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ABSTRACT

We present here some algebraic properties of PLRD functions that can be used to test whether a given bivariate density has the PLRD property with little or no differentiation.

1. Introduction

A distribution of a pair of random variables (X_1, X_2) is *positively likelihood ratio dependent* (PLRD) if its density function f satisfies

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1) \tag{1}$$

whenever $x_1 > x_2$ and $y_1 > y_2$. This is referred to as the TP_2 property in Karlin [4] (see also Tong [2, pp. 19-20] and Olkin and Liu [4]). As stated in [2], it appears to be the strongest condition for studying the positive dependence of random variables X_1, X_2. Our purpose here is to present several properties that can be used to test whether a pair of random variables satisfies PLRD. For the sake of generality, and for brevity of notation, we will say that a function f of two variables (f not necessarily a density) is PLRD when it satisfies (1). Another reason why we chose not to restrict our discussion to density functions in our definition of PLRD's is this: while some density functions are easily verified to be PLRD $-$ see for example Olkin and Riu [4]$-$ this is clearly not the case for others. To deal with such densities, one can use the criterion below (or its logarithmic version), but computing partial derivatives for complicated functions and establishing the inequality (2) below for these derivatives may prove cumbersome and time-consuming. The idea we wish to present here is to decompose a given function f into PLRD functions (keeping in mind of course that these components need not be density functions) and then use Propositions 3 and 4 below to establish that f is PLRD. The properties themselves can be proved either directly from the PLRD definition or by a straightforward application of the following criterion.

2. A PLRD Criterion

A version of this criterion (using logarithms) already appears in Holland and Wang [2]. We include it here for completeness. Let f be a positive function such that $\dfrac{\partial f(x,y)}{\partial x}$, $\dfrac{\partial f(x,y)}{\partial y}$, and $\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$ are continuous and satisfy the inequality

$$f(x,y)\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq \frac{\partial f(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y}, \tag{2}$$

and consider the function $h(y) = \dfrac{\partial f(x,y)/\partial x}{f(x,y)}$ (where x is assumed to be constant). Then the derivative $h'(y)$ is nonnegative, and so h is an increasing function (using the basic fact that a differentiable function (of a single variable) is increasing if and only if its derivative is nonnegative). Since $h(y_1) \geq h(y_2)$ whenever $y_1 > y_2$, we obtain $f(x,y_2)\dfrac{\partial f(x,y_1)}{\partial x} \geq f(x,y_1)\dfrac{\partial f(x,y_2)}{\partial x}$ for all x. This means the function $g(x) := \dfrac{f(x,y_1)}{f(x,y_2)}$ is increasing whenever $y_1 > y_2$, that is, $g(x_1) = \dfrac{f(x_1,y_1)}{f(x_1,y_2)} \geq g(x_2) = \dfrac{f(x_2,y_1)}{f(x_2,y_2)}$ whenever $x_1 > x_2$ and $y_1 > y_2$. Hence f is PLRD. Conversely, it is clear, by reversing the steps above, that if f is PLRD then it satisfies (2). Thus we have,

Proposition 1. (See [2].) *Let f be a positive function such that* $\dfrac{\partial f(x,y)}{\partial x}$, $\dfrac{\partial f(x,y)}{\partial y}$, *and* $\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$ *are continuous. Then f is PLRD if and only if* $f(x,y)\dfrac{\partial^2 f(x,y)}{\partial x \partial y} \geq \dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial f(x,y)}{\partial y}$, *for all x, y. In particular, if f is PLRD and* $\dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial f(x,y)}{\partial y} \geq 0$, *then* $\dfrac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$. ■

Note that an alternative way of stating Proposition 1 is to write $f(x,y) = e^{g(x,y)}$ and observe that $f(x,y)\dfrac{\partial^2 f(x,y)}{\partial x \partial y} \geq \dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial f(x,y)}{\partial y}$ is equivalent to having $\dfrac{\partial^2 g(x,y)}{\partial x \partial y} \geq 0$. In other words, $e^{g(x,y)}$ is PLRD precisely when $\dfrac{\partial^2 g(x,y)}{\partial x \partial y} \geq 0$. This suggests a close connection of the PLRD property to convexity (a function g of one variable is convex if $g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda)g(x_2)$ whenever $0 \leq \lambda \leq 1$). Indeed, using Proposition 1 and straightforward algebraic manipulations, one can show

Proposition 2. (i) *Let f be a function of two variables with continuous second-order partial derivatives. If $e^{f(x,y)}$ is PLRD then $f(x,x)$ is convex.*
(ii) *A twice differentiable function g is convex if and only if $e^{g(x+y)}$ is PLRD.* ■

2. Algebraic Properties

In the next propositions, we give several properties of PLRD functions. They can be used, in particular, to construct new PLRDs from old ones, and to check whether a given bivariate function is PLRD with minimal recourse to partial derivatives. To present a more coherent set of properties, let us call f NLRD (for *negatively likelihood ratio dependent)* if

$$f(x_1, y_1)f(x_2, y_2) \leq f(x_1, y_2)f(x_2, y_1) \tag{3}$$

whenever $x_1 > x_2$ and $y_1 > y_2$. Such functions are called RR_2 (for *reverse rule of order* 2) in [domma, 2009]. Clearly, a nonzero function f is PLRD if and only if $1/f$ is NLRD. We immediately have the following.

Properties

(P1) *$g(x, y)$ is PLRD (resp. NLRD) if and only if $g(y, x)$ is PLRD (resp. NLRD).*

(P2) $g(x, y)h(x, y)$ is *PLRD (resp. NLRD) whenever both g and h are PLRD (resp. NLRD). In particular, if u and v are functions of single variables then $u(x)v(y)$ is both PLRD and NLRD.*

(P3) *If g is PLRD and u and v are both increasing or both decreasing functions of single variables, then $g(u(x), v(y))$ is PLRD. In particular, $u(x) - v(y)$ and $u(x)v(y) + C$ are PLRD whenever C is a nonnegative constant and u and v are both increasing or both decreasing.*

(P4) *If g is PLRD, and if u (resp. v) is an increasing (resp. decreasing) function of a single variable, then $g(u(x), v(y))$ is NLRD. In particular, $u(x) + v(y)$ and $u(x)v(y) - C$ are NLRD whenever C is a nonnegative constant and u and v are both increasing or both decreasing functions of single variables.*

(P5) *If g is PLRD (resp. NLRD) then $g + C$ is PLRD (resp. NLRD) whenever C is a constant with* $C\dfrac{\partial^2 g(x,y)}{\partial x \partial y} \geq 0$ (resp. ≤ 0).

(P6) $(g(x, y))^a$ *is PLRD if g is PLRD and $a \geq 0$.* ■

Less obvious perhaps is the next proposition. (Note that since a number of densities involve hypergeometric functions, this result can be quite useful when checking whether they are PLRD.)

Proposition 3. *Let f be a positive PLRD such that* $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, *and* $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ *are continuous and* $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$ *have the same sign. Then, for any power series* $g(t) = \sum_{n=0}^{\infty} a_n t^n$ *with nonnegative coefficients,* $g(f(x,y))$ *is PLRD whenever* $f(x,y)$ *is in the interval of convergence of* g.

Proof. We first show that the proposition is true when g is a polynomial with nonnegative coefficients. For that we use induction on the degree of g. If g is constant, then, clearly, $g(f(x,y))$ is PLRD. Suppose now that the proposition is true for all polynomials of degree n (and with nonnegative coefficients), and that g has degree $n+1$. Then, for some nonnegative constant c and some polynomial h of degree n and with nonnegative coefficients, $g(t) = c + th(t)$. By (P2) and the induction hypothesis, $f(x,y)h(f(x,y))$ is PLRD. Since $\frac{\partial f(x,y)h(f(x,y))}{\partial x}$ and $\frac{\partial f(x,y)h(f(x,y))}{\partial y}$ are easily seen to be nonnegative (recall that all coefficients of f are nonnegative and $\frac{\partial f(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y} \geq 0$), we infer from Proposition 1 that $\frac{\partial^2 f(x,y)h(f(x,y))}{\partial x \partial y} \geq 0$. Hence $g(f(x,y)) = c + f(x,y)h(f(x,y))$ is PLRD by (P5), as required.
Finally, if $g(t) = \sum_{n=0}^{\infty} a_n t^n$ is a power series with nonnegative coefficients, then $g(f(x,y)) = \lim_{n \to \infty} \gamma_n(x,y)$, where the polynomials $\gamma_n(x,y) = \sum_{k=0}^{n} a_k (f(x,y))^k$ are the partial sums. By the first part, each $\gamma_n(x,y)$ is PLRD and hence satisfies (1). This inequality is obviously preserved when $n \to \infty$, and the proof is complete. ■

For stochastic rearrangement inequalities, an excellent reference is, for example, [1]. The proof of the result below is given here to illustrate an application of the PLRD property.

Proposition 4. *Let f be a positive PLRD. Then, for any permutation σ of $\{1, 2, ..., n\}$ and decreasing sequences $x_1 > x_2 > \cdots > x_n$, $y_1 > y_2 > \cdots > y_n$, we have* $\prod_{i=1}^{n} f(x_i, y_i) \geq \prod_{i=1}^{n} f(x_i, y_{\sigma(i)})$.

Proof. The statement is clearly true for $n = 1$. Suppose it is true for $n = k$, let σ be a permutation of $\{1, 2, ..., k, k+1\}$, and let $x_1 > x_2 > \cdots > x_{k+1}$, $y_1 > y_2 > \cdots > y_{k+1}$. If $\sigma(k+1) = k+1$, then the restriction of σ to $\{1, 2, ..., n\}$ is a permutation, and the induction hypothesis implies the required inequality $\prod_{i=1}^{k+1} f(x_i, y_i) \geq f(x_{k+1}, y_{k+1}) \prod_{i=1}^{k} f(x_i, y_{\sigma(i)})$.

Suppose now that $\sigma(k+1) = j \leq k$. Then, with the convention that empty products are equal to 1, we can apply the induction hypothesis on the sequences $x_1 > x_2 > \cdots > x_n$, $y_1 > y_2 > \cdots > y_{j-1} > y_{j+1} > \cdots > y_{k+1}$ to obtain

$$\begin{aligned}
\prod_{i=1}^{k+1} f(x_i, y_{\sigma(i)}) &\leq f(x_{k+1}, y_{\sigma(k+1)}) \prod_{i\in\{1,\dots,j-1\}} f(x_i, y_i) \prod_{i=j}^{k} f(x_i, y_{i+1}) \\
&\leq f(x_k, y_{k+1}) f(x_{k+1}, y_{\sigma(k+1)}) \prod_{i\in\{1,\dots,j-1\}} f(x_i, y_i) \prod_{i=j}^{k-1} f(x_i, y_{i+1}) \\
&\leq f(x_k, y_{k+1}) f(x_{k+1}, y_{\sigma(k+1)}) \prod_{i\in\{1,\dots,j-1\}} f(x_i, y_i) \prod_{i=j}^{k-1} f(x_i, y_{i+1}) \\
&\leq f(x_k, y_{\sigma(k+1)}) f(x_{k+1}, y_{k+1}) \prod_{i\in\{1,\dots,j-1\}} f(x_i, y_i) \prod_{i=j}^{k-1} f(x_i, y_{i+1}),
\end{aligned}$$

where the last inequality follows from the PLRD property of f applied to the sequences $x_k > x_{k+1}$, $y_{\sigma(k+1)} > y_{k+1}$. We thus obtain,

$$\prod_{i=1}^{k+1} f(x_i, y_{\sigma(i)}) \leq f(x_k, y_{\sigma(k+1)}) \prod_{i\in\{1,\dots,j-1,k+1\}} f(x_i, y_i) \prod_{i=j}^{k-1} f(x_i, y_{i+1}).$$

Continuing with this "exchange" argument, we deduce that

$$\begin{aligned}
\prod_{i=1}^{k+1} f(x_i, y_{\sigma(i)}) &\leq f(x_{j+1}, y_{\sigma(k+1)}) f(x_j, y_{j+1}) \prod_{i\in\{1,\dots,j-1,j+2,\dots,k+1\}} f(x_i, y_i) \\
&\leq f(x_j, y_{\sigma(k+1)}) f(x_{j+1}, y_{j+1})) \prod_{i\in\{1,\dots,j-1,j+2,\dots,k+1\}} f(x_i, y_i) \leq \prod_{i=1}^{k+1} f(x_i, y_i),
\end{aligned}$$

as required. ■

4. Examples

We now present a number of applications to illustrate the usefulness of our results. The first one, a variant of Hardy, Littlewood and Polya's rearrangement inequality, is given to show that the PLRD property can be used directly to derive certain inequalities.

4.1. For all $x_1 > x_2 > \cdots > x_n > 0$, $y_1 > y_2 > \cdots > y_n > 0$, and any permutation σ of $\{1, 2, ..., n\}$ we have

$$\prod_{i=1}^{n} \ln(1 + x_i y_i) \geq \prod_{i=1}^{n} \ln(1 + x_i y_{\sigma(i)}).$$

To see this, let $f(x, y) = \dfrac{1}{\ln(1 + xy)}$, so that f is positive and has continuous derivatives of all orders on the region $R = (0, \infty) \times (0, \infty)$. As is easily checked, $f(x, y)\dfrac{\partial^2 f(x, y)}{\partial x \partial y} \geq \dfrac{\partial f(x, y)}{\partial x}\dfrac{\partial f(x, y)}{\partial y}$ on R, so f is PLRD and the inequality follows from Propositions 1 and 5. (Alternatively, we could start with the function $g(x, y) = \ln(1 + xy)$ and show that $g(x, y)\dfrac{\partial^2 g(x, y)}{\partial x \partial y} \leq \dfrac{\partial g(x, y)}{\partial x}\dfrac{\partial g(x, y)}{\partial y}$.)

4.2. The following density function appears in Joarder [3].

$$f(x, y) = \frac{(xy)^{m/2-1}(1 - \rho^2)^{-m/2}}{2^m \Gamma^2(m/2)} \exp\left(-\frac{x + y}{2 - 2\rho^2}\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 xy}{(2 - 2\rho^2)^2}\right),$$

where $x > 0$, $y > 0$, $0 < \rho < 1$, $m \geq 1$, and ${}_0F_1(b; z) = \sum_{k \geq 0} \dfrac{\Gamma(b)}{\Gamma(b + k)} \dfrac{z^k}{k!}$. It is the density of random variables X and Y that have a correlated bivariate chi-square distribution, each with m degrees of freedom. Clearly, $f(x, y) = Ah(x)h(y)g(t)$, where $A = \dfrac{(1 - \rho^2)^{-m/2}}{2^m \Gamma^2(m/2)}$, $h(x) = x^{m/2-1} e^{-\frac{x}{2-2\rho^2}}$, and $g(t) = \sum_{k \geq 0} \dfrac{\Gamma(m/2)}{\Gamma(m/2 + k)} \dfrac{t^k}{k!}$. By (P1), (P4) and Proposition 3 (since all the coefficients of the series are positive), we obtain that f is PLRD.

4.3. McKay's bivariate gamma distribution (McKay [5]) has density

$$f(x, y) = \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} x^{a-1}(y - x)^{b-1} e^{-cy},$$

where $y > x > 0$, $a, b, c > 0$. Using (P1), (P2), (P4) and (P5), we obtain that f is PLRD when $b \geq 1$.

4.4. Consider the bivariate Rodriguez-Burr III density (see p.3 of [8])

$$\begin{aligned} f(x, y) = {} & \beta\lambda\gamma\delta\theta x^{-\theta-1} y^{-\delta-1}(1 + \alpha\lambda\gamma x^{-\theta} y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta})^{-\beta-2} \times \\ & \{(\beta + 1)(1 + \alpha\lambda x^{-\theta})(1 + \alpha\gamma y^{-\delta}) - \alpha\left(1 + \alpha\lambda\gamma x^{-\theta} y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta}\right)\}. \end{aligned}$$

where $\beta, \lambda, \gamma, \delta, \theta$ are all positive, $0 \leq \alpha \leq \beta + 1$, and x, y are nonnegative. As indicated in [8], Holland-Wang criterion can be used to show that f is PLRD for

$\alpha < 1$. However, without much effort (and with no differentiation), this can be deduced from the properties of PLRD functions as follows. First, notice that if $g = \beta\lambda\gamma\delta\theta x^{-\theta-1}y^{-\delta-1}$, $h = 1 + \alpha\lambda\gamma x^{-\theta}y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta}$, $u = 1 + \alpha\lambda x^{-\theta}$, and $v = 1 + \alpha\gamma y^{-\delta}$, then u, v are decreasing, $\alpha h = uv + \alpha - 1$, and

$$\begin{aligned} f(x,y) &= gh^{-\beta-2}\{(\beta+1)(1+\alpha(h-1)) - \alpha h\} \\ &= gh^{-\beta-2}(\beta uv + 1 - \alpha). \end{aligned}$$

If $\alpha = 0$, then $f(x,y) = (\beta+1)gh^{-\beta-2}$ and $h = 1 + \lambda x^{-\theta} + \gamma y^{-\delta}$. Now $1 + \lambda x^{-\theta}$ and $\gamma y^{-\delta}$ are both decreasing and NLRD by (P2), so h is NLRD by (P4) and $h^{-\beta-2}$ is PLRD. Hence, as g is PLRD (and NLRD) by (P2), we get f is PLRD. For $0 < \alpha < 1$, $h = \dfrac{uv + \alpha - 1}{\alpha}$, so h is NLRD by (P4) and hence $h^{-\beta-2}$ is PLRD, while $\beta uv + 1 - \alpha$ is PLRD by (P3). Thus, by (P2), f is PLRD.

Let us note that an alternative way to show f is PLRD when $\alpha = 0$ is to denote each f by f_α (considering α as an indexing parameter) and concluding that f_0 is PLRD from the fact that $f_0 = \lim_{\alpha \to 0^+} f_\alpha$ and that each f_α $(0 < \alpha < 1)$ is PLRD. ■

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