , \quad \vec{DC} = \langle 2, 2, 0 \rangle$$

$$\text{Volume} = | \vec{DA} \cdot (\vec{DB} \times \vec{DC}) | \quad 2 \text{pts}$$

$$\vec{DB} \times \vec{DC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \langle 0, 0, -4 \rangle \quad 1 \text{pt}$$

$$\text{Volume} = | 3 \cdot (0) + 1 \cdot (0) + 1 \cdot (-4) | = |-4| = 4 \quad 1 \text{pt}$$

7 pts.

- (b) Consider the vectors

$$\vec{a} = \langle -2, 1, 3 \rangle \text{ and } \vec{b} = \langle 3, 3, 4 \rangle.$$

If $\vec{u} = \text{proj}_{\vec{a}} \vec{b}$ and $\vec{v} = \vec{b} - \vec{u}$, find the vector \vec{w}

$$\vec{w} = \frac{1}{2} \vec{u} + (5\vec{a} \cdot \vec{v}) \vec{v} - \vec{a} \times (2\vec{u}).$$

$$\begin{aligned} \vec{u} &= \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} = \frac{-6 + 3 + 12}{\sqrt{4 + 1 + 9}} \langle -2, 1, 3 \rangle \\ &= \frac{9}{14} \langle -2, 1, 3 \rangle \quad 2 \text{pts} \end{aligned}$$

$$\vec{a} \text{ and } \vec{u} \text{ are parallel} \Rightarrow \vec{a} \times 2\vec{u} = \vec{0} \quad 2 \text{pts}$$

$$\vec{a} \text{ and } \vec{v} \text{ are perpendicular} \Rightarrow \vec{a} \cdot \vec{v} = 0. \quad 2 \text{pts}$$

$$\text{So } \vec{w} = \frac{1}{2} \vec{u} = \frac{9}{28} \langle -2, 1, 3 \rangle = \left\langle \frac{-9}{14}, \frac{9}{28}, \frac{27}{28} \right\rangle \quad 4 \text{pts}$$