King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 401

Nov 2008

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Some Characteristics of Sample Covariance

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Abstract The distribution of the sample covariance based on a sample from a bivariate normal population has been derived in the literature in a complicated way. In this paper, we present an alternative but simple derivation based on transformations on the joint distribution of sample variances and covariance. The main contribution is the direct derivation of the general moment structure of sample covariance.

AMS Mathematics Subject Classification: 62E15, 60-01, 60E05, 60E10, 33C20

Key Words and Phrases: sample covariance, Wishart distribution, Generalized hypergeometric function, Macdonald function

1. Introduction

Let $X_1, X_2, \cdots X_N \ (N > 2)$ be a two-dimensional independent normal random vectors drawn from a bivariate normal distribution with mean vector $\theta' = (\theta_1, \theta_2)'$, and covariance matrix Σ. Then $E(X_{1j}) = \theta_1, E(X_{2j}) = \theta_2, V(X_{1j}) = \sigma_1^2, V(X_{2j}) = \sigma_2^2$, $E(X_{1j} - \theta_1)(X_{2j} - \theta_2) = \rho\sigma_1\sigma_2$ $= \sigma_{12}$, where the quantity ρ is the product moment correlation coefficient. The sample mean vector $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ so that the mean-centered sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A = (a_{ik}), i = 1,2; k = 1,2$. Then

$a_{ii} = ms_i^2 = \sum_{j=1}^{N}(X_{ij} - \bar{X}_i)^2, m = N - 1, (i = 1,2)$ and $a_{12} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mrs_1s_2$.

Fisher (1915) derived the distribution of the bivariate Wishart matrix in order to study the distribution of correlation coefficient for a bivariate normal sample. Wishart (1928) obtained the distribution of Wishart matrix as the joint distribution of sample variances and covariances from multivariate normal population. The bivariate matrix A is said to have a Wishart distribution with parameters $m = N - 1$ and $\Sigma(2\times 2) > 0$, written as $A \sim W_2(m, \Sigma)$ if its probability density function is given by

$$f\,(a_{11},a_{22},a_{12}) = \frac{(1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\ \left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2} \times \exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}+\frac{\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right) \qquad (1.1)$$

$a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, m > 2, -1 < \rho < 1$ (Anderson, 2003, 123).

Because of the important role of Wishart distribution in multivariate statistical analysis, various authors have given different derivations. See the references in Gupta and Nagar (2000, 87-88) for a good update on the Wishart distribution.

The distribution of the sample covariance based on a sample from a bivariate normal population has been considered by Mahalanobis, Bose and Roy (1937), Pearson, Jeffery, and Elderton (1929), Wishart and Bartlett (1932), Hirschfeld (1937) and Springer, (1979, 343).

In this paper, we derive the density function of sample covariance by making simple transformations on bivariate Wishart distribution. The main contribution is the direct derivation of the general moment structure of sample covariance.

2. Some Preliminaries

For any nonnegative integer k , the following notations will be used in sequel:

$$a_{\{k\}} = a(a+1)(a+2)\cdots(a+k-1), \qquad (2.1)$$

$$a^{\{k\}} = a(a-1)\cdots(a-k+1). \qquad (2.2)$$

$$(2z)!\sqrt{\pi} = 2^{2z}\,z\,!\Gamma(z+\tfrac{1}{2}) \qquad (2.3)$$

The modified Bessel function of the second kind admits the following integral representation:

$$K_\alpha(y) = \tfrac{1}{2}(\tfrac{1}{2}y)^\alpha \int_0^\infty t^{-(\alpha+1)}\exp\left(-t-\frac{y^2}{4t}\right)dt \qquad (2.4)$$

(Watson, 1993, 183).

The generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q(a_1,a_2,\cdots a_p;b_1,b_2,\cdots b_q;z) = \sum_{k=0}^{\infty}\frac{(a_1)_{\{k\}}(a_2)_{\{k\}}\cdots(a_p)_{\{k\}}}{(b_1)_{\{k\}}(b_2)_{\{k\}}\cdots(b_q)_{\{k\}}}$$

(Gradshteyn and Ryzhik, 1994, 1071).

The cumulant generating function $\kappa_r(X)$ or simply κ_r of a random variable X is the coefficient of $t^r/r!$ in the Taylor series expansion of

$\ln M_X(t) = \sum_{r\geq 1} \frac{\kappa_r(X)t^r}{r!}$. It can be checked that

$$\kappa_1 = \mu,\ \kappa_2 = \mu_2,\ \kappa_3 = \mu_3,\ \kappa_4 = \mu_4 - 3\mu_2^2 \tag{2.6}$$

(Johnson, Kotz and Balakrishnan, 1993, 45).

3. The Density Function

It is known that the conditional distribution of Y_i given $(X_1, X_2, \cdots, X_N)$ is normal with expected value $\theta_2 + (\rho\sigma_2/\sigma_1)(X_j - \theta_1)$ and standard deviation $\sigma_2\sqrt{1-\rho^2}$. Then the conditional distribution of $W = a_{12}/m$, given $(X_1, X_2, \cdots, X_N)$ is normal with expected value $E(W) = \frac{\rho\sigma_2}{\sigma_1} S_1^2$ and variance $(1-\rho^2)\sigma_2^2 S_1^2/m$ where $\frac{mS_1^2}{\sigma_1^2} \sim \chi_m^2$ (Johnson, Kotz and Balakrishnan, v2, p599). Then the conditional distribution of $Z = a_{12}$, given $(X_1, X_2, \cdots, X_N)$ is normal with expected value $E(Z) = \frac{m\rho\sigma_2}{\sigma_1} S_1^2$ and variance $m(1-\rho^2)\sigma_2^2 S_1^2$ where $\frac{mS_1^2}{\sigma_1^2} \sim \chi_m^2$ (Johnson, Kotz and Balakrishnan, v2, p599). It follows from the above property that

$$f_1(z \mid s_1^2) = \frac{1}{\sigma_2 s_1 \sqrt{2\pi m(1-\rho^2)}} \exp\left[-\frac{1}{2m(1-\rho^2)\sigma_2^2 s_1^2}\left(z - \frac{m\rho\sigma_2}{\sigma_1} s_1^2\right)^2\right],$$

which implies

$$f_2(z) = \int_0^\infty \frac{1}{\sigma_2 s_1 \sqrt{2\pi m(1-\rho^2)m}} \exp\left[-\frac{1}{2m(1-\rho^2)\sigma_2^2 s_1^2 m}\left(z - \frac{m\rho\sigma_2}{\sigma_1} s_1^2\right)^2\right] h(s_1^2)$$

where $h(s_1^2)$ is the density function of S_1^2 given by

$$h(s_1^2) = \frac{1}{2^{m/2}\Gamma(m/2)}\left(\frac{ms_1^2}{\sigma_1^2}\right)^{(m-2)/2} \exp\left(-\frac{ms_1^2}{2\sigma_1^2}\right)\frac{m}{\sigma_1^2}.$$

The integral can be evaluated to be

$$f_2(z) = \frac{1}{2^{(m-1)/2}\Gamma(m/2)(\sigma_1\sigma_2)^{(m+1)/2}\sqrt{\pi(1-\rho^2)}} \exp\left[\frac{z\rho}{(1-\rho^2)\sigma_1\sigma_2}\right] |z|^{(m-1)/2} K_{(m-1)/2}\left(\frac{|z|}{(1-\rho^2)\sigma_1\sigma_2}\right)$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind (also called Macdonald function) defined in (2.4).

Hence, the density function of the covariance $W = Z / m$ is given by

$$f_2(w) = \frac{m^{(m+1)/2}}{2^{(m-1)/2}(\sigma_1\sigma_2)^{(m+1)/2}\sqrt{\pi(1-\rho^2)}\Gamma\left(\frac{m}{2}\right)} |w|^{(m-1)/2} \exp\left(\frac{\rho m w}{(1-\rho^2)\sigma_1\sigma_2}\right) K_{(m-1)/2}\left(\frac{m|w|}{(1-\rho^2)\sigma_1\sigma_2}\right). \tag{3.1}$$

The characteristic function of Z is given by

$$\begin{aligned} E(e^{itZ}) &= [1-2it\rho\sigma_1\sigma_2 + t^2(1-\rho^2)\sigma_1^2\sigma_2^2]^{-m/2} \\ &= \left[\left(1-2it\,\frac{\sigma_1\sigma_2(1+\rho)}{2}\right)\left(1+2it\,\frac{\sigma_1\sigma_2(1-\rho)}{2}\right)\right]^{-m/2} \end{aligned} \tag{3.2}$$

so that the distribution of Z is also that of $\frac{1}{2}[(1+\rho)U - (1-\rho)V]\sigma_1\sigma_2$ where U and V are two independent chi-square variables each with m degrees of freedom. The distribution of the sample covariance W is the same as that of $\frac{m}{2}[(1+\rho)U - (1-\rho)V]\sigma_1\sigma_2$. One can derive moments from it (Johnson, Kotz and Balakrishnan).

From the representation

$$Z = \frac{1}{2}[(1+\rho)U - (1-\rho)V]\sigma_1\sigma_2, \tag{3.3}$$

it follows that

$$\kappa_t(Z) = \left(\frac{1}{2}\sigma_1\sigma_2\right)^t [(1+\rho)^t + (-1)^t(1-\rho)^t]\kappa_t(\chi_m^2) \tag{3.4}$$

The representation (3.3) clearly shows how the distribution of $W = Z / m$ tends to normality as m tends to infinity.

In this paper, we derive the distribution of sample covariance $W = (Z / m) = (a_{12} / m)$ by integrating out a_{11} and a_{22} from the density function (1.1).

Theorem 3.1 Let $w = a_{12} / m$, be the sample covariance. Then the density function of W is given by

$$f_1(w) = \frac{(m/\sigma_1\sigma_2)^{(m+1)/2}}{2^{(m-1)/2}\sqrt{\pi(1-\rho^2)}\Gamma(m/2)} |w|^{(m-1)/2} \exp\left(\frac{\rho m w}{(1-\rho^2)\sigma_1\sigma_2}\right) K_{(m-1)/2}\left(\frac{m|w|}{(1-\rho^2)\sigma_1\sigma_2}\right),$$

Proof. It follows from (1.1) that the density function of $a_{12} = Z$ is given by

$$f_2(z)=\frac{(1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}\exp\left(\frac{\rho z}{(1-\rho^2)\sigma_1\sigma_2}\right)I(m,\rho),$$

$-\infty<z<\infty,\ m>2,\ -1<\rho<1$ where

$$I(m,\rho)=\iint_{z^2\le xy}\left(xy-z^2\right)^{(m-3)/2}\exp\left(-\frac{x}{2(1-\rho^2)\sigma_1^2}-\frac{y}{2(1-\rho^2)\sigma_2^2}\right)dydx\ .$$

Letting $xy-z^2=u$, i.e., $y=(z^2+u)/x$ with Jacobian $J(y\to u)=1/x$, we have

$$I(m,\rho)=\int_{x=0}^{\infty}\int_{u=0}^{\infty}u^{(m-3)/2}\exp\left(-\frac{x}{2(1-\rho^2)\sigma_1^2}-\frac{z^2+u}{2(1-\rho^2)\sigma_2^2x}\right)\frac{1}{x}dudx\,.$$

Completing the gamma integral in u, we have

$$I(m,\rho)=\Gamma\left(\frac{m-1}{2}\right)[2(1-\rho^2)\sigma_2^2]^{(m-1)/2}\int_{x=0}^{\infty}x^{(m-3)/2}\exp\left(-\frac{x}{2(1-\rho^2)\sigma_1^2}-\frac{z^2}{2(1-\rho^2)\sigma_2^2x}\right)dx\,.$$

Putting $\frac{x}{2(1-\rho^2)\sigma_1^2}=t$, and evaluating the integrals, we have

$$I(m,\rho)=2^{(m+1)/2}\Gamma\left(\frac{m-1}{2}\right)[(1-\rho^2)\sigma_1\sigma_2]^{(m-1)/2}\times|z|^{(m-1)/2}K_{(m-1)/2}\left(\frac{|z|}{(1-\rho^2)\sigma_1\sigma_2}\right),$$

so that he density function of Z is given by

$$f_2(z)=\frac{1}{2^{(m-1)/2}(\sigma_1\sigma_2)^{(m+1)/2}\sqrt{\pi(1-\rho^2)}\Gamma\left(\frac{m}{2}\right)}|z|^{(m-1)/2}\exp\left(\frac{\rho z}{(1-\rho^2)\sigma_1\sigma_2}\right)K_{(m-1)/2}\left(\frac{|z|}{(1-\rho^2)\sigma_1\sigma_2}\right).$$

Then the density function of the covariance $W=Z/m$ is given by what we have in the theorem.

Let $\upsilon=\frac{z}{(1-\rho^2)\sigma_1\sigma_2}=\frac{mw}{(1-\rho^2)\sigma_1\sigma_2}$. Then $(1-\rho^2)\sigma_1\sigma_2\upsilon/m=w$ and Υ has the following elegant distribution at υ

$$f_2(\upsilon)=\frac{(1-\rho^2)^{m/2}}{2^{(m-1)/2}\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}|\upsilon|^{(m-1)/2}e^{\rho}K_{(m-1)/2}\left(|\upsilon|\right),$$

which is what was obtained by Springer (1979, 343) by inverse Mellin transform. Some interesting forms of the density function o sample covariance is pointed by Press (1967). If $\rho = 0$, then the density function of the sample covariance is given by

$$f_4(w) = \frac{(m/\sigma_1\sigma_2)^{(m+1)/2}}{2^{(m-1)/2}\sqrt{\pi}\,\Gamma(m/2)} |w|^{(m-1)/2} K_{(m-1)/2}\left(\frac{m|w|}{\sigma_1\sigma_2}\right).$$

4. Moments of the Sample Covariance

The a-th moment of sample covariance W can be expressed in terms of the moments of sample variance and coefficient of correlation. Let $\mu'(a,b,c) = E\left[(S_1^2)^a (S_2^2)^b (R^c)\right]$ be the (a,b,c)-th moment of S_1^2, S_2^2 and R. Then indeed $E(W^h) = \mu'(h/2, h/2, h)$. It has been proved by Joarder (2008) that $\mu'(a,b,c) = E\left[(S_1^2)^a (S_2^2)^b (R^c)\right]$ is given by

$$\mu'(a,b,c;\rho) = \frac{2^{a+b-1}(1-\rho^2)^{a+b+(m/2)}}{m^{a+b}\Gamma(m/2)\sqrt{\pi}} \sigma_1^a \sigma_2^b \times \sum_{k=0}^{\infty} [1+(-1)^c(-1)^k] \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m+2a}{2}\right)\Gamma\left(\frac{k+m+2b}{2}\right) \frac{\Gamma\left(\frac{k+1+c}{2}\right)}{\Gamma\left(\frac{k+m+c}{2}\right)}. \quad (4.1)$$

where $m > 2,\ \sigma_1 > 0,\ \sigma_2 > 0,\ -1 < \rho < 1$.

The moments of sample covariance will involve infinite sum of product of gamma functions. To facilitate the calculation we have the following lemmas:

Lemma 4.1 Let $g(k) = \frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+m+2h+c}{2}\right)\Gamma\left(\frac{k+2h+d}{2}\right)$. Then

a. $\sum_{j=0}^{\infty} g(2j) = \Gamma\left(h+\frac{m+c}{2}\right)\Gamma\left(h+\frac{d}{2}\right) {}_2F_1\left(h+\frac{m+c}{2}, h+\frac{d}{2}; \frac{1}{2}\right)$

b. $\sum_{j=0}^{\infty} g(2j+1) = 2\rho\Gamma\left(h+\frac{m+c+1}{2}\right)\Gamma\left(h+\frac{d+1}{2}\right) {}_2F_1\left(h+\frac{m+c+1}{2}, h+\frac{d+1}{2}; \frac{3}{2}\right)$

where ${}_2F_1(a,b;c;z)$ is the generalized hypergeometric function defined by (2.5).

Proof.

a. By (2.3) we have

$$\sum_{j=0}^{\infty} g(2j) = \sum_{j=0}^{\infty} \frac{2^{2j}\rho^{2j}}{(2j)!}\Gamma\left(j+h+\frac{m+c}{2}\right)\Gamma\left(j+h+\frac{d}{2}\right)$$
$$= \sum_{j=0}^{\infty} \frac{\sqrt{\pi}\rho^{2j}}{j!\Gamma\left(j+\frac{1}{2}\right)}\Gamma\left(j+h+\frac{m+c}{2}\right)\Gamma\left(j+h+\frac{d}{2}\right)$$

which can be written as

$$\sum_{j=0}^{\infty} g(2j) = \sum_{j=0}^{\infty} \frac{\sqrt{\pi}\rho^{2j}}{j!\Gamma\left(j+\frac{1}{2}\right)} \frac{\Gamma\left(j+h+\frac{m+c}{2}\right)}{\Gamma\left(h+\frac{m+c}{2}\right)} \frac{\Gamma\left(j+h+\frac{d}{2}\right)}{\Gamma\left(h+\frac{d}{2}\right)}\Gamma\left(h+\frac{m+c}{2}\right)\Gamma\left(h+\frac{d}{2}\right)$$

which can be expressed as what we have in the lemma.

b. By (2.3), we have

$$\sum_{j=0}^{\infty} g(2j+1) = \sum_{j=0}^{\infty} \frac{(2)2^{2j}(\rho)\rho^{2j}}{(2j+1)(2j)!}\Gamma\left(j+h+\frac{m+c+1}{2}\right)\Gamma\left(j+h+\frac{d+1}{2}\right)$$
$$= \sum_{j=0}^{\infty} \frac{(2)\sqrt{\pi}(\rho)\rho^{2j}}{(2j+1)j!\Gamma\left(j+\frac{1}{2}\right)}\Gamma\left(j+h+\frac{m+c+1}{2}\right)\Gamma\left(j+h+\frac{d+1}{2}\right),$$

which can be written as

$$\sum_{j=0}^{\infty} g(2j+1)$$
$$= \rho\sum_{j=0}^{\infty} \frac{\sqrt{\pi}\rho^{2j}}{j!\Gamma\left(j+\frac{3}{2}\right)}\Gamma\left(j+h+\frac{m+c+1}{2}\right)\Gamma\left(j+h+\frac{d+1}{2}\right)$$
$$= 2\rho\sum_{j=0}^{\infty} \frac{\frac{1}{2}\sqrt{\pi}\rho^{2j}}{j!\Gamma\left(j+\frac{3}{2}\right)} \frac{\Gamma\left(j+h+\frac{m+c+1}{2}\right)}{\Gamma\left(h+\frac{m+c+1}{2}\right)} \frac{\Gamma\left(j+h+\frac{d+1}{2}\right)}{\Gamma\left(h+\frac{d+1}{2}\right)}\Gamma\left(h+\frac{m+c+1}{2}\right)\Gamma\left(h+\frac{d+1}{2}\right)$$

which is equivalent to what we have in lemma.

Theorem 4.1 For any real number h, the $(2h)$-th moment is given by

$$\mu'(h,h,2h;\rho)=\frac{2^{2h-1}(1-\rho^2)^{2h+(m/2)}}{m^{2h}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h}\sigma_2^{2h}$$
$$\times\left[\{1+(-1)^{2h}\}\Gamma\left(h+\frac{m}{2}\right)\Gamma\left(h+\frac{1}{2}\right){}_2F_1\left(h+\frac{m}{2},h+\frac{1}{2};\frac{1}{2};\rho^2\right)\right.$$
$$\left.+\{1-(-1)^{2h}\}2\rho\Gamma\left(h+\frac{m+1}{2}\right)\Gamma(h+1)\,{}_2F_1\left(h+\frac{m+1}{2},h+1;\frac{3}{2};\rho^2\right)\right].$$

Proof. From (4.1), we have

$$\mu'(h,h,2h;\rho)=\frac{2^{2h-1}(1-\rho^2)^{2h+(m/2)}}{m^{2h}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h}\sigma_2^{2h}\sum_{k=0}^{\infty}[1+(-1)^{2h}(-1)^k]g(k)$$

where $g(k)=\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+m+2h}{2}\right)\Gamma\left(\frac{k+1+2h}{2}\right).$

Since

$$\sum_{k=0}^{\infty}[1+(-1)^{2h}(-1)^k]g(k)=[1+(-1)^{2h}]\sum_{j=0}^{\infty}g(2j)+[1-(-1)^{2h}]\sum_{j=0}^{\infty}g(2j+1),$$

we have

$$\mu'(h,h,2h;\rho)=\frac{2^{2h-1}(1-\rho^2)^{2h+(m/2)}}{m^{2h}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h}\sigma_2^{2h}$$
$$\times[\{1+(-1)^{2h}\}\sum_{j=0}^{\infty}g(2j)+\{1-(-1)^{2h}\}\sum_{j=0}^{\infty}g(2j+1)].$$

Then using Lemma 4.1, we have the theorem.

Corollary 4.1 Let h be an integer. Then the $2h$-th moment of sample covariance is given by

$$\mu'(h,h,2h;\rho)=\frac{2^{2h}(1-\rho^2)^{(m+4h)/2}}{m^{2h}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h}\sigma_2^{2h}\Gamma\left(\frac{m}{2}+h\right)\Gamma\left(\frac{1}{2}+h\right){}_2F_1\left(\frac{1}{2}+h,\frac{m+2h}{2};\frac{1}{2};\rho^2\right).$$

Proof. If h is an integer, then $\{1+(-1)^{2h}\}=2,$ and $\{1-(-1)^{2h}\}=0$. Then the corollary follows from Theorem 4.1.

Theorem 4.2. Let h be any real number. Then the $(2h+1)$-th moment of sample covariance is given by

$$\mu'\left(\frac{2h+1}{2},\frac{2h+1}{2},2h+1;\rho\right)$$
$$=\frac{2^{2h}(1-\rho^2)^{(2h+1)+(m/2)}}{m^{2h+1}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h+1}\sigma_2^{2h+1}$$
$$\times\left[[1-(-1)^{2h}]h\Gamma\left(h+\frac{m+1}{2}\right)\Gamma(h)\,{}_2F_1\left(h+\frac{m+1}{2},h+1;\frac{1}{2};\rho^2\right)\right.$$
$$\left.+[1+(-1)^{2h}]2\rho\Gamma\left(h+\frac{m+2}{2}\right)\Gamma\left(h+\frac{3}{2}\right){}_2F_1\left(h+\frac{m+2}{2},h+\frac{3}{2};\frac{1}{2};\rho^2\right)\right].$$

Proof.

$$\mu'\left(\frac{2h+1}{2},\frac{2h+1}{2},2h+1;\rho\right)$$
$$=\frac{2^{2h}(1-\rho^2)^{(2h+1)+(m/2)}}{m^{2h+1}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h+1}\sigma_2^{2h+1}\sum_{k=0}^{\infty}[1+(-1)^k(-1)^{2h+1}]g(k)\ ,$$

where $g(k)=\dfrac{(2\rho)^k}{k!}\Gamma\left(\dfrac{k+m+2h+1}{2}\right)\Gamma\left(\dfrac{k+2h+2}{2}\right)$.

Since

$$\sum_{k=0}^{\infty}[1+(-1)^k(-1)^{2h+1}]g(k)=[1-(-1)^{2h}]\sum_{j=0}^{\infty}g(2j)+[1+(-1)^{2h}]\sum_{j=0}^{\infty}g(2j+1),$$

we have

$$\mu'\left(\frac{2h+1}{2},\frac{2h+1}{2},2h+1;\rho\right)=\frac{2^{2h}(1-\rho^2)^{(2h+1)+(m/2)}}{m^{2h+1}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h+1}\sigma_2^{2h+1}$$
$$\times\left[[1+(-1)^{2h}]\sum_{j=0}^{\infty}g(2j)+[1-(-1)^{2h}]\sum_{j=0}^{\infty}g(2j+1)\right].$$

Using Lemma 4.1, we have the theorem.

Corollary 4.2 If h is an integer, then the $(2h+1)$-th moment of sample covariance is given by

$$\mu'\left(\frac{2h+1}{2},\frac{2h+1}{2},2h+1;\rho\right)=\frac{2^{2h}(1-\rho^2)^{(2h+1)+(m/2)}}{m^{2h+1}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2h+1}\sigma_2^{2h+1}$$
$$\times[1+(-1)^{2h}]\Gamma\left(h+\frac{m+2}{2}\right)(2h+1)\Gamma\left(h+\frac{1}{2}\right)\rho\,{}_2F_1\left(h+\frac{m+2}{2},h+\frac{3}{2};\frac{1}{2};\rho^2\right)\Bigg].$$

The following corollary follow from Corollary 4.1 and Corollary 4,2.

Corollary 4.3 The first four moments of sample covariance are given by

$(i)\ E(W) = \rho\sigma_1\sigma_2$

$(ii)\ E(W^2) = \frac{1}{m}[(m+1)\rho^2 + 1]\sigma_1^2\sigma_2^2.$

$(iii)\ E(W^3) = \frac{(m+2)\rho}{m^2}[(m+1)\rho^2 + 3]\sigma_1^3\sigma_2^3.$

$(iv)\ E(W^4) = \frac{(m+2)}{m^3}[(m+1)(m+3)\rho^4 + 6(m+3)\rho^2 + 3]\sigma_1^4\sigma_2^4.$

The centered moments of sample covariance of order a is given by

$$\mu_a = E(W - \mu)^a, \quad a = 1, 2, \cdots,$$

That is the second, third and fourth order mean corrected moments are given by

$$\mu_2 = E(W^2) - \mu^2,$$
$$\mu_3 = E(W^3) - 3E(W^2)\mu + 2\mu^3,$$
$$\mu_4 = E(W^4) - 4E(W^3)\mu + 6E(W^2)\mu^2 - 3\mu^4.$$

Corollary 4.4 The first four centered moments of sample covariance are given by

$(i)\ \mu_2 = \frac{1}{m}(1+\rho^2)\sigma_1^2\sigma_2^2$

$(ii)\ \mu_3 = \frac{2}{m^2}\rho(3+\rho^2)\sigma_1^3\sigma_2^3,$

$(iii)\ \mu_4 = \frac{3}{m^3}[(m+2)\rho^4 + (2m+12)\rho^2 + (m+2)]\sigma_1^4\sigma_2^4$

which matches with (32.126b) of Johnson, Balakrishnan and Johnson (1995, 601).

The skewness and kurtosis are given by the moment ratios

$$\alpha_i(W) = \frac{\mu_i}{\mu_2^{i/2}}, \ i = 3, 4.$$

That is, they are given by

$$\alpha_3(W) = \frac{2\rho(3+\rho^2)}{\sqrt{m}(1+\rho^2)^{3/2}} \ , \text{ and } \ \alpha_4(W) = 3 + \frac{6(1+6\rho^2+\rho^4)}{m(1+\rho^2)^2}.$$

Note that the moment ratios are matching with Johnson, Kotz and Balakrishnan (1995, 601).

If $\rho = 0$, then $\alpha_3(W) = 0, \alpha_4(W) = 3.$ Moreover if $m \to \infty$, then $\alpha_3(W) = 0, \alpha_4(W) = 3.$

Draw skewness as a function of ρ, and study limiting property.

Acknowledgements

The authors gratefully acknowledge the research support provided by King Fahd University of Petroleum & Minerals through the project FT 2004-22.

References

Anderson, T.W. (2003). An *Introduction to Multivariate Statistical Analysis*. John Wiley and Sons. New York.

Fisher, R.A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, 10, 507-521.

Gradshteyn, I.S. and Ryzhik, I.M. (1994). Tables of Integrals, Series and Products, Academic Presss.

Gupta, A.K. and Nagar, D.K. (2000). *Matrix Variate Distributions*. Chapman and hall, London, UK.

Joarder, A.H. (2008). Moments of products and ratio of two correlated chi-square variables. To appear in *Statistical Papers*.

Johnson, N.L., Kotz, S. and Kemp, A.W. (1993). *Univariate Discrete Distributions*, John Wiley.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, v2. John Wiley.

Hirsfeld, H.O. (1937). The distribution of the ratio of covariance estimates in samples drawn from normal bivariate populations. *Biometrika*, 29, 65-79.

Mahalanobis, P.C., Bose, R.C. and Roy, S.N. (1937). Normalization of statistical variates and the use of rectangular coordinates in the use of sampling distributions. *Sankhya*, 3, 1-40.

Pearson, K., Jeffery, G.B. and Elderton, E.M. (1929). On the distribution of the first product moment-coefficient in samples drawn from an indefinitely large normal population,,. *Biometrika*, 21, 164-193. (tables computed by E.M. Elderton, pp 194-201).

Press, S.J. (1967). On the sample covariance from a bivarite normal distribution. *Annals of the Institute of Statistical Mathematics*, 19, 355-361.

Springer, M.D. (1979). *The Algebra of Random Variables*. John Wiley and Sons.

Watson, G.N. (1993). *A Treatise on the Theory of Bessel Functions.* Cambridge University Press.

Wishart, J. (1928). The generalized product moment distribution in samples from a normal multivariate population. *Biometrika*, A20, 32-52.

Wishart, J. anf Bartlett, M.S. (1932). The distribution of second order moment statistics in a normal system. *Proceedings of the Cambridge Philosophical Society*, 28, 455-459.

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