King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 412

Apr 2010

Regularization of Initial Inverse Problems in Heat Equation Using Fourier Transforms

M. Yousuf and K. Masood

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: c-math@kfupm.edu.sa

Regularization of Initial Inverse Problems in Heat Equation Using Fourier Transforms

M. Yousuf

Department of Mathematics and Statistics

King Fahd University of Petroleum and Minerals

Dhahran 31261, Saudi Arabia;

E-mail: myousuf@kfupm.edu.sa

K. Masood

Department of Mathematics and Statistics

King Fahd University of Petroleum and Minerals

Dhahran 31261, Saudi Arabia;

E-mail: masood@kfupm.edu.sa

Abstract: The classical inverse problem of recovering the initial temperature distribution from the final temperature distribution is extremely ill-posed. It is believed that it is very difficult to recover the initial temperature distribution in case of noisy final data by usual methods. A simple and convenient Fourier regularization method for solving one dimensional backward heat equation was proposed by Fu et. al. [4]. The method is extended to two dimensional backward heat equation problem. Some quite sharp error estimates between the approximate solution and exact solution are provided. Numerical examples show that the method works effectively.

Keywords: H eat Equation, Initial Inverse Problem, Fourier Transform, Ill-posed problems, Error Estimate.

Acknowledgment: This work is supported by the Fast Track Project # FT 080007, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

1 INTRODUCTION

The inverse heat conduction problems are extremely ill-posed [11, 2, 6] and without regularization of the inverse solution, the results are not suitable for a particular application. If solution of backward heat conduction problem (BHCP) exist, it will not be continuously dependent on the final data. Some special regularization methods are required to obtain meaningful results both by analytic and numerical methods.

Several techniques have been developed to regularize the solution of inverse heat conduction problems. Masood et.al [14, 15] developed an approach to recover the initial temperature distribution using Bessel operator. A quasi-reversibility method is applied by Lattes and Lions [10], Ames [1] and Miller [17]. Tautenhahn and Schroter [22] proposed an optimal error estimate for a particular backward problem. Seidman established an optimal filtering method [20]. Mera [16] and Jourhmane and Mera [8] used many regularization techniques and numerical methods to solve backward heat conduction problems. A mollification method has been studied by Hào [5]. Recently, Liu used a group preserving scheme to solve the backward heat equation numerically [13]. Kirkup and Wadsworth used an operator-splitting method [9].

The quasi-solution method to solve the equation of the first kind was introduced by Ivanov [7]. The essence of this method is to change the notion of solution of an ill-posed problem so that, for certain conditions, the problem of its determination will be well-posed. Tikhonov's regularization method is widely used for solving linear and nonlinear operator equations of the first kind, see Tikhonov and Arsenin [23]. Iterative methods are applied to solve different problems and particularly these methods can also be applied to solve operator equations of the first kind. Moultanovsky [19] applied such iterative method to solve an initial inverse heat transfer problem. The projective methods for solving various ill-posed problems are based on the representation of the approximate solution as a finite linear combination of a certain functional systems, see e.g. Vasin and Ageev [24]. For some recent developments in the ill posed inverse heat conduction problems, see Shidfar et al. [21], Ling et al. [12] and references there in.

2 FOURIER TRANSFORMATION

In this section we shall describe the Foruier regularization method for the inverse heat conduction problem. Fu et. al. [3] proposed a quite simple and convenient method–Fourier regularization method for one dimensional problem. Error analysis with some faster convergence error estimates are given in [3]. Especially, the convergence of the approximate solution at $t = 0$ is proved in [3]. A numerical example is also presented in [3] to demonstrate the performance of the method.

We are presenting an extension of the method to two dimensional problem. Error estimates between the approximate solution and exact solution are provided. A numerical example is also solved to show how closely the proposed method approximates the exact profile.

We consider the following two dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx} + u_{yy}, \quad -\infty < x < \infty,\ -\infty < y < \infty,\ 0 \le t < T \qquad (2.1) \\ u(x,y,T) &= \phi_T(x,y) \end{aligned}$$

The objective is to determine the temperature distribution $u(x, y, t)$, $0 \leq t < T$ from the given final data $\phi_T(x, y)$. Let $\hat{g}(\xi, \eta)$ denote the Fourier transformation of $g(x, y) \in L(\mathbf{R} \times \mathbf{R})$

$$\hat{g}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi x + \eta y)} g(x, y) dx dy. \tag{2.2}$$

Define the norm $\|g\|_{H^s}$ on the Sobolev space $H^s(\mathbf{R^2})$

$$\|g\|_{H^s} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{g}(\xi, \eta)|^2 (1 + \xi^2)^s (1 + \eta^2)^s d\xi d\eta \right)^{1/2}, \tag{2.3}$$

and $\|g\|_{H^0} = \|.\|$ reduces to the $L^2(\mathbf{R^2})$–norm. A solution of the problem (2.1) is a function $u(x, y, t)$ satisfying (2.1) in the classical sense and is twice integrable for every fixed $t \in [0, T]$. We also assume that if the solution of (2.1) exist, it must be unique. Applying the Fourier transformation to problem (2.1) with respect to the variables x and y, we can get the Fourier transform $\hat{u}(\xi, \eta, t)$ of the exact solution $u(x, y, t)$ of (2.1) as:

$$\hat{u}(\xi, \eta, t) = e^{(\xi^2 + \eta^2)(T - t)} \hat{\phi}_T(\xi, \eta) \tag{2.4}$$

and by the inverse Fourier transformation

$$u(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi x + \eta y)} e^{(\xi^2 + \eta^2)(T - t)} \hat{\phi}_T(\xi, \eta) d\xi d\eta. \tag{2.5}$$

From equation (2.4) we get,

$$\hat{u}(\xi, \eta, 0) = e^{(\xi^2 + \eta^2)T} \hat{\phi}_T(\xi, \eta). \tag{2.6}$$

The initial temperature distribution $u(x, y, 0) \in L^2(\mathbf{R^2})$, therefore there exists an apriori bound

$$\|u(x, y, 0)\| \leq E. \tag{2.7}$$

From (2.6) and (2.7) and using Parseval identity,

$$\|u(x, y, 0)\|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |e^{(\xi^2 + \eta^2)T} \hat{\phi}_T(\xi, \eta)|^2 d\xi d\eta < \infty. \tag{2.8}$$

From (2.8) we note that $e^{(\xi^2 + \eta^2)T} \to \infty$ as $|\xi|, |\eta| \to \infty$, therefore a rapid decay of $\hat{\phi}_T(\xi, \eta)$ occurs at high frequencies. As the inverse heat conduction problem is severely ill–posed, a small error in measured data $\phi_T^m(x, y)$ at $t = T$ produces a large error in the observed temperature distribution $u(x, y, t)$ for $0 \leq t < T$.

Many authors tried to recover the temperature distribution by filtering out the high frequencies. Miranker [18] considered a subspace of $L(\mathbf{R^2})$ of functions whose Fourier transform have compact support and proved that the backward problem is well posed in that subspace in the sense of Hadamard. The only draw back in Miranker's method is that it is conditionally stable and it does not take into account the noise in the measured data. In the present work, following [3], the high frequencies in the solution are eliminated by considering the inverse Fourier transform only for $|\xi| < \xi_{\max}$ and $|\eta| < \eta_{\max}$ where $\xi_{\max}$ and $\eta_{\max}$ are some suitable constants. The Fourier regularization method is simple and robust and may be applicable to a large number of inverse heat conduction problems.

3 ERROR ESTIMATES

Consider the exact data $\phi_T(x,y)$ and the measured data $\phi_T^m(x,y)$ at $t=T$, which satisfy the following condition

$$\|\phi_T - \phi_T^m\| \le \epsilon, \tag{3.1}$$

and

$$\|u(x,y,0)\|_{H^s} \le E, \quad s \ge 0. \tag{3.2}$$

Define an approximate solution $u^m_{\xi_{\max},\eta_{\max}}(x,y,t)$ of (2.1) for the measured data $\phi_T^m(x,y)$ as

$$u^m_{\xi_{\max},\eta_{\max}}(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y)} e^{(\xi^2+\eta^2)(T-t)} \hat{\phi}_T^m(\xi,\eta)\chi_{\xi_{\max}}\chi_{\eta_{\max}} d\xi d\eta, \tag{3.3}$$

where $\chi_{\xi_{\max}}$ and $\chi_{\eta_{\max}}$ are characteristics functions on the intervals $[-\xi_{\max},\xi_{\max}]$ and $[-\eta_{\max},\eta_{\max}]$ respectively. The solution given in (3.3) is Fourier regularized solution for suitable choices of $\xi_{\max}$ and $\eta_{\max}$. Error estimates and choices of appropriate constants $\xi_{\max}$ and $\eta_{\max}$ are described in the next theorem.

Theorem 3.1. *: Let $u(x,y,t)$ be the exact solution and $u^m_{\xi_{\max},\eta_{\max}}$ be the Fourier regularized solution of (2.1) for $0 \le t < T$. Suppose conditions (3.1) and (3.2) hold. If we choose*

$$\xi_{\max} = \left[\frac{1}{2}\ln\left(\left(\frac{E}{\epsilon}\right)^{\frac{1}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{-s}{2T}}\right)\right]^{\frac{1}{2}}, \tag{3.4}$$

$$\eta_{\max} = \left[\frac{1}{2}\ln\left(\left(\frac{E}{\epsilon}\right)^{\frac{1}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{-s}{2T}}\right)\right]^{\frac{1}{2}}, \tag{3.5}$$

then the following logarithmic stability estimate holds,

$$\|u(x,y,t) - u^m_{\xi_{\max}\eta_{\max}}(x,y,t)\| \le E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{s(t-T)}{2T}}\left[\left(\frac{2\sqrt{\ln(\frac{E}{\epsilon})}}{\frac{1}{T}\ln(\frac{E}{\epsilon})+\ln(\ln(\frac{E}{\epsilon}))^{\frac{-s}{2T}}}\right)^{s}+1\right] \tag{3.6}$$

Proof. Using (2.5),(2.6),(3.1),(3.2), (3.3) and Parseval formula

$$
\begin{aligned}
& \|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \\
= \; & \|e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}_T(\xi,\eta) - e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}^m_T(\xi,\eta)\chi_{\xi_{\max}}\chi_{\eta_{\max}}\| \\
\leq \; & \|e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}_T(\xi,\eta) - e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}_T(\xi,\eta)\chi_{\xi_{\max}}\chi_{\eta_{\max}}\| \\
+ \; & \|e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}_T(\xi,\eta)\chi_{\xi_{\max}}\chi_{\eta_{\max}} - e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}^m_T(\xi,\eta)\chi_{\xi_{\max}}\chi_{\eta_{\max}}\| \\
= \; & \left(\int_{|\xi|>\xi_{\max}}\int_{|\xi|>\xi_{\max}} |e^{(\xi^2+\eta^2)(T-t)}\hat{\phi}_T(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
+ \; & \left(\int_{|\xi|\leq\xi_{\max}}\int_{|\eta|\leq\eta_{\max}} |e^{(\xi^2+\eta^2)(T-t)}\left(\hat{\phi}_T(\xi,\eta) - \hat{\phi}^m_T(\xi,\eta)\right)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
= \; & \left(\int_{|\xi|>\xi_{\max}}\int_{|\eta|>\eta_{\max}} |e^{(\xi^2+\eta^2)(T-t)}e^{-(\xi^2+\eta^2)T}\hat{u}(\xi,\eta,0)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
+ \; & \left(\int_{|\xi|\leq\xi_{\max}}\int_{|\eta|\leq\eta_{\max}} |e^{(\xi^2+\eta^2)(T-t)}\left(\hat{\phi}_T(\xi,\eta) - \hat{\phi}^m_T(\xi,\eta)\right)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
= \; & \left(\int_{|\xi|>\xi_{\max}}\int_{|\eta|>\eta_{\max}} |e^{-(\xi^2+\eta^2)t}\hat{u}(\xi,\eta,0)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
+ \; & \left(\int_{|\xi|\leq\xi_{\max}}\int_{|\eta|\leq\eta_{\max}} |e^{(\xi^2+\eta^2)(T-t)}\left(\hat{\phi}_T(\xi,\eta) - \hat{\phi}^m_T(\xi,\eta)\right)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
\leq \; & \sup_{\xi_{\max},\eta_{\max}} \frac{e^{-t(\xi^2+\eta^2)}}{(1+\xi^2)^{s/2}(1+\eta^2)^{s/2}} \left(\int_{|\xi|>\xi_{\max}}\int_{|\eta|>\eta_{\max}} |\hat{u}(\xi,\eta,0)|^2(1+\xi^2)^s(1+\eta^2)^s d\xi d\eta\right)^{1/2} \\
+ \; & \sup_{\xi_{\max},\eta_{\max}} e^{(\xi^2+\eta^2)(T-t)} \left(\int_{|\xi|\leq\xi_{\max}}\int_{|\eta|\leq\eta_{\max}} |\hat{\phi}_T(\xi,\eta) - \hat{\phi}^m_T(\xi,\eta)|^2 d\xi d\eta\right)^{1/2} \\
\leq \; & \sup_{\xi_{\max},\eta_{\max}} \frac{e^{-t(\xi^2+\eta^2)}}{|\xi|^s|\eta|^s}E + \sup_{\xi_{\max},\eta_{\max}} e^{(\xi^2+\eta^2)(T-t)}\epsilon \\
\leq \; & \frac{e^{-t\ln\left(\frac{E}{\epsilon}\right)^{1/T}(\ln(\frac{E}{\epsilon})^{-s/2T})}}{\left[\frac{1}{2}\ln\left((\frac{E}{\epsilon})^{1/T}\ln(\frac{E}{\epsilon})^{-s/2T}\right)\right]} + e^{(T-t)\ln\left(\frac{E}{\epsilon}\right)^{1/T}(\ln(\frac{E}{\epsilon}))^{-s/2T}}\epsilon \\
= \; & \left(\frac{E}{\epsilon}\right)^{t/T}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{st/2T}\left(\frac{2\sqrt{\ln\left(\frac{E}{\epsilon}\right)}}{\frac{1}{T}\ln\left(\frac{E}{\epsilon}\right) + \ln\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{-s/2T}}\right)^s \left(\ln\left(\frac{E}{\epsilon}\right)\right)^{-s/2} E \\
+ \; & \left(\frac{E}{\epsilon}\right)^{\frac{T-t}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{-s(T-t)}{2T}}\epsilon \\
= \; & E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)^{\frac{s(t-T)}{2T}}\right)\left[\left(\frac{2\sqrt{\ln\left(\frac{E}{\epsilon}\right)}}{\frac{1}{T}\ln\left(\frac{E}{\epsilon}\right) + \ln\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{-s/2T}}\right)^s + 1\right].
\end{aligned}
$$

■

In the estimate (3.6), the term $\frac{\sqrt{\ln(\frac{E}{\epsilon})}}{\frac{1}{T}\ln(\frac{E}{\epsilon})+\ln(\ln(\frac{E}{\epsilon}))^{-s/2T}}$ is bounded as $\epsilon \to 0$. For $s = 0$, estimate (3.6) can be written as

$$\|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \leq 2E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}} \tag{3.7}$$

Remark. *From the estimate (3.7), it is clear that as $t \to 0$, the error bound is $2E$, i.e. the convergence at $t = 0$ cannot be obtained. However from (3.6), for $t = 0$,*

$$\|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \leq E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{s(t-T)}{2T}}\left[\left(\frac{2\sqrt{\ln(\frac{E}{\epsilon})}}{\frac{1}{T}\ln(\frac{E}{\epsilon})+\ln(\ln(\frac{E}{\epsilon}))^{\frac{-s}{2T}}}\right)^s + 1\right]$$
$$\to 0, \text{ as } \epsilon \to 0 \text{ and } s > 0$$

Remark. *Note that* $\frac{\sqrt{\ln(\frac{E}{\epsilon})}}{\frac{1}{T}\ln(\frac{E}{\epsilon})+\ln(\ln(\frac{E}{\epsilon}))^{-s/2T}} \to 0$ *as $\epsilon \to 0$. Therefore (3.6) can be written as*

$$\|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \leq E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\ln\left(\frac{E}{\epsilon}\right)\right)^{\frac{s(t-T)}{2T}}[1+o(1)], \text{ as } \epsilon \to 0 \tag{3.8}$$

It is clear form (3.8) that accuracy of the estimate increases with decreasing T.

Remark. *In actual practice, the bound on E is not known, and therefore, we assume*

$$\xi_{\max} = \left[\frac{1}{2}\ln\left(\left(\frac{1}{\epsilon}\right)^{\frac{1}{T}}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{-s}{2T}}\right)\right]^{\frac{1}{2}}, \tag{3.9}$$

$$\eta_{\max} = \left[\frac{1}{2}\ln\left(\left(\frac{1}{\epsilon}\right)^{\frac{1}{T}}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{-s}{2T}}\right)\right]^{\frac{1}{2}}, \tag{3.10}$$

and also assume that as estimate on ϵ is available for practical applications. In this case the estimate (3.5) takes the form

$$\|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \leq \epsilon^{\frac{t}{T}}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{s(t-T)}{2T}}\left[E\left(\frac{2\sqrt{\ln(\frac{1}{\epsilon})}}{\frac{1}{T}\ln(\frac{1}{\epsilon})+\ln(\ln(\frac{1}{\epsilon}))^{\frac{-s}{2T}}}\right)^s + 1\right]$$

or

$$\|u(x,y,t) - u^m_{\xi_{\max},\eta_{\max}}(x,y,t)\| \leq \epsilon^{\frac{t}{T}}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{s(t-T)}{2T}}[1+o(1)], \text{ as } \epsilon \to 0. \tag{3.11}$$

4 NUMERICAL EXPERIMENTS

Numerical experiments are performed on a two dimensional problem for various values of ϵ, $\xi_{\max}$ and $\eta_{\max}$. To test the method, Matlab function *rand* is used to generate the noisy (measured) data

$$\phi^m_T(x,y) = \phi_T(x,y) + \epsilon\, rand(\phi_T(x,y)), \tag{4.1}$$

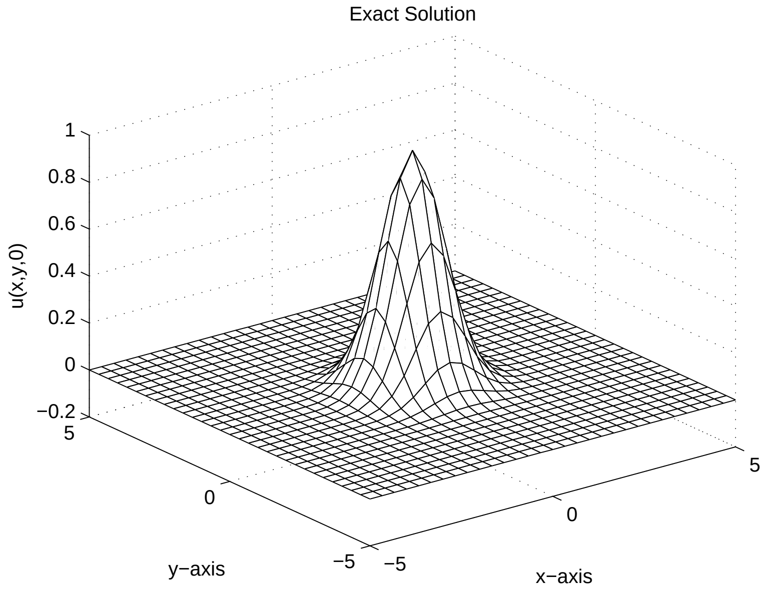

FIG. 1. Exact solution at t=0.

where $\phi_T(x, y)$ is the exact data, $rand(\phi_T(x, y))$ is an $n \times n$ matrix of random numbers in $[0, 1]$ and ϵ is the magnitude of the random noise.

Example 1. Consider the function

$$u(x, y, t) = \frac{1}{1+4t} e^{\frac{-(x^2+y^2)}{1+4t}}. \tag{4.2}$$

as solution of the following initial value problem

$$u_t = u_{xx} + u_{yy}, \;\; x, y \in \mathbf{D}, \;\; t \in (0, T] \tag{4.3}$$

with initial condition

$$u(x, y, 0) = e^{-(x^2+y^2)}, \;\; x, y \in \mathbf{D}, \tag{4.4}$$

where the domain $\mathbf{D}$ *is taken as a rectangle* $[-5, 5] \times [-5, 5]$. *The solution (4.2) is also the solution of the following backward heat equation for* $0 \le t < T$

$$u_t = u_{xx} + u_{yy}, \;\; x, y \in \mathbf{D}, \;\; t \in (0, T] \tag{4.5}$$

with initial condition

$$u(x, y, T) = \frac{1}{1+4T} e^{\frac{-(x^2+y^2)}{1+4T}}. \tag{4.6}$$

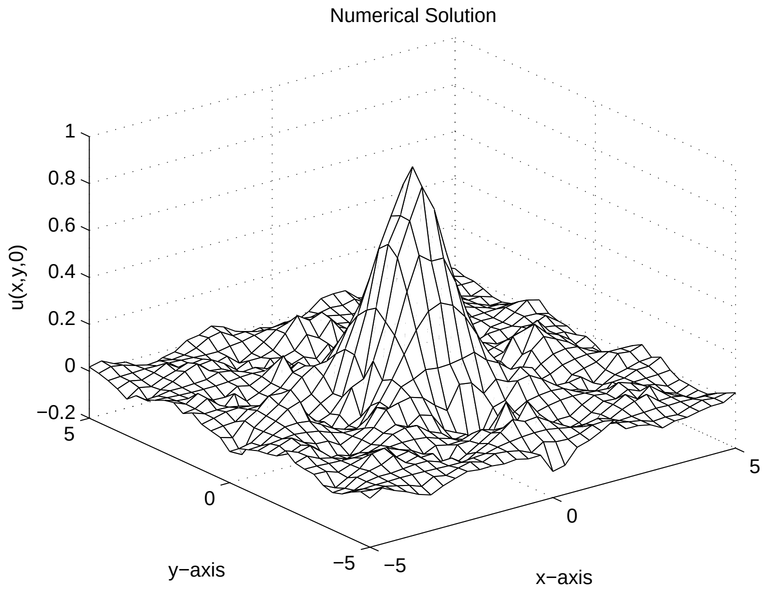

FIG. 2. Numerical solution at t=0 for $\epsilon = 0.001$, $\xi_{\max} = 2.0$ and $\eta_{\max} = 2.125$.

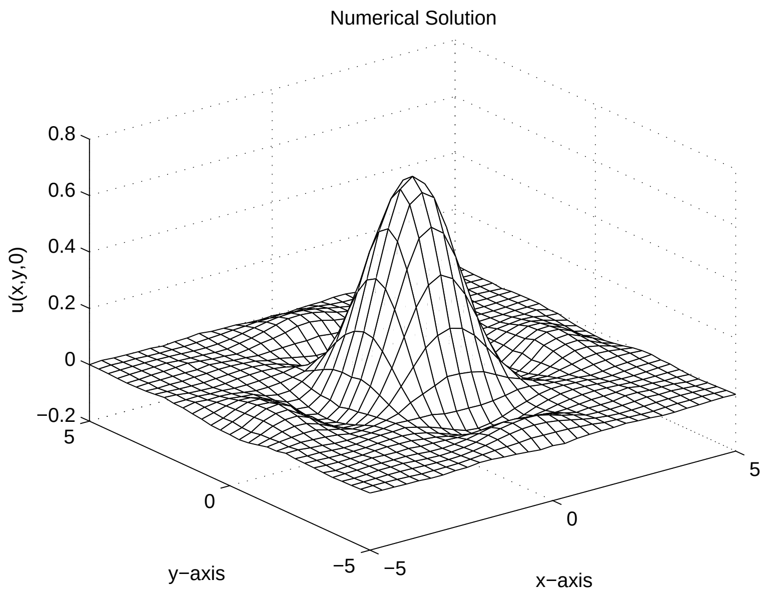

FIG. 3. Numerical solution at t=0 for $\epsilon = 0.0001$, $\xi_{\max} = 2.00$ and $\eta_{\max} = 2.125$.

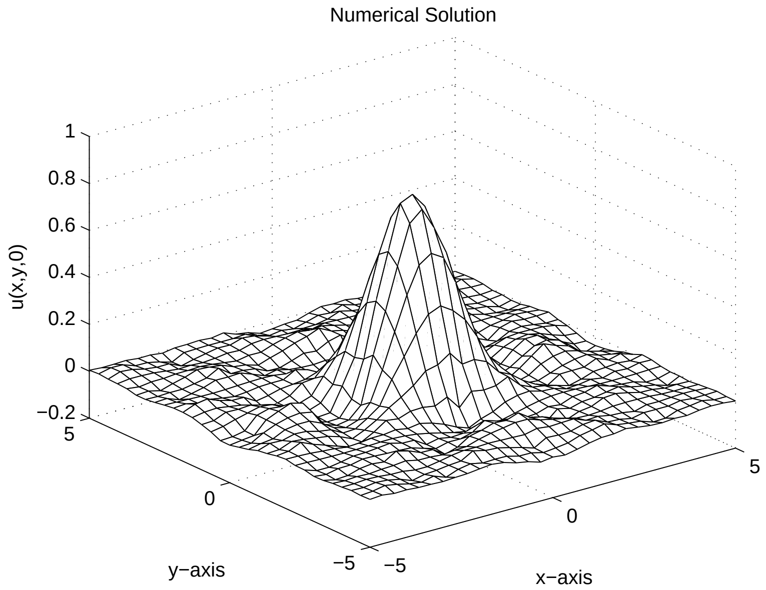

FIG. 4. Numerical solution at t=0 for $\epsilon = 0.0001$, $\xi_{\max} = 2.25$ and $\eta_{\max} = 2.25$.

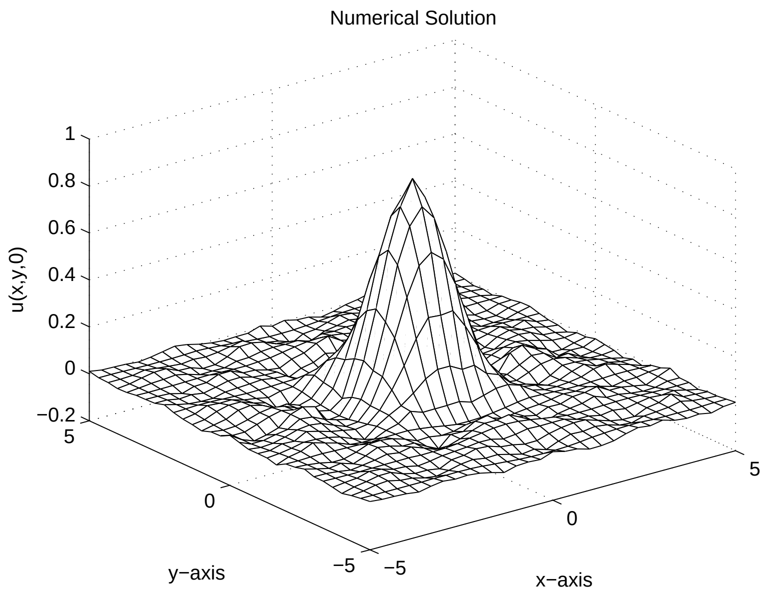

FIG. 5. Numerical solution at t=0 for $\epsilon = 0.00001$, $\xi_{\max} = 2.50$ and $\eta_{\max} = 2.50$.

REFERENCES

1. K.A. Ames, W.C. Gordon, J.F. Epperson, S.F. Oppenhermer, *A comparison of regularizations for an ill-posed problem*, Math. Comput. 67 (1998) 1451–1471.
2. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996, 31–42.
3. C.L. Fu, *Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation*, J. Comput. Appl. Math. 167 (2004) 449-463.
4. C.L. Fu, X.T. Xiong, Z. Qian, *Fourier regularization method for a backward heat equation*, J. Math. Anal. Appl. 331 (2007) 472-480.
5. D.N. Hào, *A mollification method for ill-posed problems*, Numer. Math. 68 (1994) 469–506.
6. V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, New York, 1998.
7. Ivanov, V. K., 1963, *On ill-posed problems*, Mat. Sb., 61(2), pp. 211-223 (in Russian).
8. M. Jourhmane, N.S. Mera, *An iterative algorithm for the backward heat conduction problem based on variable relaxation factors*, Inverse Problems in Engineering 10 (2002) 293–308.
9. S.M. Kirkup, M. Wadsworth, *Solution of inverse diffusion problems by operator-splitting methods*, Appl. Math. Modelling 26 (10) (2002) 1003–1018.
10. R. Lattes, J.L. Lions, *Methode de Quasi-Reversibility et Applications*, Dunod, Paris, 1967 (English translation: R. Bellman, Elsevier, New York, 1969).
11. M.M. Lavrentév, V.G. Romanov, S.P. Shishat'fskiĭ, *Ill-posed Problems of Mathematical Physics and Analysis*, Amer. Math. Soc., Providence, RI, 1986.
12. Ling, L., Yamamoto, M., Hon, Y. C., and Takeuchi, T., 2006, *Identification of source locations in two dimensional heat equations*, Inverse Problems, 22(4), pp. 1289-1305.
13. C.S. Liu, *Group preserving scheme for backward heat conduction problems*, Internat. J. Heat Mass Transfer 47 (2004) 2567–2576.
14. K. Masood, S. Messaoudi, and F. D. Zaman, *Initial Inverse Problem in Heat Equation with Bessel Operator*, Int. Journal of Heat and Mass Transfer, 45(14)(2002), pp. 2959-2965.
15. K. Masood, F. D. Zaman, *Investigation of the Initial Inverse Problem in the Heat Equation*, Journal of Heat Transfer, 126(2), (2004), 294–296.
16. N.S. Mera, L. Elliott, D.B. Ingham, D. Lesnic, *An iterative boundary element method for solving the onedimensional backward heat conduction problem*, Internat. J. Heat Mass Transfer 44 (2001) 1937–1946.
17. K. Miller, *Stabilized quasireversibility and other nearly best possible methods for non-well-posed problems*, in: Symposium on Non-Well-Posed Problems and Logarithmic Convexity, in: Lecture Notes in Math., vol. 316, Springer-Verlag, Berlin, 1973, pp. 161–176.
18. W.L. Miranker, *A well posed problem for the backward heat equation*, Proc. Amer. Math. Soc. 12 (2) (1961) 243-274.

19. Moultanovsky, A. V., 2002, *Mobile HVAC system evaporator optimization and cooling Capacity estimation by means of inverse problem solution*, Inverse Problems in Engng., 10(1), pp. 1-18.

20. T.I. Seidman, *Optimal filtering for the backward heat equation*, SIAM J. Numer. Anal. 33 (1996) 162–170.

21. A. Shidfar, and K. Tavakoli, *An inverse heat conduction problem*, Southeat Asian Bulletin of Mathematics, 26(3)(2003), pp. 503-507.

22. U. Tautenhahn, T. Schroter, *On optimal regularization methods for the backward heat equation*, Z. Anal. Anwendungen 15 (1996) 475–493.

23. A. N. Tikhonov and V. Ya. Arsenin, *Solution of Ill-Posed Problems*, 1977, John Wiley, New York.

24. V. V. Vasin and A. L. Ageev, *Ill-Posed Problems with a Priori Information*, 1995, VSP, Utrecht.