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Abstract. Benefit, premium, and risk in insurance have traditionally been calculated based on fixed interest rates. When investment rates are not fixed and/or future rates are unknown, several researchers have suggested modeling of these rates prior to actuarial present value calculations. Among these are methods that assume auto-regressive models and likely scenarios for interest or investment rates. In this article, on the other hand, we explore these calculations from the perspective of nonparametric permutation of investment yields. Results, advantages, and disadvantages of this method are outlined in this article.

1. **Introduction**

Insurance valuation has traditionally been done based on fixed interest rates. Present value and future value formula in Actuarial literature assume interest rates that are fixed (see for example Bowers et. al.[1]). If interest rates are not fixed and/or future rates are not known, several researchers have suggested alternative methods for Actuarial present value calculations where rates are treated as random. For investments especially, return rates are not fixed and future return rates are not yet known.

When rates are not fixed, Bowers et. al.[1] suggested calculating actuarial present values for each of k judgment-based likely scenarios of periodic interest profiles and averaging across these. Boyle[3] instead used auto-regressive models of order 1 to model interest rates and Panjer and Bellhouse[8] and Bellhouse and Panjer[4] use similar models

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to calculate moments of insurance and annuity functions. Wilkie[10] used Gaussian random walk to model random interest rates. Waters[9] calculated actuarial functions with interest rates that were assumed to independently come from identical Gaussian distributions and then approximated limiting distibutions of insurance policies with Pearson's curves. Dhaene[5] used ARIMA(p,d,q) process to model the force of interest. Hoedemakers, Darkiewicz, and Goovaerts[7] used the theory of comonotonic risks developed by Dhaene et. al.[6] to obtain reliable approximations of the underlying distribution functions for accurate estimates of upper quantiles and stop-loss premiums.

All these methods provide a rich ensemble of alternatives to stochastic valuation of investment rates for the purpose of assessing actuarial functions. However, each method has its own assumptions that must be met for its accurate use. Autoregressive approaches to interest-rates assume a dependence structure among past periodic interest profile holds in future periods and this model can be used to derive actuarial present values. Auto-regressive models also rely on the availability of large time series data for stability of the model. Judgment-based likely scenarios, on the other hand, assumes each of k competing scenarios of periodic interest rate profiles are the only possible profiles and are as likely as the other. Thus, calculations are performed for each interest rate profile and the resulting average is used as the actuarial present value.

In this paper, we suggest another alternative method: a method based on permutations of an investment return profile. This method neither assume the availability of large time-series data for investment rates nor restrict assessment to only a handful of investment profiles. It, however, uses the investment profile from the previous k periods as the basis of the permutated investment returns for the future k years from which actuarial present values are assessed.

In section 2, we will review the classical method of present value calculations and provide explicit formulas for present value calculations under non-fixed investment returns. In section 3, we will introduce an alternative permutation-based calculation of actuarial present values. In the conclusion section, limitations and strengths of this new method will be addressed.

2. **Classical Method of Present Value Calculations**

In this section, although we consider life insurance contracts, the discussion also applies to other kinds of insurance such as pension, property, or casualty insurance. We will assume without loss of generality an n-year term insurance with benefits payable to the beneficiary upon death of the insured. For this type of insurance, the following theorem from Bowers et. al.[1] provides a classical method of the actuarial present value calculations when the interest rate is fixed.

Theorem 1. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with fixed investment return $r_i = r$ $(i \geq 1)$ taking effect over the insurance contract period, the actuarial present value is given as*

$$A_{\substack{b \\ x:\overline{n}|}} = b\sum_{k=0}^{n-1}({}_kp_x q_{x+k})\upsilon^{k+1} \tag{2.1}$$

where
${}_kp_x$ =*probability of a person age x surviving k years*
q_{x+k} =*probability of dying after surviving x+k years*
b =*benefit amounts*
υ =*a discount factor* $1/(1+r)$ *and*
r =*fixed interest rate.*

Some may argue that this fixed interest rate, r, is defensible and is based on the geometric mean of a profile of past interest rates and can actually be used in lieu of future interest rates. This argument may be tenable for very stable investments. However, unlike most interest rates, investment returns may be variable from one period to the next. Thus, often an investment rate may not be fixed from one period to the next. For example, the following table from Broverman[2] provides investment returns for AltaMira Investment company in Canada from year 1998 to 2002.

Table 1. Yearly Returns for AltaMira Investment Co.

Year	Investment Return
1998	7.8%
1999	-3.0%
2000	9.4%
2001	6.4%
2002	6.9%

For this, the geometric mean of investment returns can be calculated to be $\sqrt[5]{(1+r_1)(1+r_2)...(1+r_5)} - 1$ which is equal to an $r = 0.054$. However, if the geometric mean r is far away from the actual investment returns, the long run use of this index as a basis for a fixed rate actuarial present value calculations may lead to some inaccurate results. Thus, a more general discounting factor based on variable rate investment returns is given by the following lemma.

Lemma 1. *For an investment with return rates r_i, $i \geq 1$, the discounting factor for the k^{th} year is given as*
$$\upsilon_k = \frac{1}{(1+r_1)(1+r_2)...(1+r_k)} = \prod_{i=1}^{k} \frac{1}{(1+r_i)}$$
where $r_o = 0$.

In many textbooks, actuarial present values (APV) are calculated on the basis of a simplifying assumption that investment rates are fixed. However, in this paper, we will consider the same calculations when investment rates are variable to reflect the true nature of uncertainty in actuarial work.

The APV based on non-constant investment returns can be given by the following theorem.

Theorem 2. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with variable investment returns r_i ($i \geq 1$) taking effect over the insurance contract period, the actuarial present value is given as*

$$A_{\substack{b \\ x:\overline{n}|}} = b \sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1} ({}_k p_x q_{x+k}) \tag{2.2}$$

where ${}_k p_x$ =probability of a person age x surviving k years
q_{x+k} =probability of dying after surviving $x+k$ years
b =premium benefits
υ_j =a discount factor $1/(1+r_j)$ for time period j and
r_j =a variable investment return rate for time period j.

Remark 1. *Equation (2.2) reduces to equation (2.1) when $r_i = r$ for all i.*

Equation (2.2) in the Theorem can be used as a basis of calculation with any non-fixed investment return profile. If the investment return rates follows an auto-regressive (AR) model, then equation (2.2) can be used with investment rates r_j following this model.

Theorem 3. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with variable investment returns r_i ($i \geq 1$) taking effect over the insurance contract period, the second moment of the actuarial present value is given as*

$$ {}^2A_{\substack{b \\ x:\overline{n}|}} = b^2 \sum_{k=0}^{n-1} \prod_{j=0}^{k} \left(\upsilon_{j+1}\right)^2 ({}_k p_x q_{x+k}) \tag{2.3}$$

and the variance of the actuarial present value, $Var(Z)$, is given as

(2.4)
$$V(Z) = b^2 \left(\sum_{k=0}^{n-1} \prod_{j=0}^{k} \left(\upsilon_{j+1}\right)^2 \left({}_k p_x q_{x+k}\right) - \left(\sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1} \left({}_k p_x q_{x+k}\right) \right)^2 \right)$$

where ${}_k p_x$ *=probability of a person age x surviving k years*
q_{x+k} *=probability of dying after surviving x+k years*
b *=premium benefits*
υ_j *=a discount factor* $1/(1+r_j)$ *for time period* j *and*
r_j *=a variable investment return rate for time period* j.

Proof. ${}^2A_{\substack{b\\x:\overline{n}|}} = E[Z^2] = \sum_{k=0}^{n-1} \prod_{j=0}^{k} \left(b\upsilon_{j+1}\right)^2 \left({}_k p_x q_{x+k}\right)$
which can be expressed as equation (2.3). For the variance of Z, $V(Z) = {}^2A_{\substack{b\\x:\overline{n}|}} - \left(A_{\substack{b\\x:\overline{n}|}}\right)^2$
$$= b^2 \sum_{k=0}^{n-1} \prod_{j=0}^{k} \left(\upsilon_{j+1}\right)^2 \left({}_k p_x q_{x+k}\right) - \left(b \sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1} \left({}_k p_x q_{x+k}\right) \right)^2$$
$$= b^2 \sum_{k=0}^{n-1} \prod_{j=0}^{k} \left(\upsilon_{j+1}\right)^2 \left({}_k p_x q_{x+k}\right) - b^2 \left(\sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1} \left({}_k p_x q_{x+k}\right) \right)^2$$
which can be expressed as equation (2.4). □

Remark 2. *When* $r_i = r$ *for all* i, *equations (2.3) and (2.4) reduce to their fixed rate analogues discussed in numerous actuarial modeling textbooks such as on page 110 of Bowers et. al.*[1].

3. **Permutation-Based Method of Present Value Calculations**

In this section, we provide an alternative method for assessing actuarial present values. But we will first provide some results for investment discount factors on the basis of permutation.

Theorem 4. *For an investment with return rates* r_i, $i \geq 1$, *the discounting factor for the* k^{th} *year in the* s^{th} *permutation*

(3.1)
$$\upsilon_{s,k} \neq \upsilon_k$$

where $\upsilon_k = \prod_{i=1}^{k} \frac{1}{(1+r_i)}$ *and* $r_{s,o} = 0$.

Proof. $\upsilon_{s,k} = \dfrac{1}{(1+r_{s,1})(1+r_{s,2})...(1+r_{s,k})} = \prod_{j=1}^{k} \dfrac{1}{(1+r_{s,j})}$.
But $\upsilon_k = \prod_{j=1}^{k} \dfrac{1}{(1+r_j)}$.
Because in general $r_{s,j} \neq r_j$, then $\upsilon_{s,k} \neq \upsilon_k$.

Remark 3. *Generally $\upsilon_{s,k} \neq \upsilon_k$ since there are exactly C_k^n ways k rates taking effects over the first k investment years can be obtained from n distinct rates. However, the following corollary provides an exception to the theorem as there is exactly $C_n^n = 1$ way n rates can be chosen from n distinct rates.*

Corollary 1. *For an investment with return rates r_i, $i \geq 1$, the discounting factor for the n^{th} year in the s^{th} permutation*

$$\upsilon_{s,n} = \upsilon_n$$

where $\upsilon_k = \prod_{i=1}^{k} \dfrac{1}{(1+r_i)}$ *and* $r_{s,o} = 0$.

□

We now turn to the actuarial present value calculations based on permutations of investment return rates.

Theorem 5. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with variable investment return r_i $(i \geq 1)$ taking effect over the insurance contract period, the actuarial present value on the basis of permutations of investment return yields is given as*

$$E[Z] = \frac{b}{n!} \sum_{s=1}^{n!} \sum_{k=0}^{n-1} ({}_k p_x q_{x+k}) \prod_{j=0}^{k} \frac{1}{(1+r_{sj})^{j+1}} \tag{3.2}$$

where r_{sj} = the s^{th} permutated investment rate at time j.

Proof. The s^{th} permutation of interest return yield produces a $n \times 1$ vector with elements $(r_{s,k})$. Because there are n time periods with investment returns $r_{s,k}$ $(k = 1, ...n)$, the number of permutations of the $n \times 1$ vector of investment returns is equal to $n!$. Calculating actuarial present value for each permutation s and averaging across permutations yield equation (3.2).

Remark 4. *The theorem provides the average of the actuarial present values across all permutations. Equation (3.2) is more general than calculating APV with judgement-based likely scenarios of investment profiles as it includes all permutations of investment profiles rather than just a handful of profiles judged to be likely.*

□

Corollary 2. *For an n-term life insurance with fixed investment return r, the actuarial present value on the basis of permutation of investment return yields is given as*

$$E[Z] = \sum_{k=0}^{n-1} ({}_kp_x q_{x+k}) b_k \left(\frac{1}{(1+r)} \right)^{k+1} \tag{3.3}$$

Proof. Because all investment return rates are equal to r, there is exactly only 1 permutation. Thus, equation (3.2) reduces to equation (3.3) in the corollary, which coincides with the formula in equation (2.1) for the well known case with fixed r. □

Now, we are also interested in the second moment and variability measures of the actuarial present values on the basis of permutations.

Theorem 6. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with investment returns r_i $(i \geq 1)$ taking effect over the insurance contract period, the second moment of the actuarial present value is given as*

$$E[Z^2] = \frac{b^2}{n!} \sum_{s=1}^{n!} \sum_{k=0}^{n-1} \prod_{j=0}^{k} \frac{1}{(1+r_{sj})^{2(j+1)}} ({}_kp_x q_{x+k}) \tag{3.4}$$

and the variance of the actuarial present value, $V(Z)$, is given as
(3.5)

$$V(Z) = \frac{b^2}{n!} \left(\sum_{s=1}^{n!} \sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{s,j+1}^2 ({}_kp_x q_{x+k}) - \frac{1}{n!} \left(\sum_{s=1}^{n!} \sum_{k=0}^{n-1} ({}_kp_x q_{x+k}) \prod_{j=0}^{k} \upsilon_{s,j+1}^2 \right)^2 \right)$$

Proof. $E[Z^2] = \frac{1}{n!} \sum_{k=1}^{n!} \sum_{k=0}^{n-1} \prod_{j=0}^{k} (b\upsilon_{s,j+1})^2 ({}_kp_x q_{x+k})$
which can be expressed as equation (3.4). For the variance of Z,
$V(Z) = E[Z^2] - (E[Z])^2$

$$= \frac{b^2}{n!} \sum_{s=1}^{n!} \sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1}^2 ({}_kp_x q_{x+k}) - \left(\frac{b}{n!} \sum_{s=1}^{n!} \sum_{k=0}^{n-1} ({}_kp_x q_{x+k}) \prod_{j=0}^{k} \upsilon_{j+1} \right)^2$$

$$= \frac{b^2}{n!} \sum_{s=1}^{n!} \sum_{k=0}^{n-1} \prod_{j=0}^{k} \upsilon_{j+1}^2 ({}_kp_x q_{x+k}) - \left(\frac{b}{n!} \right)^2 \left(\sum_{s=1}^{n!} \sum_{k=0}^{n-1} ({}_kp_x q_{x+k}) \prod_{j=0}^{k} \upsilon_{j+1} \right)^2$$

which can be expressed as equation (3.5). □

Corollary 3. *For an n-year term insurance payable to the beneficiary at the end of the year of death, with fixed investment return r, the* *second moment of the actuarial present value is given as*

$$E[Z^2] = b^2 \sum_{k=0}^{n-1} \frac{1}{(1+r)^{2(k+1)}} ({}_kp_x q_{x+k}) \tag{3.6}$$

and the variance of the actuarial present value is given as

$$V(Z) = b^2 \sum_{k=0}^{n-1} \left(\frac{1}{(1+r)^{2(k+1)}} ({}_kp_x q_{x+k}) \right) \tag{3.7}$$

Proof. Since $n \times 1$ vector of investment returns with fixed rate r can only have $n = 1$ permutation, the results in the Corollary are clear by substituting $n!$=1 in the theorem. □

The corollary coincides with known results in actuarial modeling textbooks. Interested readers can refer to Bowers et. al.[1] for example.

4. Conclusion

Results in this paper can be used for insurance contracts with investment rates that are not necessarily fixed. The derivation of the permutation-based actuarial present values are nonparametric in nature and do not assume any particular model of future investment rates. Since the calculations are done over all permutations of an investment yield profile, the method actually assumes any permutation of an investment rate profile is equally likely to occur in the future. This is in contrast to judgemental-based likely scenario of investment rates which restricts attention to just a handful of investment rate profiles.

Since this is one of the first papers to deal with permutations of investment yields as a basis for insurance calculation, no work has been done on comparing this method with other methods such as those based on autoregressive models. The empirical utility of this method over the autoregressive model has yet to be determined. Nevertheless, the nonparametric nature of the method clearly offers a logical alternative when parametric-based modeling of investment rates such as the autoregressive method do not hold.

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