King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
2010-2011 (101)

Tuesday, December 7, 2010 **Allowed Time: 2 hours**

Name: Solution KEY

ID Number: ___________

Section Number:___________ **Serial Number:**___________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 10 different problems (6 pages + cover page)

Page Total	Grade	Maximum Points
Page 1		13
Page 2		18
Page 3		20
Page 4		21
Page 5		11
Page 6		17
Total		100

1. (7-points) Find the average value of the function $f(x) = 45 - 10\cos\left(\frac{\pi x}{12}\right)$ over the interval $[0, 24]$.

$$f_{ave} = \frac{1}{24-0}\int_0^{24}\left[45 - 10\cos\left(\frac{\pi x}{12}\right)\right]dx \quad ②$$

$$= \frac{1}{24}\left[45x - 10\cdot\frac{12}{\pi}\sin\left(\frac{\pi x}{12}\right)\right]\Big|_0^{24} \quad (1+2)$$

$$= \frac{1}{24}\left[(45\cdot 24 - 0) - 0\right]$$

$$= 45 \quad ②$$

2. (6 points) Write out the form of the partial fraction decomposition of $\frac{6x^3+3x^2+2x+3}{x^5+10x^3}$.
Do not determine the numerical values of the coefficients.

$$\frac{6x^3+3x^2+2x+3}{x^5+10x^3} = \frac{6x^3+3x^2+2x+3}{x^3(x^2+10)} \sim ① \text{ Factoring Deno.}$$

$$= \underbrace{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}}_{③} + \underbrace{\frac{Dx+E}{x^2+10}}_{②}$$

3. (8 points) Find $\int (\sec\theta - \sin\theta)^2\, d\theta$.

$$\int (\sec\theta - \sin\theta)^2\, d\theta = \int [\sec^2\theta - 2\tan\theta + \sin^2\theta]\, d\theta \qquad 2$$

$$= \tan\theta - 2\ln|\sec\theta| + \int \sin^2\theta\, d\theta \qquad 1+1$$

$$= \tan\theta - 2\ln|\sec\theta| + \tfrac{1}{2}\int [1 - \cos(2\theta)]\, d\theta \qquad 2$$

$$1+1 \longrightarrow \quad = \tan\theta - 2\ln|\sec\theta| + \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin(2\theta)\right] + C$$

$$= \tan\theta - 2\ln|\sec\theta| + \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta) + C$$

4. (10 points) Find $\int x^3(\ln x)^2 dx$.

Integration by Parts

$$u = (\ln x)^2 \qquad dv = x^3\, dx$$

$$du = \frac{2\ln x}{x}\, dx \qquad v = \frac{1}{4}x^4$$

$$\int x^3 (\ln x)^2\, dx = \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\int x^3 \ln x\, dx \qquad 4$$

By Parts again

$$u = \ln x \qquad dv = x^3\, dx$$

$$du = \frac{1}{x}\, dx \qquad v = \frac{1}{4}x^4$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\, dx\right] \qquad 4$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C \qquad 2$$

5. (10 points) Find $\int \sec^6(t)\tan^3(t)dt$.

$$\int \sec^6 t \tan^3 t \, dt =$$

$$= \int \sec^5 t \tan^2 t \cdot \sec t \tan t \, dt \quad (2)$$

$$= \int \sec^5 t \cdot (\sec^2 t - 1) \cdot \sec t \tan t \, dt \quad (2)$$

$$= \int (\sec^7 t - \sec^5 t) \cdot \sec t \tan t \, dt$$

$$u = \sec t \Rightarrow du = \sec t \tan t \, dt \quad (2)$$

$$= \int (u^7 - u^5) \, du \quad (2)$$

$$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C \quad (1)$$

$$= \frac{1}{8}\sec^8 t - \frac{1}{6}\sec^6 t + C \quad (1)$$

OR $\int \sec^6 t \tan^3 t \, dt$

$$= \int \sec^4 t \cdot \tan^3 t \cdot \sec^2 t \, dt \quad (2)$$

$$= \int (1 + \tan^2 t)^2 \cdot \tan^3 t \cdot \sec^2 t \, dt \quad (2)$$

$$= \int (\tan^3 t + 2\tan^5 t + \tan^7 t) \cdot \sec^2 t \, dt \quad (2)$$

$$u = \tan t \Rightarrow du = \sec^2 t \, dt \quad (2)$$

$$= \int (u^3 + 2u^5 + u^7) \, du \quad (1)$$

$$= \frac{1}{4}u^4 + \frac{1}{3}u^6 + \frac{1}{8}u^8 + C$$

$$= \frac{1}{4}\tan^4 t + \frac{1}{3}\tan^6 t + \frac{1}{8}\tan^8 t + C \quad (1)$$

6. (10 points) Find $\int \frac{1}{1 - 5\sin x}\, dx$. (Hint: Use the substitution $t = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$).

- $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2}{1+t^2}\, dt \quad (2)$

- $\int \frac{1}{1 - 5\sin x}\, dx = \int \frac{1}{1 - 5 \cdot \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

$$= \int \frac{2}{t^2 - 10t + 1}\, dt \quad (2)$$

$$= \int \frac{2}{(t-5)^2 - 24}\, dt \quad (2)$$

$$= \frac{2}{2\sqrt{24}} \ln\left|\frac{(t-5) - \sqrt{24}}{(t-5) + \sqrt{24}}\right| + C \quad (3)$$

$$= \frac{1}{\sqrt{24}} \ln\left|\frac{\tan(\frac{x}{2}) - 5 - \sqrt{24}}{\tan(\frac{x}{2}) - 5 + \sqrt{24}}\right| + C. \quad (1)$$

7. (10 points) Find $\displaystyle\int \frac{\sin^{-1} x}{x^2}\, dx$.

- By Parts: $u = \sin^{-1}x \qquad dv = \frac{1}{x^2}\,dx$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = \frac{-1}{x}$$

$$\int \frac{\sin^{-1}x}{x^2}\,dx = -\frac{\sin^{-1}x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,dx \qquad 3$$

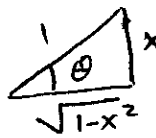
. Trig. Sub.: $x = \sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\Rightarrow dx = \cos\theta\, d\theta$ 2

$$= -\frac{\sin^{-1}x}{x} + \int \frac{1}{\sin\theta \cdot \cos\theta} \cdot \cos\theta\, d\theta$$

$$= -\frac{\sin^{-1}x}{x} + \int \csc\theta\, d\theta \qquad 1$$

$$= -\frac{\sin^{-1}x}{x} + \ln\left|\csc\theta - \cot\theta\right| + C \qquad 2$$

$$= -\frac{\sin^{-1}x}{x} + \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C \qquad 2$$



8. (11 points) Find $\displaystyle\int \frac{x^2}{(9+x^2)^{5/2}}\, dx$.

Trig. Sub: $x = 3\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\Rightarrow dx = 3\sec^2\theta\, d\theta$ 2

$$\int \frac{x^2}{(9+x^2)^{5/2}}\,dx = \int \frac{9\tan^2\theta}{(9\sec^2)^{5/2}} \cdot 3\sec^2\theta\, d\theta$$

$$= \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta \qquad 3$$

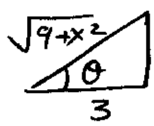
$$= \frac{1}{9}\int \sin^2\theta\cos\theta\, d\theta \qquad 2$$

$$= \frac{1}{9}\cdot\frac{1}{3}\sin^3\theta + C \qquad 2$$

$$= \frac{1}{27}\cdot\left(\frac{x}{\sqrt{9+x^2}}\right)^3 + C$$

or $\Big\}$ 2

$$= \frac{1}{27}\cdot\frac{x^3}{(9+x^2)^{3/2}} + C$$



9. (11 points) Find $\int \frac{x^3+x^2}{x^2-5x+6}\,dx$.

- By Long division, we get

$$\frac{x^3+x^2}{x^2-5x+6} = x+6 + \frac{24x-36}{x^2-5x+6} \qquad 3$$

- Decompose the fraction

$$\frac{24x-36}{x^2-5x+6} = \frac{24x-36}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \qquad 2$$

$$\Rightarrow \quad 24x-36 = A(x-3) + B(x-2) \qquad 1$$

$$x=3: \quad 72-36 = 0+B \Rightarrow B = 36 \qquad 1$$

$$x=2: \quad 48-36 = -A+0 \Rightarrow A = -12 \qquad 1$$

. Integrate

$$\int \frac{x^3+x^2}{x^2-5x+6}\,dx = \int x+6 + \frac{-12}{x-2} + \frac{36}{x-3} \quad dx$$

$$= \frac{1}{2}x^2+6x \;-12\ln|x-2| \;+36\ln|x-3| \;+C$$

$$1 \qquad 1 \qquad 1$$

10. Determine whether the integral is convergent or divergent. If it is convergent, find its value.

(a) (7 points) $\int_{-\infty}^{0} \frac{e^x}{2+e^x}\,dx$.

$$\int_{-\infty}^{0} \frac{e^x}{2+e^x}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{e^x}{2+e^x}\,dx \quad ②$$

$$= \lim_{t\to-\infty} \ln(2+e^x)\Big|_t^0 \quad ②$$

$$= \lim_{t\to-\infty} [\ln 3 - \ln(2+e^t)] \quad ①$$

$$= \ln 3 - \ln 2 \quad \text{or} \quad ①$$

$$= \ln(3/2)$$

Thus the Integral is Convergent & Its value is $\ln(3/2)$

①

(b) (10 points) $\int_0^2 \frac{1}{x^p}\,dx$, where p is a constant such that $p > 1$.

$$\int_0^2 \frac{1}{x^p}\,dx = \lim_{t\to0^+} \int_t^2 x^{-p}\,dx \quad ②$$

$$= \lim_{t\to0^+} \frac{x^{-p+1}}{-p+1}\Big|_t^2 \quad ③$$

$$= \lim_{t\to0^+} \frac{1}{-p+1}\left[2^{-p+1} - t^{-p+1}\right]$$

$$= \lim_{t\to0^+} \frac{1}{-p+1}\left[\frac{1}{2^{p-1}} - \frac{1}{t^{p-1}}\right] \quad ①$$

$$= \frac{1}{-p+1}\left[\frac{1}{2^{p-1}} - \infty\right] \quad \text{since } p>1 \quad ②$$

$$= \infty \quad \text{since } p>1 \quad ①$$

So the Integral is divergent

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