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Correlated Sample Variances

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Abstract The ratio of correlated sample variances is important in testing equality of variances under correlation. In this technical note, we study the raw moments, mean centered moments, coefficient of variation, and coefficient of skewness and kurtosis of the ratio of correlated sample variances. Special cases of the results for the uncorrelated case match with that of the ratio of independent sample variances.

1. Introduction

Let S_1^2 and S_2^2 be sample variances based on a sample of size $N = m + 1$ from a bivariate normal distribution with unknown means, unknown variances σ_1^2 and σ_2^2 , and correlation coefficient ρ ($-1 < \rho < 1$). The ratio of sample variances S_1^2 / S_2^2 or $H = U / V$ where $U = mS_1^2 / \sigma_1^2$ and $V = mS_2^2 / \sigma_2^2$ is important in testing equality of true variances under correlation. In this technical report, we study we study the raw moments, mean centered moments, coefficient of skewness and kurtosis of H .

2. The Density Function

The following density function of H was derived by Bose (1935) and Finney (1938).

Theorem 2.1 The density function of $H = U / V$ is given by

$$f_H(h) \propto \frac{h^{(m-2)/2}}{(1+h)^m} \left(1 - \frac{4\rho^2 h}{(1+h)^2} \right)^{-(m+1)/2}, \quad h > 0, \quad (2.1)$$

where $m > 2$ and $-1 < \rho < 1$. We will denote the distribution of H by $F(m, m; \rho)$.

In what follows we will be using the following formulae.

The product of k consecutive integers is denoted by the following:

$$k_{\{a\}} = k(k+1)\cdots(k+a-1), \quad k^{\{a\}} = k(k-1)\cdots(k-a+1), \quad (2.2)$$

with $k_{\{0\}} = 1$ and $k^{\{0\}} = 1$. The hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is defined by

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_{\{k\}}(a_2)_{\{k\}} \cdots (a_p)_{\{k\}} z^k}{(b_1)_{\{k\}}(b_2)_{\{k\}} \cdots (b_q)_{\{k\}} k!}, \quad (2.3)$$

3. Moments, Skewness and Kurtosis of the Distribution

Theorem 3.1 Let H have a correlated F distribution with density function (2.1). Then the a -th moment of H is given by

$$E(H^a) = \frac{\Gamma(a + (m/2))\Gamma(-a + (m/2))}{\Gamma^2(m/2)} {}_2F_1(-a, a; m/2; \rho^2), \quad (3.1)$$

where $m > 2a$, $-1 < \rho < 1$ and ${}_2F_1(a, b; c; \rho^2)$ is defined by (2.3).

Proof. By definition, it follows from (2.1) that

$$E(H^a) = \frac{2^{m-1}(1-\rho^2)^{m/2}}{B\left(\frac{1}{2}, \frac{m}{2}\right)} \int_0^{\infty} \frac{h^{a+(m/2)-1}}{(1+h)^m} \left(1 - \frac{4\rho^2 h}{(1+h)^2}\right)^{-(m+1)/2} dh.$$

The proof is completed by expanding the last term in the above integrand by binomial expansion, followed by simple definite integrals based on the usual F -distribution, and then by using the hypergeometric function (2.3).

Corollary 3.1 Let H have a correlated F distribution with density function (2.1). Then for nonnegative integers a , the a -th moment of H is given by

$$E(H^a) = \frac{(m/2)_{\{a\}}}{((m/2)-1)_{\{a\}}} \sum_{k=0}^a \binom{a}{k} \frac{(1-a-k)_{\{k\}}}{(m/2)_{\{k\}}} \rho^{2k}, \quad (3.2)$$

$m > 2a$, $-1 < \rho < 1$, $k^{\{a\}}$ and $k_{\{a\}}$ are defined by (2.2).

In particular, the first four raw moments are given by

$$E(H) = \frac{m - 2\rho^2}{m - 2}, \quad m > 2, \quad (3.3)$$

$$E(H^2) = \frac{1}{(m-2)(m-4)} \left(24\rho^4 - 8(m+2)\rho^2 + m(m+2) \right), \quad m > 4, \quad (3.4)$$

$$E(H^3) = \frac{m^3 - (18\rho^2 - 6)m^2 + 4(3\rho^2 - 2)(12\rho^2 - 1)m - 48\rho^2(10\rho^4 - 12\rho^2 + 3)}{(m-2)(m-4)(m-6)}, \quad m > 6 \quad (3.5)$$

and

$$E(H^4) = \frac{1}{(m-2)(m-4)(m-6)(m-8)} \times \left(m^4 - 4(8\rho^2 - 3)m^3 + 4(120\rho^4 - 96\rho^2 + 11)m^2 - 16(240\rho^6 - 300\rho^4 + 88\rho^2 - 3)m + 384(35\rho^6 - 60\rho^4 + 30\rho^2 - 4)\rho^2 \right), \quad m > 8. \quad (3.6)$$

If $\rho = 0$, then

$$E(H) = \frac{m}{m-2}, \quad m > 2,$$

$$E(H^2) = \frac{m(m+2)}{(m-2)(m-4)}, \quad m > 2,$$

$$E(H^3) = \frac{m^3 + 6m^2 + 8m}{(m-2)(m-4)(m-6)} = \frac{m(m+2)(m+4)}{(m-2)(m-4)(m-6)}, \quad m > 6,$$

$$E(H^4) = \frac{m^4 + 12m^3 + 44m^2 + 48m}{(m-2)(m-4)(m-6)} = \frac{m(m+2)(m+4)(m+6)}{(m-2)(m-4)(m-6)(m-8)}, \quad m > 8,$$

which are all matching with the moments of $F(m, m)$, central F -distribution.

The mean corrected moments of a random variable X are given by $\mu_a = E(X - \mu)^a$, $a = 1, 2, \dots$. The corrected moments of H are given by the following theorem:

Theorem 3.2 Let H have a correlated F distribution with density function (2.1). Then the second to the fourth centered moment of H are respectively given by

$$\mu_2 = \frac{4(1-\rho^2)(m^2 - m + 8\rho^2 - 5m\rho^2)}{(m-2)^2(m-4)}, \quad m > 4, \quad (3.7)$$

$$\mu_3 = \frac{16(1-\rho^2)^2(m-1)(3m^2 - 2m(1+11\rho^2) + 36\rho^2)}{(m-2)^3(m-4)(m-6)}, \quad m > 6 \quad (3.8)$$

$$\begin{aligned} \mu_4 = & \frac{48(1-\rho^2)^2}{(m-2)^4(m-4)(m-6)(m-8)} \\ & \times \left(m^5 + m^4(10 - 30\rho^2) + m^3(211\rho^4 - 38\rho^2 - 29) \right. \\ & \left. - 2m^2(447\rho^4 - 202\rho^2 - 13) + 128\rho^2(2 - 5\rho^2) \right), \quad m > 8, \end{aligned} \quad (3.9)$$

where $-1 < \rho < 1$.

Corollary 3.2 Let H have a correlated F distribution with density function (2.1). Then The Coefficient of Variation $CV(H) = \frac{\sigma}{\mu}$ is given by

$$CV(H) = \frac{2\sqrt{(1-\rho^2)(m^2 - m + \rho^2(8-5m))}}{(m-2\rho^2)\sqrt{m-4}}, \quad (3.10)$$

where $m > 4$ and $-1 < \rho < 1$.

Proof. From (3.3) and (3.7), we have (3.10).

The skewness and kurtosis of a random variable X are given by the moment ratios

$$\alpha_i(X) = \mu_2^{-i/2} \mu_i, \quad i = 3, 4, \quad (3.11)$$

where $\mu_i = E(X - E(X))^i, i = 1, 2, 3, \dots$.

Corollary 3.3 The coefficient of skewness and kurtosis of H are respectively given by

$$\alpha_3(H) = \frac{2(m-1)\sqrt{(m-4)(1-\rho^2)}(3m^2 - 2m + \rho^2(36 - 22m))}{(m-6)(m(m-1) + \rho^2(8-5m))^{3/2}}, \quad m > 6 \quad (3.12)$$

and

$$\alpha_4(H) = \frac{3(m-4)}{(m-6)(m-8)(m(m-1) + (8-5m)\rho^2)^2} \times [(m^5 + 10m^4(1-3\rho^2) + m^3(211\rho^4 - 38\rho^2 - 29) + 2m^2(447\rho^4 - 202\rho^2 - 13) + 8m(161\rho^4 - 74\rho^2 - 1) + 128\rho^2(2-5\rho^2)), \quad m > 8 \quad (3.13)$$

where $-1 < \rho < 1$.

Note that when m increases indefinitely, the skewness and kurtosis parameters converge asymptotically to 0 and 3 respectively which are parameters of the normal distribution.

Corollary 3.4 Let $\rho = 0$. Then, the coefficient of skewness and kurtosis of H are respectively given by

$$\alpha_3(H) = \frac{2(3m-2)\sqrt{m-4}}{(m-6)\sqrt{m(m-1)}}, \quad m > 6, \quad (3.14)$$

and

$$\alpha_4(H) = 3 \times \frac{(m-4)(m^3 + 11m^2 - 18m + 8)}{m(m-1)(m-6)(m-8)}, \quad m > 8 \quad (3.15)$$

respectively.

Note that (3.14) and (3.15) provide the skewness and kurtosis for the regular F distribution with m and m degrees of freedom.

Corollary 3.5 Let $\rho = 1$. Then, the coefficient of skewness and kurtosis of H are respectively given by

$$\alpha_3(H) = 0, \quad (3.16)$$

and

$$\alpha_4(H) = \frac{3(m-2)}{m-4}, \quad m > 4, \quad (3.17)$$

respectively.

We remark that (3.16) and (3.17) respectively represent the skewness and kurtosis parameters of the t -distribution with m degrees of freedom.

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