King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101-Calculus I
Exam I
Term (101)

Tuesday November 2, 2010 **Allowed Time: 2 hours**

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Section Number: 25 **Serial Number:** ____________

Instructions:

1. Write neatly and legibly. You may lose points for messy work.
2. **Show all you work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed in this exam.**
4. Make sure that you have 6 pages of problems (**Total of 12 Problems**)

Page Total	Grade	Maximum Points
Page 1	10	16
Page 2	11	16
Page 3	7	18
Page 4	14	16
Page 5	11	16
Page 6	8	18
Total	61	100

1. (8-points) The displacement (in meters) of a particle moving in straight line is given by $s(t) = t - \frac{1}{t}$, where t is measured in seconds. Use limits to find the instantaneous velocity of the particle at $t = 1$.

we must get $s(t)'$

we are supposing that the limit of $s(t)$ is exist

$\lim_{t\to 1} \frac{s(t)-s}{t-1}$ or $\lim_{h\to 0} \frac{s(t...)}{h}$

$$S(t)' = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \Rightarrow \lim_{t\to 1} \cancel{\frac{(t+h) - \frac{1}{(t+h)} - (t - \frac{1}{t})}{h}}$$

$$\lim_{t\to 1} \frac{1}{h}\left[\frac{(t+h)^2 - 1}{(t+h)} - \frac{t^2-1}{t}\right] \Rightarrow \lim_{t\to 1} \frac{1}{h}\left[\frac{t^2+2th+h^2-1}{t+h} - \frac{(t-1)(t+1)}{t}\right]$$

$$\lim_{t\to 1} \frac{1}{h}\left[\frac{t^3+2t^2h+th^2-t-t^2+1-t-1}{t^2+th}\right] \Rightarrow \lim_{t\to 1} \frac{1}{h}\left[\frac{t(t^2+2ht+h-2)}{t(t+h)}\right]$$

$$\lim_{t\to 1} \frac{1}{h}\left[\frac{(t^2+h+h)-2}{t+h}\right] \Rightarrow \lim_{t\to 1}\left[\frac{(t+h)(t+h)-2}{t+h}\right]$$

$$\lim_{t\to 1} \frac{1}{h}(t+h-2)$$

3

2. (8-points) Use continuity to evaluate the limit

$$\lim_{x\to -3} \arctan\left(\frac{x^2+7x+12}{x^2+5x+6}\right).$$

domain tan (∞, ∞)

① as we know that $\tan^{-1}$ is continuity for any numpe
So we can use this meothed:

$$\arctan\left(\lim_{t\to -3} \frac{(x+3)(x+4)}{(x+3)(x+2)}\right)$$

$$\arctan\left(\lim_{x\to -3} \frac{x+4}{x+2}\right) = \tan^{-1}\left(\frac{-3+4}{-3+2}\right) = \tan^{-1}(-1)$$

$$= \frac{2\pi}{3} \quad -\pi/4$$

$$②\ f(3) = \tan^{-1}\left(\frac{(x+3)(x+4)}{(x+3)(x+2)}\right) = \tan^{-1}\frac{x+4}{x+2} = \tan^{-1}\frac{-3+4}{-3+2} = \tan^{-1} -1$$

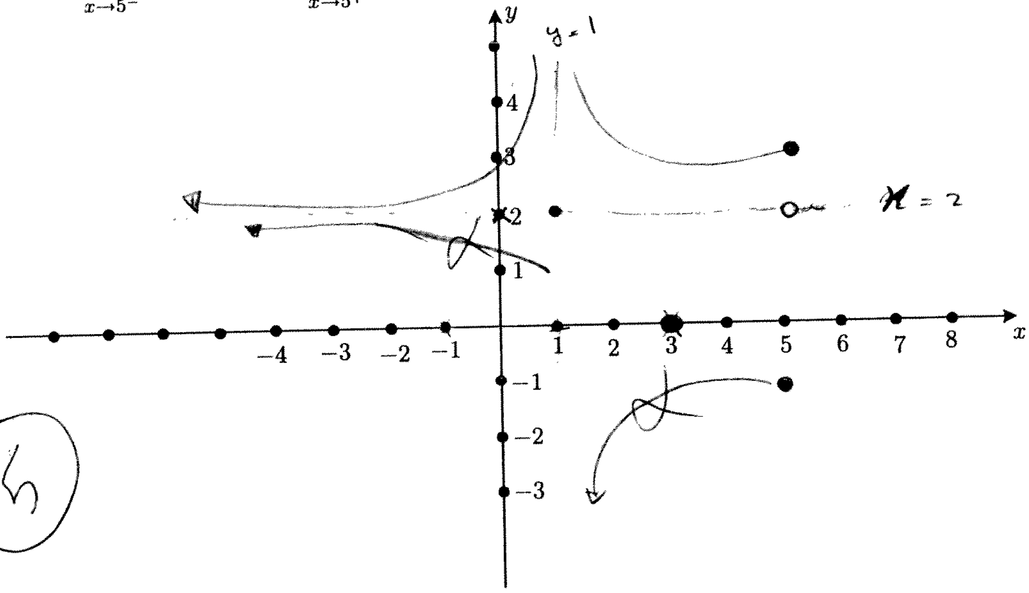
$$= \frac{2\pi}{3}$$

7

3. (8-points) Sketch the graph of a function f that satisfies the following conditions

$$\lim_{x\to-\infty} f(x) = 2\ ,\quad \lim_{x\to1^-} f(x) = \infty\ ,\quad \lim_{x\to1^+} f(x) = 2\ ,\quad f'(3) = 0\ ,$$

$$\lim_{x\to5^-} f(x) = 3\ ,\quad \lim_{x\to5^+} f(x) = -1\ ,\quad f(5) = 2\ ,\quad \lim_{x\to\infty} f(x) = -\infty\ .$$



6

4. (8-points) Use the Squeeze Theorem to show that $\lim_{x\to0^-} x^3 \sin\frac{\pi}{\sqrt[3]{x}} = 0.$

we know that

$$-1 \leqslant \sin\frac{\pi}{\sqrt[3]{x}} \leqslant 1$$

$$-x^3 \leqslant x^3 \sin\frac{\pi}{\sqrt[3]{x}} \leqslant x^3 .$$

2

$$\therefore \lim_{x\to0} -x^3 = 0 = \lim_{x\to0} x^3$$

$$\text{so: } \lim_{x\to0} x^3 \sin\frac{\pi}{\sqrt[3]{x}} = 0$$

by SqueezeTheorem ..

6

5. Given the function $f(x) = \dfrac{\sqrt{1+x^2} - \sqrt{1-x}}{x}$.

(a) (3-points) Find the domain of f in interval notation.

1 $\sqrt{1+x^2} \Rightarrow x^2+1 \geq 0 \Rightarrow x^2 \geq -1 \Rightarrow$ domain is: $(-\infty,\infty)$

2 $\sqrt{1-x} \Rightarrow -x+1 \geq 0 \Rightarrow x \leq 1 \Rightarrow$ domain is : $(-\infty, 1]$

3 x should not be equal zero $\Rightarrow (-\infty,0)\cup(0,\infty)$

So: The $f(x)$ domain is : $(-\infty, 0)\cup(0,1]$

(b) (8-points) Find the horizontal asymptotes to the graph of f.

using limits

$$\lim_{x\to\infty} \frac{(\sqrt{1+x^2} - \sqrt{1-x})(\sqrt{1+x^2}+\sqrt{1-x})}{x(\sqrt{1+x^2}+\sqrt{1-x})} \Rightarrow \lim_{x\to\infty} \frac{1+x^2 - 1-x}{x(\sqrt{1+x^2}+\sqrt{1-x})}$$

$$\lim_{x\to\infty} \frac{\cancel{x}(x-1)}{\cancel{x}(\sqrt{1+x^2}+\sqrt{1-x})} \Rightarrow \lim_{x\to\infty} \frac{x-1}{\sqrt{x^2+1}+\sqrt{1-x}}$$

Not in the domain

2

$\lim_{x\to-\infty}$

6. (7-points) Use limits to discuss the continuity of the greatest integer function $f(x) = [x]$ on the interval $[1,2]$.

$$\lim_{x\to 1^-} [x] = 0$$

$$\lim_{x\to 1^+} [x] = 1$$

DNE

$$\lim_{x\to 2^-} [x] = 1$$

$$\lim_{x\to 2^+} [x] = 2$$

DNE

The function does not Exist at the point 1 and does not exist at 2

2

7. (8-points) Let $f(x) = 5x^3 - 4x^2 + 5$ and $g(x) = x^3 + 2x^2 - 3x + 7$. Use the Intermediate Value Theorem to show that the equation $f(x) = g(x)$ has a solution between 1 and 2.

$f(x)$ and $g(x)$

Their domains are $(-\infty, \infty)$ what can't

becous they are a polynomial functions

$f(x) = 5x^3 - 4x^2 + 5$	$g(x) = x^3 + 2x^2 + 7$

$f(x) = g(x)$

$5x^3 - 4x^2 + 5 = x^3 + 2x^2 - 3x + 7$

$k(x) = 4x^3 - 6x^2 + 3x - 2 \Rightarrow (-\infty, \infty)$ also.

$k(1) = 4 - 6 + 3 - 2 = -1 < 0$

$k(2) = 40 - 24 + 6 - 2 = 20 > 0$

6

So: There are $c \in (1, 2)$

which make $\lim_{x \to c} k(c) = c$

* by Intermediate value Theorem

8. (8-points) Use the ϵ, δ definition of limit to prove that $\lim_{x \to 6} \left(\frac{x}{4} + 3\right) = \frac{9}{2}$.

$a = 6$ $\quad x = \left(\frac{x}{4} + 3\right)$

$L = \frac{9}{2}$

$|x - L| < \varepsilon$ if $0 < |x - a| < \delta$

$\left|\frac{x}{4} + 3 - \frac{9}{2}\right| < \varepsilon$ if $0 < |x - 6| < \delta$

$\left|\frac{x}{4} + \frac{-3}{2}\right| < \varepsilon$

$\frac{1}{4}|x - 6| < \varepsilon$

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$|x - 6 < 4\varepsilon$

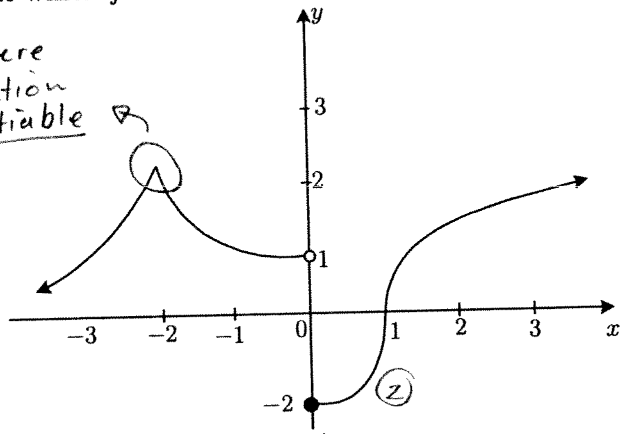
So: $\delta = |x - 6| = 4\varepsilon$

$\therefore \delta = 4\varepsilon$

9. (6-points) Use the given graph of a function f to state **with reasons**, the numbers at which f is not differentiable.

The corner here means the function is not differentiable

at $f(-1)$



The function here is not differentiable because the function is not continuous at the point $\underline{0}$

at $f(0)$

4

10. (10-points) Let $f(x) = \begin{cases} 5 - x\,, & \text{if } x < 4 \\ \dfrac{1}{5-x}\,, & \text{if } x \geq 4 \end{cases}$. Use limits to determine whether f is differentiable or not at 4 $\left[\text{Hint: Find } f'_-(4) \text{ and } f'_+(4)\right]$.

$$f'_+(4) = \lim_{x \to 4^+} \frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{5-(x+h)} - \frac{1}{5-x}}{h}$$

$$= \lim_{x \to 4^+} \frac{\frac{1}{5-x-h} - \frac{1}{5-x}}{h} = \lim_{x \to 4^+} \frac{\frac{5-x - 5-x-h}{(5-x-h)(5-x)}}{}$$

$$\lim_{x \to 4^+} \frac{-2x+h}{25-5x-5h-5x+2x-hx} = \lim_{x \to 4^+} \frac{-2x+h}{-hx-5h+x^2-10x+25} = 1$$

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$f'_-(4) \Rightarrow$ easy

we can solve it directly $= -1$

OR by lim

$$\lim \frac{f(x+h) - f(x)}{} = \frac{5-(x+h)-5+x}{} = 5-x-h-5+x$$

11. (8-points) Find, if any, all the vertical asymptotes to the graph of the function $f(x) = \dfrac{|x-1|}{x^3 - x^2 + x - 1}$. Use limits to justify your answer.

$$\lim_{x \to 1} \frac{|x-1|}{x^3 - x^2 + x - 1} \Rightarrow \lim_{x \to 1}$$

0

12. (10-points) Find the values of a so that the given function is continuous or has a removable discontinuity. and why?

$$f(x) = \begin{cases} a(a+2), & \text{if } x = 1 \\ a^3 x, & \text{if } x > 1 \\ 3a^2x^2 - 2ax, & \text{if } x < 1, \end{cases}$$

8

$f(1) = a(a+2)$

$\lim_{x \to 1^+} a^3 x = a^3$

$\lim_{x \to 1^-} 3a^2x^2 - 2ax = 3a^2 - 2a$

if it is continuous: $\lim_{x\to 1^-} = \lim_{x \to 1^+}$

So: $a^3 = 3a^2 - 2a$

$a^3 - 3a^2 + 2a = 0$

$a(a^2 - 3a + 2) = 0$

$a(a-1)(a-2) = 0$

$a = 0$, $a = 1$, $a = 2$

So: $f(x)$ is continuous

if $a = 0$

or

$a \cancel{=} 1$