King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101-Calculus I
Exam I
Term (101)

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Tuesday November 2, 2010 Allowed Time: 2 hours

Name: Solutions

ID Number: KEY

Section Number:________________ Serial Number:____________________

Instructions:

1. Write neatly and legibly. You may lose points for messy work.
2. **Show all you work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed in this exam.**
4. Make sure that you have 6 pages of problems **(Total of 12 Problems)**

Page Total	Grade	Maximum Points
Page 1		16
Page 2		16
Page 3		18
Page 4		16
Page 5		16
Page 6		18
Total		100

1. (8-points) The displacement (in meters) of a particle moving in straight line is given by $s(t) = t - \frac{1}{t}$, where t is measured in seconds. Use limits to find the instantaneous velocity of the particle at $t = 1$.

$$v\Big|_{t=1} = \lim_{h\to 0} \frac{s(1+h)-s(1)}{h} \; ① $$

$$= \lim_{h\to 0} \frac{1}{h}\left[1+h-\frac{1}{1+h}-0\right] \; ② \quad \Big\} \; \text{3 pts}$$

$$= \lim_{h\to 0} \frac{1}{h}\,\frac{(1+2h+h^2-1)}{1+h} \; ①$$

$$= \lim_{h\to 0} \frac{h(2+h)}{h(1+h)} \; ①$$

$$= \lim_{h\to 0} \frac{2+h}{1+h} \; ①$$

$$= 2 \text{ m/sec} \; ② \quad \Big\} \; \text{5 pts}$$

2. (8-points) Use continuity to evaluate the limit

$$\lim_{x\to -3} \arctan\left(\frac{x^2+7x+12}{x^2+5x+6}\right).$$

$$\lim_{x\to -3} \frac{x^2+7x+12}{x^2+5x+6} \; ① = \lim_{x\to -3} \frac{(x+3)(x+4)}{(x+3)(x+2)} \; ①$$

$$= \lim_{x\to -3} \frac{x+4}{x+2} = -1 \; ① \quad \Big\} \; \text{3 pts}$$

Also, arctan is continuous at -1 ..... 2 pts

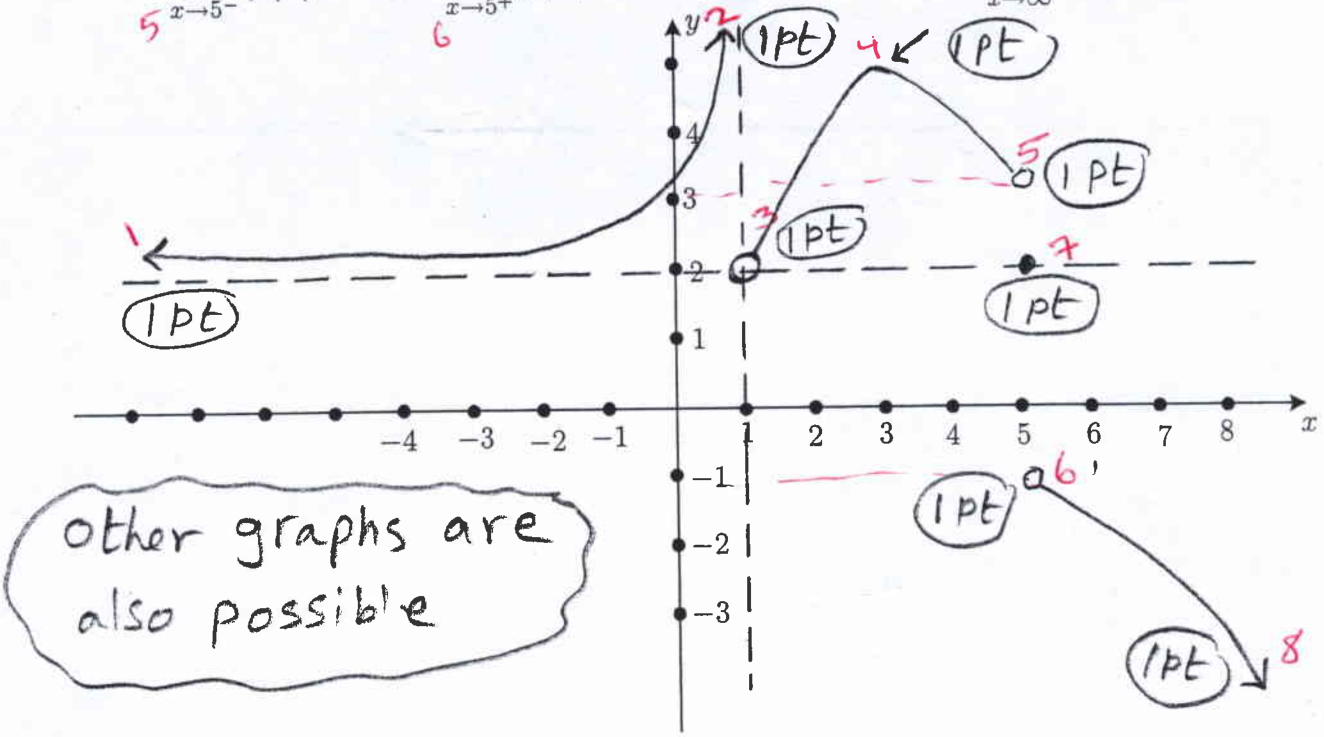
$$\Rightarrow \lim_{x\to -3} \arctan\left(\frac{x^2+7x+12}{x^2+5x+6}\right) = \arctan\left(\lim_{x\to -3} \frac{x^2+7x+12}{x^2+5x+6}\right) \; ①$$

$$= \arctan(-1) \; ① = -\frac{\pi}{4} \; ① \quad \Big\} \; \text{3 pts}$$

3. (8-points) Sketch the graph of a function f that satisfies the following conditions

$$\lim_{x\to-\infty} f(x)=2\,,\quad \lim_{x\to1^-} f(x)=\infty\,,\quad \lim_{x\to1^+} f(x)=2\,,\quad f'(3)=0\,,$$

$$\lim_{x\to5^-} f(x)=3\,,\quad \lim_{x\to5^+} f(x)=-1\,,\quad f(5)=2\,,\quad \lim_{x\to\infty} f(x)=-\infty\,.$$



4. (8-points) Use the Squeeze Theorem to show that $\displaystyle\lim_{x\to0^-} x^3\sin\frac{\pi}{\sqrt[3]{x}}=0.$

We know that $-1 \leq \sin\dfrac{\pi}{\sqrt[3]{x}} \leq 1$, $x\neq 0$, (1pt)

and $x\to 0^- \Rightarrow x<0$ (1) $\Longrightarrow$ (1)

$$-x^3 \geq x^3\sin\frac{\pi}{\sqrt[3]{x}} \geq x^3 \quad \text{----- (2 pts)}$$

(−2 pts if the above inequality is reversed)

But $\displaystyle\lim_{x\to0^-}(-x^3)=0$ (1) and $\displaystyle\lim_{x\to0^-} x^3=0$ (1) ---- (2 pt)

$\Rightarrow \displaystyle\lim_{x\to0^-} x^3\sin\frac{\pi}{\sqrt[3]{x}}=0$ (1)

by the squeezing Theorem (2) } (3 pts)

5. Given the function $f(x) = \dfrac{\sqrt{1+x^2} - \sqrt{1-x}}{x}$.

(a) (3-points) Find the domain of f in interval notation.

Must have $x \leq 1$ (1) and $x \neq 0$ (1)

$\Rightarrow$ domain $f = (-\infty, 0) \cup (0, 1]$ (1) } (3 pts)

(b) (8-points) Find the horizontal asymptotes to the graph of f.

Rationalize

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(1+x^2)-(1-x)}{x\left[\sqrt{1+x^2}+\sqrt{1-x}\right]} = \lim_{x\to-\infty} \frac{x(x+1)}{x\left[\sqrt{x^2+1}+\sqrt{1-x}\right]}$$ (3 pts)

Take common fac

$$= \lim_{x\to-\infty} \frac{x\left(1+\frac{1}{x}\right)}{|x|\left[\sqrt{1+\frac{1}{x^2}}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right]} = \lim_{x\to-\infty} \frac{-\left(1+\frac{1}{x}\right)}{\left[\sqrt{x^2+1}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right]}$$ (3 pts)

$= -1 \Rightarrow$ $y = -1$ is the only horizontal asymptote (2 pts)

−2 pts if they calculate $\lim_{x\to\infty} f(x)$

6. (7-points) Use limits to discuss the continuity of the greatest integer function $f(x) = [x]$ on the interval $[1, 2]$.

- $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} [\![x]\!] = 1 = f(1)$ (1)
  $\Rightarrow f$ is continuous from the right at 1 (1) } (2 pts)
- For $1 < c < 2 \Rightarrow \lim_{x\to c} f(x) = \lim_{x\to c} [\![x]\!] = 1 = f(c)$ (1)
  $\Rightarrow f$ is continuous on the open interval $(0,1)$ (1). } (2 pts)
- $\lim_{x\to2^-} f(x) = \lim_{x\to2^-} [\![x]\!] = 1 \neq f(2) = 2$ (1) (1)
  $\Rightarrow f$ is discontinuous from the left at 2 (1) } (3 pts)

7. (8-points) Let $f(x) = 5x^3 - 4x^2 + 5$ and $g(x) = x^3 + 2x^2 - 3x + 7$. Use the Intermediate Value Theorem to show that the equation $f(x) = g(x)$ has a solution between 1 and 2.

$g(x) - f(x) = 2 - 3x + 6x^2 - 4x^3$

Let $h(x) = f(x) - g(x) = 4x^3 - 6x^2 + 3x - 2$. 2 pts

h is continuous on the interval $[1, 2]$, 1 pt

and $h(1) = -1$, $h(2) = 12$ ------ 1 pt

Since $-1 < 0 < 12$, there is a number c in $(1, 2)$ such that $h(c) = 0$ by the Intermediate Value Theorem. 2 pts

Thus, there is a root of the equation $f(x) - g(x) = 0$, or $f(x) = g(x)$ in the interval $(1, 2)$ 2 pts

8. (8-points) Use the ϵ, δ definition of limit to prove that $\lim_{x \to 6} \left(\frac{x}{4} + 3\right) = \frac{9}{2}$.

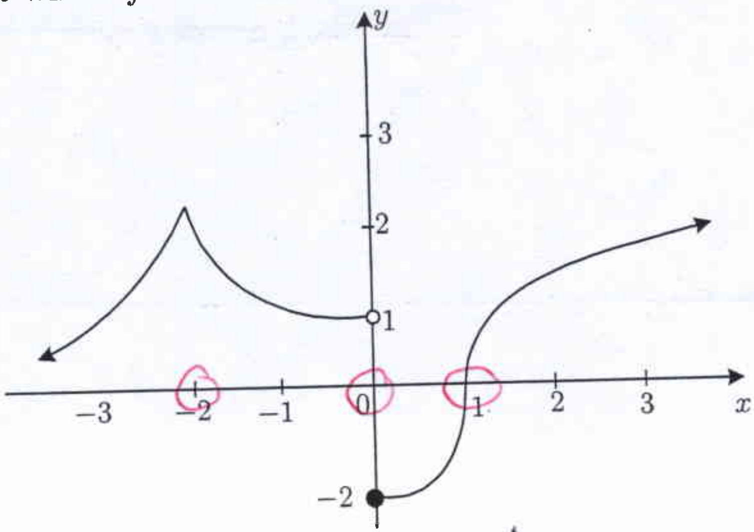
1

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 6| < \delta$, then $\left|\left(\frac{x}{4} + 3\right) - \frac{9}{2}\right| < \varepsilon$

$\iff \left|\frac{x}{4} - \frac{3}{2}\right| < \varepsilon \iff |x - 6| < 4\varepsilon$ 5 pts

Thus if we choose $0 < \delta \leq 4\varepsilon$, then the required result follows 3 pts

9. (6-points) Use the given graph of a function f to state **with reasons**, the numbers at which f is not differentiable.



f is not differentiable at :

(2) -2 (corner)

(2) 0 (discontinuity)

(2) 1 (vertical tangent)

2 pts each

10. (10-points) Let $f(x) = \begin{cases} 5-x\,, & \text{if } x < 4 \\ \dfrac{1}{5-x}\,, & \text{if } x \geq 4 \end{cases}$. Use limits to determine whether f is differentiable or not at 4 [Hint: Find $f'_-(4)$ and $f'_+(4)$].

$$f'_-(4) = \lim_{h\to 0^-} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-} \frac{(5-(4+h))-1}{h}$$

$$= \lim_{h\to 0^-}\left(-\frac{h}{h}\right) = \lim_{h\to 0^-}(-1) = -1$$

(1) (1) (1) (1) — 4 pts

$$f'_+(4) = \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{5-(4+h)} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{1-(1-h)}{1-h}}{h} = \lim_{h\to 0^+} \frac{1}{1-h} = 1$$

(1) (1) (1) (1) — 4 pts

$\Longrightarrow$ (1) $f'_-(4) \neq f'_+(4)$ which means that f is (1) not differentiable at 4 — 2 pts

11. (8-points) Find, if any, all the vertical asymptotes to the graph of the function $f(x) = \dfrac{|x-1|}{x^3 - x^2 + x - 1}$. Use limits to justify your answer.

$$f(x) = \frac{|x-1|}{x^2(x-1)+(x-1)} = \frac{|x-1|}{(x-1)(x^2+1)} \quad 1$$

$\Rightarrow$ f is discontinuous at 1 1 $\notin$ dom f } 2 pts

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{-(x-1)}{(x-1)(x^2+1)} = \lim_{x\to1^-} \frac{-1}{x^2+1} = -\frac{1}{2} \quad 1 \quad 2\text{ pts}$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{(x-1)}{(x-1)(x^2+1)} = \lim_{x\to1^+} \frac{1}{x^2+1} = \frac{1}{2} \quad 1 \quad 2\text{ pts}$$

$\Rightarrow$ f has no infinite discontinuity at 1

$\Rightarrow$ no vertical asymptotes to the graph of f. } 2 pts

12. (10-points) Find the values of a so that the given function is continuous or has a removable discontinuity

$$f(x) = \begin{cases} a(a+2)\,, & \text{if } x = 1 \\ a^3 x\,, & \text{if } x > 1 \\ 3a^2x^2 - 2ax\,, & \text{if } x < 1, \end{cases}$$

1

$$\lim_{x\to1^-} f(x) \overset{1}{=} \lim_{x\to1^+} f(x) \Leftrightarrow 3a^2 - 2a = a^3 \Leftrightarrow a(a^2-3a+2)=0,$$

$$\Leftrightarrow a(a-1)(a-2)=0 \Leftrightarrow \boxed{a = 0, 1, 2} \quad 2 \quad 4\text{ pts}$$

$\boxed{\text{For } a=0}$ $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 0 = f(1)$ } 2 pts

$\Rightarrow$ f is continuous for $a=0$

$\boxed{\text{For } a=1}$ $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 1 \neq f(1) = 3$

$\Rightarrow$ f has a removable discontinuity for $a=1$ } 2 pts

$\boxed{\text{For } a=2}$ $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = 8 = f(2) \Rightarrow$ f is continuous for $a=2$ } 2 pts