6 pts 1. (a) Eliminate the parameter to find a cartesian (rectangular) equation of the curve whose parametric equations are $x = 2\sin^2 t$ and $y = 4\sec t$ , $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

$y = \frac{4}{\cos t}$.

$\cos t = \sqrt{1 - \sin^2 t} \Rightarrow$

$y = \frac{4}{\sqrt{1-\sin^2 t}}$ 2 pts

$\sin^2 t = \frac{x}{2} \Rightarrow$

$y = \frac{4}{\sqrt{1 - \frac{x}{2}}}$ , 3 pts

$0 \le x < 2$ 1 pt

or

$y \cos t = 4$

$y^2 \cos^2 t = 16 \Rightarrow$ 1 pt

$y^2(1 - \sin^2 t) = 16 \Leftrightarrow$ 1 pt

$y^2(1 - \frac{x}{2}) = 16 \Leftrightarrow$ 2 pts

$\underbrace{x = 2 - \frac{32}{y^2}}_{1\text{ pt}}$ , $\underbrace{y > 0}_{1\text{ pt}}$.

8 pts (b) Find an equation of the tangent line to the curve in part (a) at $t = \frac{\pi}{4}$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sec t \tan t}{4 \cos t \sin t} = \frac{\sec t \tan t}{\cos t \sin t}$ 2 pts

at $t = \frac{\pi}{4}$, $x = 1$ and $y = 4\sqrt{2}$ 2 pts

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{2/\sqrt{2}}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$. 2 pts

The equation of the tangent line is:

$y = 2\sqrt{2}(x-1) + 4\sqrt{2}$
$= 2\sqrt{2}x + 2\sqrt{2}$ 2 pts

8 pts 2. (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve given by

$$x = t + t^2 \text{ and } y = t - t^2.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-2t}{1+2t} \quad 2\text{pts}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{-2(1+2t)-2(1-2t)}{(1+2t)^3} \quad 4\text{pts}$$

$$= \frac{-4}{(1+2t)^3} \quad 2\text{pts}$$

6 pts (b) Find the values of $t$ for which the curve given in part (a) is concave down.

The graph is concave down if

$$\underbrace{\frac{d^2y}{dx^2} < 0}_{3\text{pts}} \iff \underbrace{\frac{-4}{(1+2t)^3} < 0}_{2\text{pts}} \iff$$

$$1 + 2t > 0 \iff \underbrace{t > -\frac{1}{2}}_{1\text{pt}}$$

10 pts 3. (a) Find the area of the surface obtained by rotating about the $x$-axis the curve given by

$$x = \sqrt{7}\cos^3\theta \text{ and } y = \sqrt{7}\sin^3\theta \quad 0 \le \theta \le \frac{\pi}{2}.$$

$$A = 2\pi \int_0^{\frac{\pi}{2}} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad 2 \text{ pts}$$

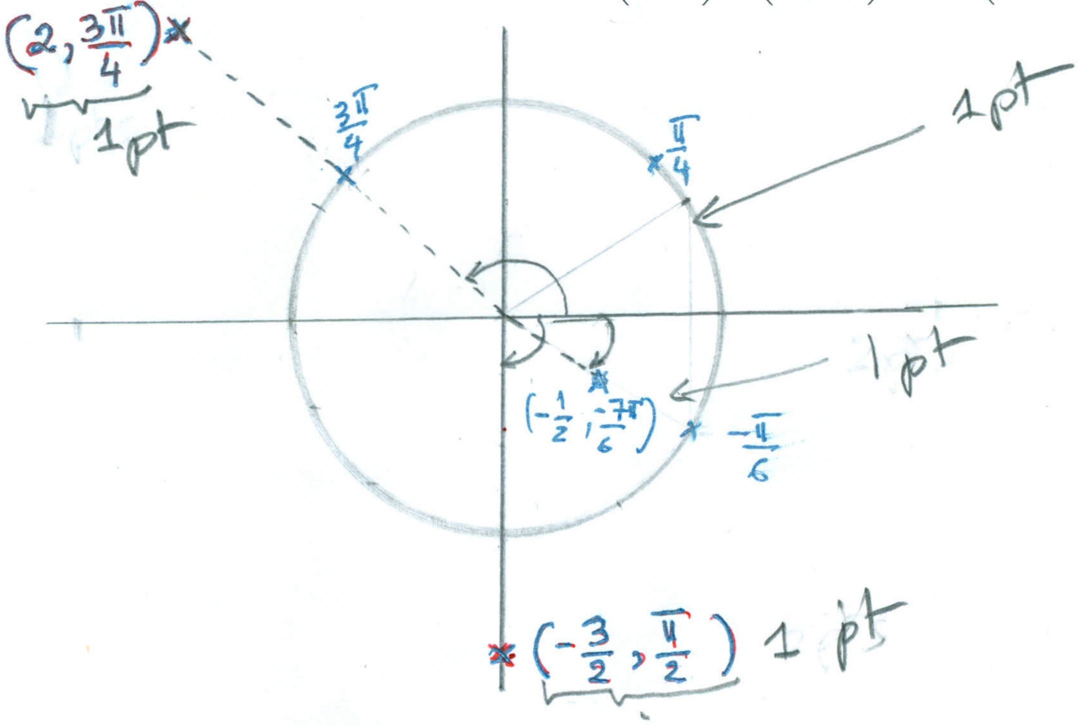
$$= 2\pi \int_0^{\frac{\pi}{2}} \sqrt{7}\sin^3\theta \sqrt{(3\sqrt{7})^2\cos^4\theta\sin^2\theta + (3\sqrt{7})^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 21\sin^3\theta \sqrt{\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 42\pi \int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta \qquad 6 \text{ pts}$$

$$= 42\pi \left( \left.\frac{\sin^5\theta}{5}\right|_0^{\frac{\pi}{2}} \right) = \frac{42\pi}{5} \qquad 2 \text{ pts}$$

4 pts (b) On the same polar coordinate system, draw the unit circle centered at the origin and plot the points whose polar coordinates are $\left(2, \frac{3\pi}{4}\right)$, $\left(-\frac{3}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{-1}{2}, -\frac{7\pi}{6}\right)$.

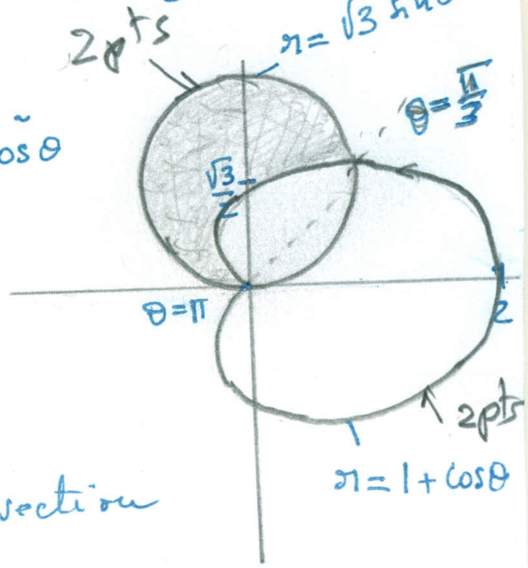


8 pts 4. (a) Sketch on the same polar coordinate system the curves

$$r = \sqrt{3}\sin\theta \quad \text{and} \quad r = 1 + \cos\theta.$$

and determine polar coordinates of the intersecting point(s) if any.

$r = \sqrt{3}\sin\theta$ is the circle centered at $(0, \frac{\sqrt{3}}{2})$ with radius $\frac{\sqrt{3}}{2}$.




Intersection points

1pt $\left\{ \sqrt{3}\sin\theta = 1 + \cos\theta \Rightarrow 3\sin^2\theta = 1 + 2\cos\theta + \cos^2\theta \right.$

$\Leftrightarrow 3(1 - \cos^2\theta) = 1 + 2\cos\theta + \cos^2\theta \Leftrightarrow$

$4\cos^2\theta + 2\cos\theta - 2 = 0 \Rightarrow$

$\cos\theta = -1$ or $\cos\theta = \frac{1}{2} \Rightarrow$

$\theta = \pi$ or $\theta = \frac{\pi}{3}$. 1pt

The polar coordinates of the intersection points are:

$\underbrace{(0, \pi)}_{1pt}$ and $\underbrace{\left(\frac{3}{2}, \frac{\pi}{3}\right)}_{1pt}$

8 pts (b) Find the area of the region that lies inside the curve $r = \sqrt{3}\sin\theta$ and outside the curve $r = 1 + \cos\theta$.

$A = \pi\left(\frac{\sqrt{3}}{2}\right)^2 - \text{Area inside both curves}$ 2pts

$= \frac{3\pi}{4} - \left[ \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{3}} 3\sin^2\theta \, d\theta}_{2pts} + \underbrace{\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1 + \cos\theta)^2 \, d\theta}_{2pts} \right]$

$= \frac{3\pi}{4} - \left( \frac{\pi}{4} - \frac{3\sqrt{3}}{16} + \frac{\pi}{2} - \frac{9\sqrt{3}}{16} \right)$ 1pt

$= \frac{3\pi}{4} - \frac{3\pi}{4} + \frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$ 1pt

8 pts

5. (a) Find an equation of the sphere whose diameter has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

center: $\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3, 2, 7)$ 2pts

Radius: $R = \sqrt{(2-3)^2 + (1-2)^2 + (4-7)^2}$
$= \sqrt{1+1+9} = \sqrt{11}$ 2pts

The equation of the sphere is

$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$ 4pts

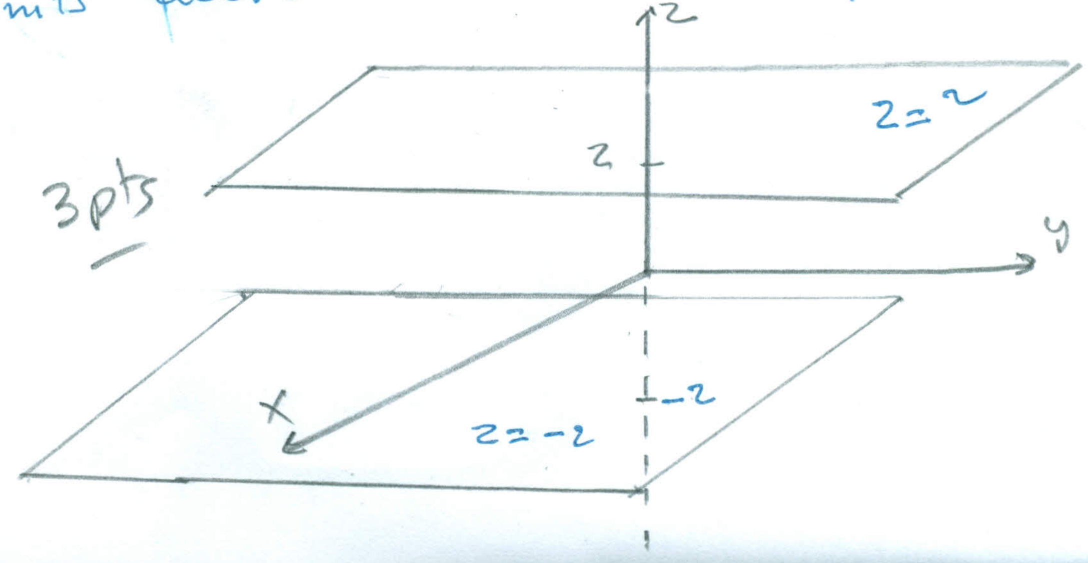
6pts

(b) Describe in words and sketch the region of the three-dimensional space represented by $z^2 = 4$.

3pts

$z^2 = 4$ means $z = 2$ or $z = -2$. The region is the union of the two planes $z = 2$ and $z = -2$. The planes are both parallel to the xy-planes and are two units above and below it respectively.

3pts



7pts

6. (a) Given the points $A(1,0,-1)$, $B(2,1,0)$, find the coordinates of the point $C$ such that $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have opposite directions and $|\overrightarrow{AC}| = 3\,|\overrightarrow{AB}|$.

We have $\underbrace{\overrightarrow{AC} = -3\,\overrightarrow{AB}}$, 2pts so if $C$ has coordinates $(x, y, z)$ then

$\langle x-1, y, z+1\rangle = -3\langle 1, 1, 1\rangle$ 2pts

i.e.: $\left.\begin{array}{l} x-1=-3 \\ y=1 \\ z+1=-3 \end{array}\right\} \Rightarrow \begin{array}{l} x=-2 \text{ 1pt} \\ y=1 \text{ 1pt} \\ z=-4 \text{ 1pt} \end{array}$

The point $C$ is given by $C(-2, 1, -4)$.

7pts (b) Consider the vectors

$$\vec{u} = \langle \sin x, 2, 1\rangle \text{ and } \vec{v} = \langle 1, \cos x, \sin x\rangle\,, \quad 0 \le x \le 2\pi.$$

Find the values of $x$ for which the two vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

2pts $\left\{ \begin{array}{l} \vec{u} \text{ and } \vec{v} \text{ are orthogonal if and only if} \\ \vec{u}\cdot\vec{v} = 0 \iff \underbrace{\sin x + 2\cos x + \sin x = 0} \iff \end{array}\right.$ 3pts

$\sin x = -\cos x \iff \tan x = -1 \Rightarrow x = \underbrace{\frac{3\pi}{4}}_{\text{1pt}}$ or $\underbrace{\frac{7\pi}{4}}_{\text{1pt}}$

$x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

7pts

7. (a) Find the volume of the parallelepiped with adjacent edges $DA$, $DB$, and $DC$, where $A(3,0,1), B(1,2,0), C(2,1,0), D(0,-1,0)$.

$\underbrace{\overrightarrow{DA} = \langle 3,1,1 \rangle}_{1pt}$ , $\underbrace{\overrightarrow{DB} = \langle 1,3,0 \rangle}_{1pt}$ , $\underbrace{\overrightarrow{DC} = \langle 2,2,0 \rangle}_{1pt}$

$\text{Volume} = \left| \overrightarrow{DA} \cdot (\overrightarrow{DB} \times \overrightarrow{DC}) \right|$ 2pts

$\overrightarrow{DB} \times \overrightarrow{DC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \langle 0, 0, -4 \rangle$ 1pt

$\text{Volume} = \left| 3\cdot(0) + 1(0) + 1(-4) \right| = |-4| = 4$ 1pt

7pts.

(b) Consider the vectors

$$\vec{a} = \langle -2, 1, 3 \rangle \text{ and } \vec{b} = \langle 3, 3, 4 \rangle.$$

If $\vec{u} = \mathbf{proj}_{\vec{a}}\vec{b}$ and $\vec{v} = \vec{b} - \vec{u}$, find the vector $\vec{w}$

$$\vec{w} = \frac{1}{2}\vec{u} + (5\vec{a}\cdot\vec{v})\vec{v} - \vec{a} \times (2\vec{u}).$$

$\vec{u} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} = \dfrac{-6+3+12}{\sqrt{4+1+9}}\langle -2,1,3 \rangle$

$= \dfrac{9}{14}\langle -2,1,3 \rangle$ 2pts

$\vec{a}$ and $\vec{u}$ are parallel $\Rightarrow \vec{a} \times 2\vec{u} = \vec{0}$ 2pts

$\vec{a}$ and $\vec{v}$ are perpendicular $\Rightarrow \vec{a}\cdot\vec{v} = 0$. 2pts

So $\vec{w} = \frac{1}{2}\vec{u} = \underbrace{\frac{9}{28}\langle -2,1,3 \rangle}_{1pt} = \langle -\frac{9}{14}, \frac{9}{28}, \frac{27}{28} \rangle$.