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Contents

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Notation

T	:=	associative ring;
$\sigma[_T M]$	:=	Wisbauer's category of M-subgenerated T-modules
R	:=	commutative ring;
A, B	:=	R-algebras;
$_A M$ (M_A)	:=	left (right) A-module;
$_A M_B$	:=	(A, B)-bimodule;
$_A \mathbb{M}$ ($\mathbb{M}_A$)	:=	category of left (right) A-modules;
$_A \mathbb{M}_B$	:=	(A, B)-bimodules;
$\mathcal{C}$	:=	A-coring;
$\mathcal{D}$	:=	B-coring;
$\mathbb{M}^{\mathcal{C}}$ ($^{\mathcal{C}}\mathbb{M}$)	:=	category of right (left) $\mathcal{C}$-comodules;
$^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$	:=	category of $(\mathcal{D}, \mathcal{C})$-bicomodules;
$\mathrm{E}_M^{\mathcal{C}}$	:=	endomorphism ring of $M \in \mathbb{M}^{\mathcal{C}}$;
$^{\mathcal{C}}_M\mathrm{E}$	:=	endomorphism ring of $M \in {}^{\mathcal{C}}\mathbb{M}$;
$^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$	:=	endomorphism ring of $M \in {}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$;
$\mathcal{R}(\mathcal{C})$ ($\mathcal{R}_{f.i.}(\mathcal{C})$)	:=	class of (fully invariant) right $\mathcal{C}$-coideals;
$\mathcal{L}(\mathcal{C})$ ($\mathcal{L}_{f.i.}(\mathcal{C})$)	:=	class of (fully invariant) left $\mathcal{C}$-coideals
$\mathcal{I}_r(\mathcal{C}^*)$ ($\mathcal{I}_{t.s.}(\mathcal{C}^*)$)	:=	class of right (two-sided) ideals of $\mathcal{C}^*$;
$\mathcal{I}_l({}^*\mathcal{C})$ ($\mathcal{I}_{t.s.}({}^*\mathcal{C})$)	:=	class of left (two-sided) ideals of $^*\mathcal{C}$;
$\mathcal{B}(\mathcal{C})$	:=	class of $\mathcal{C}$-bicoideals;
$\mathcal{B}^l$ ($\mathcal{B}^r$)	:=	The $\mathcal{C}$-bicoideal $\mathcal{B}$ considered in $^{\mathcal{C}}\mathbb{M}$ ($\mathbb{M}^{\mathcal{C}}$);

Abstract

Prime rings (prime modules) were defined as generalization of simple rings (simple modules). In this report we introduce and investigate what turns out to be a suitable generalization of simple corings (simple comodules), namely *fully coprime corings* (*fully coprime comodules*). Moreover, we consider several *primeness* and *coprimeness* notions for comodules of a given coring and investigate their relations with the fully coprimeness and the simplicity of these comodules. These notions are applied then to study primeness and coprimeness properties of a given coring, considered as an object in its category of right (left) comodules.

We also introduce and investigate *top (bi)comodules* of corings, that can be considered as dual to top (bi)modules of rings. The *fully coprime spectrum* of a top (bi)comodule attains a *Zariski topology*, defined in a way dual to that of defining the Zariski topology on the prime spectrum of a (commutative) ring. We restrict our attention here to duo (bi)comodules (satisfying suitable conditions) and study the interplay between the coalgebraic properties of such (bi)comodules and the introduced Zariski topology. In particular, we apply our results to introduce a Zariski topology on the fully coprime spectrum of a given non-zero coring considered canonically as a duo bicomodule.

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Introduction

Prime ideals play a central role in the theory of rings. In particular, *localization of commutative rings* at prime ideals is a powerful tool in commutative algebra. One goal of this project is to introduce a suitable dual notion of *coprimeness* for corings over arbitrary (not necessarily commutative) ground rings as a first step towards developing a theory of (*co*)*localization of corings*, which seems till now to be far from reach.

The classical notion of a prime ring was generalized, in different ways, to introduce prime objects in the category of modules of a given ring (see [Wis1996, Section 13]). A main goal of this project is to introduce *coprime* comodules (coprime corings), which generalize simple comodules (simple corings). As there are several *primeness* properties of modules of a given ring, we are led as well to several *primeness* and *coprimeness* properties of comodules of a coring. We investigate these different properties and clarify the relations between them.

Coprime subcoalgebras (of cocommutative) coalgebras over base fields were introduced first by M. Takeuchi [Tak1974] and studied recently by R. Nekooei and L. Torkzadeh in [NT2001] as a generalization of simple coalgebras: simple coalgebras are coprime; and finite dimensional coprime coalgebras are simple. These coalgebras, which we call here *fully coprime,* were defined using the so called *wedge product* of subcoalgebras and can be seen as dual to prime algebras: a coalgebra C over a base field is coprime if and only if its dual algebra C^* is prime. Coprime coalgebras were considered also by P. Jara et. al. in their study of representation theory of coalgebras and path coalgebras [JMNR].

For a coring $\mathcal{C}$ over a QF ring A such that ${}_A\mathcal{C}$ $(\mathcal{C}_A)$ is projective, we observe in Proposition 3.1.12 that if $K, L \subseteq \mathcal{C}$ are any A-subbimodules that are right (left) $\mathcal{C}$-coideals as well and satisfy suitable purity conditions, then the *wedge product* $K \wedge L$, in the sense of [Swe1969], is nothing but their *internal coproduct* $(K :_{\mathcal{C}^r} L)$ $((K :_{\mathcal{C}^l} L))$ in the category of right (left) $\mathcal{C}$-comodules, in the sense of [RRW2005]. This observation suggests extending the notion of fully coprime coalgebras over base fields to *fully coprime corings* over arbitrary ground rings by replacing the wedge product of subcoalgebras with the internal coproduct of subbicomodules. We also extend that notion to *fully coprime comodules* using the internal coproduct of fully invariant subcomodules. Using the internal coproduct of a bicoideal of a coring (a fully invariant subcomodule of a comodule) with itself enables us to introduce *fully cosemiprime corings* (*fully cosemiprime comodules*). Dual to prime radicals of rings (modules), we introduce and investigate the *fully coprime coradicals* of corings (comodules).

Several papers considered the so called *top modules,* i.e. modules (over commutative rings) whose spectrum of *prime submodules* attains a Zariski topology, e.g. [Lu1999], [MMS1997], [Zha1999]. Dually, we introduce and investigate *top (bi)comodules* for corings and study their properties (restricting our attention here to duo (bi)comodules satisfying suitable conditions). In particular, we extend results of [NT2001] on the topology defined on the spectrum of (*fully*) *coprime subcoalgebras* of a given coalgebra over a base field to the general situation of a topology on the *fully coprime spectrum* of a given non-zero bicomodule over a given pair of non-zero corings.

Throughout, R is a commutative ring with $1_R \neq 0_R$, A, B are arbitrary but fixed unital R-algebras, $\mathcal{C}$ is a non-zero A-coring and $\mathcal{D}$ is a non-zero B-coring. With $\mathbb{M}^{\mathcal{C}}$ (resp. ${}^{\mathcal{C}}\mathbb{M}$, ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$) we denote the category of right $\mathcal{C}$-comodules (resp. left $\mathcal{C}$-comodules, $(\mathcal{D}, \mathcal{C})$-bicomodules). By $\mathcal{C}^r$ $(\mathcal{C}^l)$ we mean the coring $\mathcal{C}$, considered as an object in $\mathbb{M}^{\mathcal{C}}$ $({}^{\mathcal{C}}\mathbb{M})$. For a right (left) $\mathcal{C}$-comodule M we denote with $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op}$ $({}_M^{\mathcal{C}}\mathrm{E} := {}^{\mathcal{C}}\mathrm{End}(M))$ the ring of all $\mathcal{C}$-colinear endomorphisms of M with multiplication the opposite (usual) composition of maps, and call an R-submodule $X \subseteq M$ *fully invariant,* iff $f(X) \subseteq X$ for every $f \in \mathrm{E}_M^{\mathcal{C}}$ $(f \in {}_M^{\mathcal{C}}\mathrm{E})$.

All rings have unities preserved by morphisms of rings and all modules are unital. Let T be a ring and denote with ${}_T\mathbb{M}$ $(\mathbb{M}_T)$ the category of left (right) T-modules. For a left (right) T-module M, we denote with $\sigma[{}_TM] \subseteq {}_T\mathbb{M}$ $(\sigma[M_T \subseteq \mathbb{M}_T])$ Wisbauer's category of M*-subgenerated* left (right) T-modules (see [Wis1991] and [Wis1996]). With *locally projective* modules, we mean those in the sense of [Z-H1976] (see also [Abu2006]).

This report is divided as follows: after this introduction, we give a brief literature review. The first chapter is devoted to preliminaries including some definitions and results theory of rings and modules as well as from the theory of corings and comodules that are recalled later.

In Chapter 2, we investigate *(Co)Primeness of Comodules* for corings. As a coalgebra C over a base field is fully coprime if and only if its dual algebra $C^* \simeq \mathrm{End}^C(C)^{op}$ is prime (see [Tak1974, 1.4.2.] and [NT2001, Proposition 1.2]), we devote Section 2.1 to the study of primeness properties of the ring of $\mathcal{C}$-colinear endomorphisms $\mathrm{E}^{\mathcal{C}}_M := \mathrm{End}^{\mathcal{C}}(M)^{op}$ of a given right $\mathcal{C}$-comodule M of a coring $\mathcal{C}$. Given a coring $\mathcal{C}$, we say a non-zero right $\mathcal{C}$-comodule M is E-*prime* (respectively E-*semiprime, completely* E-*prime, completely* E-*semiprime*), provided the ring $\mathrm{E}^{\mathcal{C}}_M := \mathrm{End}^{\mathcal{C}}(M)^{op}$ is prime (respectively semiprime, domain, reduced). In case M is *self-cogenerator*, Theorem 2.1.17 provides sufficient and necessary conditions for M to be E-prime (respectively E-semiprime, completely E-prime, completely E-semiprime). Under suitable conditions, we clarify in Theorem 2.1.29 the relation between E-prime and subdirectly irreducible comodules. In Section 2.2 we study *fully coprime* (*fully cosemiprime*) *comodules* using the *internal coproduct* of fully invariant subcomodules. Let $\mathcal{C}$ be a coring and M be a non-zero right $\mathcal{C}$-comodule. A fully invariant non-zero $\mathcal{C}$-subcomodule $K \subseteq M$ will be called fully M-coprime (fully M-cosemiprime), iff for any (equal) fully invariant $\mathcal{C}$-subcomodules $X, Y \subseteq M$ with $K \subseteq (X :^{\mathcal{C}}_M Y)$, we have $K \subseteq X$ or $K \subseteq Y$, where $(X :^{\mathcal{C}}_M Y)$ is the internal coproduct of X, Y in the category of right $\mathcal{C}$-comodules. We call the non-zero right $\mathcal{C}$-comodule M *fully coprime* (*fully cosemiprime*), iff M is fully M-coprime (fully M-cosemiprime). The notion of fully coprimeness (fully cosemiprimeness) in the category of left $\mathcal{C}$-comodules is defined analogously. Theorem 2.2.11 clarifies the relation between fully coprime (fully cosemiprime) and E-prime (E-semiprime) comodules under suitable conditions. We define the *fully coprime spectrum* of M as the class of all fully M-coprime $\mathcal{C}$-subcomodules of M and the *fully coprime coradical* of M as the sum of all fully M-coprime $\mathcal{C}$-subcomodules. In Proposition 2.2.12 we clarify the relation between the fully coprime coradical of M and the prime radical of $\mathrm{E}^{\mathcal{C}}_M$, in case M is intrinsically injective self-cogenerator and $\mathrm{E}^{\mathcal{C}}_M$ is right Noetherian. Fully coprime comodules turn to be a generalization of simple comodules: simple comodules are trivially fully coprime; and Theorem 2.2.16 (2) shows that if the ground ring A is right Artinian and ${}_A\mathcal{C}$ is locally projective, then a non-zero finitely generated self-injective self-cogenerator right $\mathcal{C}$-comodule M is fully coprime if and only if M is simple as a $({}^*\mathcal{C}, \mathrm{E}_M^{\mathcal{C}})$-bimodule. Under suitable conditions, we clarify in Theorem 2.2.21 the relation between fully coprime and subdirectly irreducible comodules. In Section 2.3 we clarify the relations between these primeness properties and the ring structure of ${}^*\mathcal{C}$ and $\mathrm{E}_M^{\mathcal{C}}$.

In Chapter 3, we introduce and study several primeness and coprimeness properties of a non-zero coring $\mathcal{C}$, considered as an object in the category $\mathbb{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules and as an object in the category ${}^{\mathcal{C}}\mathbb{M}$ of left $\mathcal{C}$-comodules. We define the internal coproducts of $\mathcal{C}$-bicoideals, i.e. $(\mathcal{C}, \mathcal{C})$-subbicomodules of $\mathcal{C}$, in $\mathbb{M}^{\mathcal{C}}$ and in ${}^{\mathcal{C}}\mathbb{M}$ and use them to introduce the notions of *fully coprime* (*fully cosemiprime*) *$\mathcal{C}$-bicoideals* and *fully coprime* (*fully cosemiprime*) *corings*. Moreover, we introduce and study the *fully coprime spectrum* and the *fully coprime coradical* of $\mathcal{C}$ in $\mathbb{M}^{\mathcal{C}}$ (in ${}^{\mathcal{C}}\mathbb{M}$) and clarify their relations with the prime spectrum and the prime radical of $\mathcal{C}^*$ (${}^*\mathcal{C}$). We investigate several notions of coprimeness (cosemiprimeness) and primeness (semiprimeness) for $\mathcal{C}$ and clarify their relations with the simplicity (semisimplicity) of the coring under consideration. In Theorems 3.1.1 we give sufficient and necessary conditions for the dual ring $\mathcal{C}^*$ (${}^*\mathcal{C}$) of $\mathcal{C}$ to be prime (respectively semiprime, domain, reduced). In case the ground ring A is a QF ring, ${}_A\mathcal{C}, \mathcal{C}_A$ are locally projective and $\mathcal{C}^*$ is right Artinian, ${}^*\mathcal{C}$ is left Artinian, we show in Theorem 3.2.10 that $\mathcal{C}^r$ is fully coprime if and only if $\mathcal{C}$ is simple if and only if $\mathcal{C}^l$ is fully coprime.

In Chapter 4, we introduce a Zariski topology for bicomodules, whose properties turn out to be dual to those of the classical Zariski topology on the prime spectrum of commutative rings (e.g. [AM1969], [Bou1998]). Let M be a given non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule and consider the *fully coprime spectrum*

$$\mathrm{CPSpec}(M) := \{K \mid K \subseteq M \text{ is a } \textit{fully } M\textit{-coprime } (\mathcal{D}, \mathcal{C})\text{-subbicomodule}\}.$$

For every $(\mathcal{D}, \mathcal{C})$-subbicomodule $L \subseteq M$, set

$$\mathcal{V}_L := \{K \in \mathrm{CPSpec}(M) \mid K \subseteq L\} \text{ and } \mathcal{X}_L := \{K \in \mathrm{CPSpec}(M) \mid K \nsubseteq L\}.$$

As in the case of the spectra of prime submodules of modules over (commutative) rings (e.g. [Lu1999], [MMS1997], [Zha1999]), the class of *varieties* $\xi(M) := \{\mathcal{V}_L \mid L \subseteq M \text{ is a } (\mathcal{D}, \mathcal{C})\text{-subbicomodule}\}$ satisfies all axioms of closed sets in a topological space with the exception that $\xi(M)$ is *not necessarily* closed under finite unions. We say M is a *top bicomodule*, iff $\xi(M)$ is closed under finite unions, equivalently iff

$$\tau_M := \{\mathcal{X}_L \mid L \subseteq M \text{ is a } (\mathcal{D}, \mathcal{C})\text{-subbicomodule}\}$$

is a topology (in this case we call $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$ a *Zariski topology* of M). We then restrict our attention to the case in which M is a *duo bicomodule* (i.e. every subbicomodule of M is fully invariant) satisfying suitable conditions. For such a bicomodule M we study the interplay between the coalgebraic properties of M and the topological properties of $\mathbf{Z}_M$. In Section 4.3, we give some applications and examples (mainly to non-zero corings which turn out to be *duo* bicomodules in the canonical way).

Brief Literature Review

- The notion of a coprime coalgebra is due to M. Takeuchi [Tak1974, 1.4.2.][2] who defined: a subcoalgebra D of a *cocommutative* coalgebra C over a base field is coprime, iff for any subcoalgebras X and Y of C,

$$D \subseteq X \wedge Y \Rightarrow D \subseteq X \text{ or } D \subseteq Y,$$

  where

$$X \wedge Y := \mathrm{Ker}(C \overset{\Delta}{\longrightarrow} C \otimes_R C \overset{\pi_X \otimes_R \pi_Y}{\longrightarrow} C/X \otimes_R C/Y)$$

  is the so called *wedge product* of X and Y. Takeuchi proved also that a subcoalgebra $D \subseteq C$ is coprime if and only if $D^{\perp} \lhd C^*$ is a prime ideal, where C^* is the dual algebra of C and $D^{\perp} := \{f \in C^* \mid f(D) = 0\}$. The authors of [JMNR] notices also that his proof is actually valid for non necessarily cocommutative coalgebras.

- In [NT2001][3], the authors studied the notion of coprime (sub)coalgebras over fields, where the coalgebra C is said to be coprime, iff C is coprime in C. In Proposition 1.2. of [NT2001] they reproved Takeuchi's characterization of coprime subcoalgebra (in particular that C is a coprime coalgebra if and only if C^* is a prime algebra).

- For a coalgebra C, let X be the set of all coprime subcoalgebras of C. The authors of [NT2001] proved then that every simple coalgebra is coprime, and that finite dimensional coprime coalgebras are necessarily simple.

[2] this was noticed by P. Jara et al. [JMNR]

[3] This paper contains several typos and mistakes that we clarified in Remark 4.3.10.

- In [JMNR], P. Jara et. al. studied coprime subcoalgebras of path coalgebras over base fields. After defining the path coalgebra C associated to a graph, they show in particular that the path coalgebra defined by a graph (V, E) is coprime if and only if the graph is strongly connected (Theorem 3.4). The problem of characterizing coprime subcoalgebras of path coalgebras is reduced then to the case of path coalgebras with at most two vertices (Theorem 4.3.).

- In [XLZ1992], the authors use the structure of a given coalgebra C over a base field to describe some properties of the dual algebra C^*. In particular they gave sufficient and necessary conditions for the dual algebra of coalgebra to be prime (Theorem 3), domain or reduced (Corollary, page 509).

- In her Ph.D. thesis, V. Rodrigues studied prime (semiprime) comodules and prime (semiprime) coalgebras over base fields (the main results are included in [FR2005]): a right comodule M of a given coalgebra C over a base filed is said to be prime provided ${}_{C^*}M$ is a prime module, and a coalgebra C was said to be prime provided ${}_{C^*}C$ is a prime module. Observing that for any comodule of a coalgebra C over a base field, the algebra is left Artinian, prime (semiprime) comodules were characterized as those that are direct sums of simple (prime) subcomodules. Investigating prime coalgebras carefully it turned out that these are just the simple coalgebras (i.e. finite dimensional coprime coalgebras in the sense of [Tak1974]).

- Given an A-coring $\mathcal{C}$, M. Ferrero and V. Rodrigues studied in [FR2005] prime and semiprime right $\mathcal{C}$-comodules considered as rational left ${}^*\mathcal{C}$-modules in the canonical way. Although *prime coalgebras* over perfect ground commutative rings turned out to be simple, a full description of the structure of prime (semiprime) right comodules of a left locally projective coring over a left perfect (right Artinian) ground ring was obtained.

- A different approach has been taken in the recent work [Wij2006] by I. Wijayanti, where several primeness and coprimeness conditions are studied in categories of modules and then applied to categories of comodules of *locally projective* coalgebras over commutative rings.

Chapter 1

Preliminaries

In this section we introduce some definitions, remarks and lemmas to which we refer later.

1.1 Prime and coprime modules

Definition 1.1.1. Let T be a ring. A proper ideal $P \lhd T$ is called

prime, iff for any two ideals $I, J \lhd T$ with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$;

semiprime, iff for any ideal $I \lhd T$ with $I^2 \subseteq P$, we have $I \subseteq P$;

completely prime, iff for any $f, g \in P$ with $fg \in P$, either $f \in P$ or $g \in P$;

completely semiprime, iff for any $f \in T$ with $f^2 \in P$, we have $f \in P$.

The ring T is called *prime* (respectively *semiprime*, *domain*, *reduced*), iff $(0_T) \lhd T$ is prime (respectively semiprime, completely prime, completely semiprime).

1.1.2. Let T be a ring. With $\mathrm{Max}(T)$ (resp. $\mathrm{Max}_r(T)$, $\mathrm{Max}_l(T)$) we denote the class of maximal two-sided T-ideals (resp. maximal right T-ideals, maximal left T-ideals) and with $\mathrm{Sepc}(T)$ the prime spectrum of T consisting of all prime ideals of T. The *Jacobson radical* of T is denoted by $\mathrm{Jac}(T)$ and the *prime radical* of T by $\mathrm{Prad}(T)$. Notice that the ring T is semiprime if and only if $\mathrm{Prad}(T) = 0$.

Definition 1.1.3. A ring T is called

semiprimitive, iff $\mathrm{Jac}(T) = 0$;

semiprimary, iff $T/\mathrm{Jac}(T)$ is semisimple and $\mathrm{Jac}(T)$ is nilpotent.

There are various notions of prime and coprime modules in the literature; see [Wis1996, Section 13] for more details. In this paper we adopt the notion of *prime modules* due to R. Johnson [Joh1953] and its dual notion of *coprime modules* considered recently by S. Annin [Ann].

Definition 1.1.4. Let T be a ring. A non-zero T-module M will be called

prime, iff $\mathrm{ann}_T(K) = \mathrm{ann}_T(M)$ for every non-zero T-submodule $0 \neq K \subseteq M$;

coprime, iff $\mathrm{ann}_T(M/K) = \mathrm{ann}_T(M)$ for every proper T-submodule $K \subsetneqq M$;

diprime, iff $\mathrm{ann}_T(K) = \mathrm{ann}_T(M)$ or $\mathrm{ann}_T(M/K) = \mathrm{ann}_T(M)$ for every non-trivial T-submodule $0 \neq K \subsetneqq M$;

strongly prime, iff $M \in \sigma[K]$ for every non-zero T-submodule $0 \neq K \subseteq M$;

semiprime, iff $M/\mathcal{T}_K(M) \in \sigma[K]$ for every cyclic T-submodule $K \subseteq M$, where

$$\mathcal{T}_K(M) := \sum\{U \subseteq M \mid \mathrm{Hom}_T(U, \widehat{\mathrm{End}_T(\widehat{M})}K = 0\}$$

and $\widehat{M}$ denoted the self-injective hull of M;

strongly semiprime, iff $M/\mathcal{T}_K(M) \in \sigma[K]$ for every T-submodule $K \subseteq M$.

It's well known that for every prime (coprime) T-module M, the associated quotient ring $\overline{T} := T/\mathrm{ann}_T(M)$ is prime. In fact we have more:

Proposition 1.1.5. ([Lom2005, Proposition 1.1]) *Let T be a ring and M be a non-zero T-module. Then the following are equivalent:*

1. $\overline{T} := T/\mathrm{ann}_T(M)$ *is a prime ring;*
2. M *is diprime;*
3. *For every fully invariant T-submodule $K \subseteq M$ that is M-generated as an* $\mathrm{End}_T(M)$*-module,* $\mathrm{ann}_T(K) = \mathrm{ann}_T(M)$ *or* $\mathrm{ann}_T(M/K) = \mathrm{ann}_T(M)$.

Remark 1.1.6. Let T be a ring and consider the following conditions for a non-zero T-module M :

$$\mathrm{ann}_T(M/K) \neq \mathrm{ann}_T(M) \text{ for every non-zero } T\text{-submodule } 0 \neq K \subseteq M \ (*)$$

$$\mathrm{ann}_T(K) \neq \mathrm{ann}_T(M) \text{ for every proper } T\text{-submodule } K \subsetneqq M \ (**).$$

We introduce condition (**) as dual to condition (*), which is due to Wisbauer [Wis1996, Section 13]. Modules satisfying either of these conditions allow further conclusions from the primeness (coprimeness) properties: by Proposition 1.1.5, a T-module M satisfying condition (*) (condition (**)) is prime (coprime) if and only if $\overline{T} := T/\mathrm{ann}_T(M)$ is prime.

Proposition 1.1.7. (*See* [FR2005, Theorem 2.5, Corollary 2.7]) *Let T be a ring and M be a non-zero left (right) T-module, for which the ring $T/\mathrm{ann}_T(M)$ is left (right) Artinian. Then the following are equivalent:*

1. *M is a prime T-module;*
2. *$T/\mathrm{ann}_T(M)$ is simple;*
3. *M is a strongly prime T-module;*
4. *$M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, a direct sum of isomorphic simple T-submodules;*
5. *$M = \sum_{\lambda\in\Lambda} M_\lambda$, a sum of isomorphic simple T-submodules;*
6. *M is generated by each of its non-zero T-submodules;*
7. *M has no non-trivial fully invariant T-submodules;*
8. *For any pretorsion class $\mathcal{T}$ in $\sigma[M]$, $\mathcal{T}(M) = 0$ or $\mathcal{T}(M) = M$.*

Proposition 1.1.8. (*See* [FR2005, Theorem 2.9, Corollary 2.10]) *Let T be a ring and M be a non-zero left (right) T-module, for which the quotient ring $\overline{T} := T/\mathrm{ann}_T(M)$ is left (right) Artinian. Then the following are equivalent:*

1. *M is a semiprime T-module;*
2. *M is a semisimple T-module;*
3. *M is a strongly semiprime T-module;*
4. *$M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, a direct sum of prime T-submodules;*
5. *$M = \sum_{\lambda\in\Lambda} M_\lambda$, a sum of prime T-submodules;*
6. *Any semiprime T-submodule of M is a direct summand.*

1.2 Corings and comodules

In module categories, monomorphisms are injective maps. In comodule categories this is not the case in general. In fact we have:

Remark 1.2.1. For any coring $\mathcal{C}$ over a ground ring A, the module ${}_A\mathcal{C}$ is flat if and only if every monomorphism in $\mathbb{M}^{\mathcal{C}}$ is injective (e.g. [Abu2003, Proposition 1.10]). In this case, $\mathbb{M}^{\mathcal{C}}$ is a Grothendieck category with kernels formed in the category of right A-modules and given a right $\mathcal{C}$-comodule M, the intersection $\bigcap_{\lambda\in\Lambda} M_\lambda \subseteq M$ of any family $\{M_\lambda\}_\Lambda$ of $\mathcal{C}$-subcomodules of M is again a $\mathcal{C}$-subcomodule.

Definition 1.2.2. Let ${}_A\mathcal{C}$ ($\mathcal{C}_A$) be flat. We call a non-zero right (left) $\mathcal{C}$-comodule M
simple, iff M has no non-trivial $\mathcal{C}$-subcomodules;
semisimple, iff $M = \mathrm{Soc}(M)$ where

$$\mathrm{Soc}(M) := \bigoplus\{K \subseteq M \mid K \text{ is a simple } \mathcal{C}\text{-subcomodule}\}. \tag{1.1}$$

The right (left) $\mathcal{C}$-subcomodule $\mathrm{Soc}(M) \subseteq M$ defined in (1.1) is called the *socle* of M.We call a non-zero right (left) $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$ *essential in* M, and write $K \lhd_e M$, provided $K \cap \mathrm{Soc}(M) \neq 0$.

Lemma 1.2.3. ([Abu2003, Proposition 1.10]) *If A_A is injective (cogenerator) and N is a right A-module, then the canonical right $\mathcal{C}$-comodule $M := (N \otimes_A \mathcal{C}, id \otimes_A \Delta_{\mathcal{C}})$ is injective (cogenerator) in $\mathbb{M}^{\mathcal{C}}$. In particular, if A_A is injective (cogenerator) then $\mathcal{C} \simeq A \otimes_A \mathcal{C}$ is injective (cogenerator) in $\mathbb{M}^{\mathcal{C}}$.*

For an A-coring $\mathcal{C}$, the dual module ${}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ ($\mathcal{C}^* := \mathrm{Hom}_{-A}(\mathcal{C}, A)$) of left (right) A-linear maps from $\mathcal{C}$ to A is a ring under the so called *convolution product.* We remark here that the multiplications used below are opposite to those in previous papers of the author, e.g. [Abu2003], and are consistent with the ones in [BW2003].

1.2.4. Dual rings of corings. Let $(\mathcal{C}, \Delta, \varepsilon)$ be an A-coring. Then ${}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ (respectively $\mathcal{C}^* := \mathrm{Hom}_{-A}(\mathcal{C}, A)$) is an A^{op}-ring with multiplication

$$(f *^l g)(c) = \sum f(c_1 g(c_2)) \text{ (respectively } (f *^r g)(c) = \sum g(f(c_1)c_2)$$

and unity ε. The coring $\mathcal{C}$ is a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule through the left ${}^*\mathcal{C}$-action (respectively the right $\mathcal{C}^*$-action):

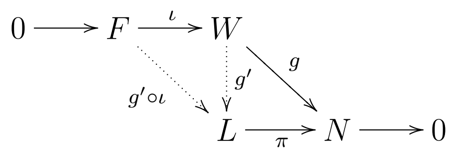
$$\begin{aligned} f \rightharpoonup c &:= \sum c_1 f(c_2) \text{ for all } f \in {}^*\mathcal{C}; \\ c \leftharpoonup g &:= \sum g(c_1)c_2 \text{ for all } g \in \mathcal{C}^*. \end{aligned}$$

1.2.5. Let M be a right (left) $\mathcal{C}$-comodule. Then M is a left ${}^*\mathcal{C}$-module (a right $\mathcal{C}^*$-module) under the left (right) action

$$\begin{aligned} f \rightharpoonup m &:= \sum m_{<0>} f(m_{<1>}) \text{ for all } f \in {}^*\mathcal{C}; \\ m \leftharpoonup g &:= \sum g(m_{<-1>})m_{<0>} \text{ for all } g \in \mathcal{C}^*. \end{aligned}$$

Notice that M is a $({}^*\mathcal{C}, \mathrm{E}_M^{\mathcal{C}})$-bimodule (a $(\mathcal{C}^*, {}_M^{\mathcal{C}}\mathrm{E})$-bimodule) in the canonical way. A right (left) $\mathcal{C}$-subcomodule $K \subseteq M$ is said to be *fully invariant*, provided K is a $({}^*\mathcal{C}, \mathrm{E}_M^{\mathcal{C}})$-subbimodule ($(\mathcal{C}^*, {}_M^{\mathcal{C}}\mathrm{E})$-subbimodule) of M. Since $\mathbb{M}^{\mathcal{C}}$ (${}^{\mathcal{C}}\mathbb{M}$) has cokernels, we conclude that for any $f \in \mathrm{E}_M^{\mathcal{C}}$ (any $g \in {}_M^{\mathcal{C}}\mathrm{E}$), $Mf := f(M) \subseteq M$ ($gM := g(M) \subseteq M$) is a right (left) $\mathcal{C}$-subcomodule and that for any right ideal $I \lhd_r \mathrm{E}_M^{\mathcal{C}}$ (left ideal $J \lhd_l {}_M^{\mathcal{C}}\mathrm{E}$) we have a fully-invariant right (left) $\mathcal{C}$-subcomodule $MI \subseteq M$ ($JM \subseteq M$).

1.2.6. ([Z-H1976]) An A-module W is called *locally projective* (in the sense of B. Zimmermann-Huisgen [Z-H1976]), if for every diagram



with exact rows and F f.g.: for every A-linear map $g : W \to N$, there exists an A-linear map $g' : W \to L$, such that the entstanding parallelogram is commutative. Note that every projective A-module is locally projective. Moreover, every locally projective A-module is flat and A-cogenerated.

Proposition 1.2.7. ([Abu2003, Theorems 2.9, 2.11]) *For any A-coring $\mathcal{C}$ we have*

1. $\mathbb{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{{}^*\mathcal{C}^{op}}] \simeq \sigma[{}_{{}^*\mathcal{C}}\mathcal{C}]$ *if and only if ${}_A\mathcal{C}$ is locally projective.*
2. ${}^{\mathcal{C}}\mathbb{M} \simeq \sigma[{}_{\mathcal{C}^{*op}}\mathcal{C}] \simeq \sigma[\mathcal{C}_{\mathcal{C}^*}]$ *if and only if $\mathcal{C}_A$ is locally projective.*

Notation. Let M be a right $\mathcal{C}$-comodule. We denote with $\mathcal{C}(M)$ ($\mathcal{C}_{f.i.}(M)$) the class of (fully invariant) $\mathcal{C}$-subcomodules of M and with $\mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}})$ ($\mathcal{I}_{t.s.}(\mathrm{E}_M^{\mathcal{C}})$) the class of right (two-sided) ideals of $\mathrm{E}_M^{\mathcal{C}}$. For $\varnothing \neq K \subseteq M$, $\varnothing \neq I \subseteq \mathrm{E}_M^{\mathcal{C}}$ set

$$\mathrm{An}(K) := \{f \in \mathrm{E}_M^{\mathcal{C}} \mid f(K) = 0\},\ \mathrm{Ke}(I) := \bigcap\{\mathrm{Ker}(f) \mid f \in I\}.$$

The following notions for right $\mathcal{C}$-comodules will be used in the sequel. The analogous notions for left $\mathcal{C}$-comodules can be defined analogously:

Definition 1.2.8. Let ${}_A\mathcal{C}$ be flat. We say a right $\mathcal{C}$-comodule M is

self-injective, iff for every $\mathcal{C}$-subcomodule $K \subseteq M$, every $\mathcal{C}$-colinear morphism $f \in \mathrm{Hom}^{\mathcal{C}}(K, M)$ extends to a $\mathcal{C}$-colinear endomorphism $\widetilde{f} \in \mathrm{End}^{\mathcal{C}}(M)$;

semi-injective, iff for every monomorphism $0 \longrightarrow N \xrightarrow{h} M$ in $\mathbb{M}^{\mathcal{C}}$, where N is a factor $\mathcal{C}$-comodule of M, and every $f \in \mathrm{Hom}^{\mathcal{C}}(N, M)$, $\exists\ \widetilde{f} \in \mathrm{End}^{\mathcal{C}}(M)$ such that $\widetilde{f} \circ h = f$;

self-projective, iff for every $\mathcal{C}$-subcomodule $K \subseteq M$, and $g \in \mathrm{Hom}^{\mathcal{C}}(M, M/K)$, $\exists\ \widetilde{g} \in \mathrm{End}^{\mathcal{C}}(M)$ such that $\pi_K \circ \widetilde{g} = g$;

self-cogenerator, iff M cogenerates all of its factor $\mathcal{C}$-comodules;

self-generator, iff M generates each of its $\mathcal{C}$-subcomodules;

coretractable, iff $\mathrm{Hom}^{\mathcal{C}}(M/K, M) \neq 0$ for every proper $\mathcal{C}$-subcomodule $K \subsetneqq M$;

retractable, iff $\mathrm{Hom}^{\mathcal{C}}(M, K) \neq 0$ for every non-zero $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$;

intrinsically injective, iff $\mathrm{AnKe}(I) = I$ for every f.g. right ideal $I \lhd \mathrm{E}_M^{\mathcal{C}}$;

subdirectly irreducible[1], iff M has a *unique* simple $\mathcal{C}$-subcomodule that is contained in every $\mathcal{C}$-subcomodule of M (equivalently, iff the intersection of all non-zero $\mathcal{C}$-subcomodules of M is again non-zero).

The following result follows immediately from ([Wis1991, 31.11, 31.12]) and Proposition 1.2.7:

Proposition 1.2.9. *Let ${}_A\mathcal{C}$ be locally projective, M be a non-zero right $\mathcal{C}$-comodule and consider the ring* $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op} = \mathrm{End}({}_{^*\mathcal{C}}M)^{op}$.

1. *If M is Artinian and self-injective, then $\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian.*

[1] *Subdirectly irreducible* comodules were called *irreducible* in [Abu2006]. However, we observed that such a terminology may cause confusion, so we choose to change it in this paper to be consistent with the terminology used for modules (e.g. [Wis1991, 9.11., 14.8.]).

2. *If M is Artinian, self-injective and self-projective, then $\mathrm{E}_M^{\mathcal{C}}$ is right Artinian.*

3. *If M is semi-injective and satisfies the ascending chain condition for annihilator $\mathcal{C}$-subcomodules, then $\mathrm{E}_M^{\mathcal{C}}$ is semiprimary.*

1.3 Annihilator conditions for comodules

Analogous to the *annihilator conditions* for modules (e.g. [Wis1991, 28.1]), the following result gives some annihilator conditions for comodules.

1.3.1. Let ${}_A\mathcal{C}$ be flat, M be a right $\mathcal{C}$-comodule and consider the order-reversing mappings

$$\mathrm{An}(-) : \mathcal{C}(M) \to \mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}}) \text{ and } \mathrm{Ke}(-) : \mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}}) \to \mathcal{C}(M). \qquad (1.2)$$

1. For every $K \in \mathcal{C}_{f.i.}(M)$ $(I \in \mathcal{I}_{t.s.}(\mathrm{E}_M^{\mathcal{C}}))$, we have $\mathrm{An}(K) \in \mathcal{I}_{t.s.}(\mathrm{E}_M^{\mathcal{C}})$ $(\mathrm{Ke}(I) \in \mathcal{C}_{f.i.}(M))$.Setting

$$\begin{array}{lcl} \mathcal{A}(\mathrm{E}_M^{\mathcal{C}}) & := & \{\mathrm{An}(K) |\ K \in \mathcal{C}(M)\}; \\ \mathcal{K}(M) & := & \{\mathrm{Ke}(I) |\ I \in \mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}})\}; \\ \mathcal{A}_{t.s.}(\mathrm{E}_M^{\mathcal{C}}) & := & \{\mathrm{An}(K) |\ K \in \mathcal{C}_{f.i.}(M)\}; \\ \mathcal{K}_{f.i.}(M) & := & \{\mathrm{Ke}(I) |\ I \in \mathcal{I}_{t.s.}(\mathrm{E}_M^{\mathcal{C}})\}, \end{array}$$

we see that $\mathrm{An}(-)$ and $\mathrm{Ke}(-)$ induce bijections

$$\mathcal{A}(\mathrm{E}_M^{\mathcal{C}}) \longleftrightarrow \mathcal{K}(M) \text{ and } \mathcal{A}_{t.s.}(\mathrm{E}_M^{\mathcal{C}}) \longleftrightarrow \mathcal{K}_{f.i.}(M).$$

2. For any $\mathcal{C}$-subcomodule $K \subseteq M$ we have

$$\mathrm{KeAn}(K) = K \text{ if and only if } M/K \text{ is } M\text{-cogenerated.}$$

3. If M is self-injective, then

   (a) $\mathrm{An}(\bigcap_{i=1}^{n} K_i) = \sum_{i=1}^{n} \mathrm{An}(K_i)$ for any finite set of $\mathcal{C}$-subcomodules $K_1, ..., K_n \subseteq M$.

   (b) M is intrinsically injective.

Remarks 1.3.2. let ${}_A\mathcal{C}$ be flat and M be a right $\mathcal{C}$-comodule.

1. If M is self-injective (self-cogenerator), then every *fully invariant* $\mathcal{C}$-subcomodule of M is also self-injective (self-cogenerator).

2. If M is self-injective, then M is semi-injective. If M is self-generator (self-cogenerator), then it is obviously retractable (coretractable).

3. If M is self-cogenerator (M is intrinsically injective and $\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian), then the mapping

$$\mathrm{An}(-) : \mathcal{C}(M) \to \mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}}) \ (\mathrm{Ke}(-) : \mathcal{I}_r(\mathrm{E}_M^{\mathcal{C}}) \to \mathcal{C}(M))$$

   is injective.

4. Let M be self-injective. If $H \subsetneqq K \subseteq M$ are $\mathcal{C}$-subcomodules with K coretractable and fully invariant in M, then $\mathrm{An}(K) \subsetneqq \mathrm{An}(H)$: since M is self-injective and $K \subseteq M$ is fully invariant, we have a surjective morphism of R-algebras $\mathrm{E}_M^{\mathcal{C}} \to \mathrm{E}_K^{\mathcal{C}} \to 0,\ f \mapsto f_{|K}$, which induces a bijection $\mathrm{An}(H)/\mathrm{An}(K) \longleftrightarrow \mathrm{An}_{\mathrm{E}_K^{\mathcal{C}}}(H) \simeq \mathrm{Hom}^{\mathcal{C}}(K/H, K) \neq 0$.

1.3.3. (e.g. [BW2003, 17.8.]) We have an isomorphism of R-algebras

$$\phi_r : \mathcal{C}^* \to \mathrm{End}^{\mathcal{C}}(\mathcal{C})^{op},\ f \mapsto [c \mapsto c \leftharpoonup f := \sum f(c_1)c_2]$$

with inverse map $\psi_r : g \mapsto \varepsilon \circ g$, and there is a ring morphism $\iota_r : A \longrightarrow (\mathcal{C}^*)^{op}$, $a \mapsto \varepsilon(a-)$.

Similarly, we have an isomorphism of R-algebras

$$\phi_l : {}^*\mathcal{C} \to {}^{\mathcal{C}}\mathrm{End}(\mathcal{C}),\ f \mapsto [c \mapsto f \rightharpoonup c := \sum c_1 f(c_2)]$$

with inverse map $\psi_l : g \mapsto \varepsilon \circ g$, and there is a ring morphism $\iota_l : A \longrightarrow ({}^*\mathcal{C})^{op}$, $a \mapsto \varepsilon(-a)$.

Definition 1.3.4. 1. We call a right (left) A-submodule $K \subseteq \mathcal{C}$ a *right (left) $\mathcal{C}$-coideal*, iff K is a right (left) $\mathcal{C}$-subcomodule of $\mathcal{C}$ with structure map the restriction of $\Delta_{\mathcal{C}}$ to K.

2. We call an (A, A)-subbimodule $B \subseteq \mathcal{C}$ a *$\mathcal{C}$-bicoideal,* iff B is a $\mathcal{C}$-subbicomodule of $\mathcal{C}$ with structure map the restriction of $\Delta_{\mathcal{C}}$ to B;

3. We call an (A,A)-subbimodule $\mathcal{D} \subseteq \mathcal{C}$ an A-*subcoring,* iff $\mathcal{D}$ is an A-coring with structure maps the restrictions of $\Delta_{\mathcal{C}}$ and $\varepsilon_{\mathcal{C}}$ to $\mathcal{D}$.

Notation. With $\mathcal{R}(\mathcal{C})$ ($\mathcal{R}_{f.i.}(\mathcal{C})$) we denote the class of (fully invariant) right $\mathcal{C}$-coideals and with $\mathcal{I}_r(\mathcal{C}^*)$ ($\mathcal{I}_{t.s.}(\mathcal{C}^*)$) the class of right (two-sided) ideals of $\mathcal{C}^*$. Analogously, we denote with $\mathcal{L}(\mathcal{C})$ ($\mathcal{L}_{f.i.}(\mathcal{C})$) the class of (fully invariant) left $\mathcal{C}$-coideals and with $\mathcal{I}_l({}^*\mathcal{C})$ ($\mathcal{I}_{t.s.}({}^*\mathcal{C})$) the class of left (two-sided) ideals of ${}^*\mathcal{C}$. With $\mathcal{B}(\mathcal{C})$ we denote the class of $\mathcal{C}$-bicoideals and for each $B \in \mathcal{B}(\mathcal{C})$ we write B^r (B^l) to indicate that we consider B as an object in the category of right (left) $\mathcal{C}$-comodules.

Remarks 1.3.5. For $\varnothing \neq I \subseteq \mathcal{C}^*$ ($\varnothing \neq I \subseteq {}^*\mathcal{C}$) and $\varnothing \neq K \subseteq \mathcal{C}$, set

$$I^{\perp(\mathcal{C})} := \bigcap_{f \in I} \{c \in \mathcal{C} \mid f(c) = 0\}$$

and

$$K^{\perp({}^*\mathcal{C})} := \{f \in {}^*\mathcal{C} \mid f(K) = 0\}; \qquad K^{\perp(\mathcal{C}^*)} := \{f \in \mathcal{C}^* \mid f(K) = 0\}.$$

1. If ${}_A\mathcal{C}$ is flat, then a right A-submodule $K \subseteq \mathcal{C}$ is a right $\mathcal{C}$-coideal, iff $\Delta(K) \subseteq K \otimes_A \mathcal{C}$.

   If $\mathcal{C}_A$ is flat, then a left A-submodule $K \subseteq \mathcal{C}$ is a left $\mathcal{C}$-coideal, iff $\Delta(K) \subseteq \mathcal{C} \otimes_A K$.

   If ${}_A\mathcal{C}$ and $\mathcal{C}_A$ are flat, then an A-subbimodule $B \subseteq \mathcal{C}$ is a $\mathcal{C}$-bicoideal, iff $\Delta(B) \subseteq (B \otimes_A \mathcal{C}) \cap (\mathcal{C} \otimes_A B)$.

   If ${}_A\mathcal{C}$ and $\mathcal{C}_A$ are flat, then an A-subbimodule $\mathcal{D} \subseteq \mathcal{C}$ is a subcoring, iff $\Delta(\mathcal{D}) \subseteq \mathcal{D} \otimes_A \mathcal{D}$.

2. Every A-subcoring $\mathcal{D} \subseteq \mathcal{C}$ is a $\mathcal{C}$-bicoideal in the canonical way.

   If $B \subseteq \mathcal{C}$ is a $\mathcal{C}$-bicoideal that is pure as a left and as a right A-submodule, then we have by [BW2003, 40.16]:

   $$\Delta(B) \subseteq (B \otimes_A \mathcal{C}) \cap (\mathcal{C} \otimes_A B) = B \otimes_A B,$$

   i.e. $B \subseteq \mathcal{C}$ is an A-subcoring.

3. If $\mathcal{C}_A$ (respectively ${}_A\mathcal{C}$) is locally projective, then $\mathcal{R}_{f.i.}(\mathcal{C}) = \mathcal{B}(\mathcal{C})$ (respectively $\mathcal{L}_{f.i.}(\mathcal{C}) = \mathcal{B}(\mathcal{C})$): if $B \subseteq \mathcal{C}$ is a fully invariant right (left) $\mathcal{C}$-coideal, then $B \subseteq \mathcal{C}$ is a right $\mathcal{C}^*$-submodule (left ${}^*\mathcal{C}$-submodule) and it follows by Proposition 1.2.7 that $B \subseteq \mathcal{C}$ is a $\mathcal{C}$-subbicomodule with structure map the restriction of $\Delta_{\mathcal{C}}$ to B, i.e. B is a $\mathcal{C}$-bicoideal.

4. Let $\mathcal{C}_A$ $({}_A\mathcal{C})$ be locally projective. If $P \lhd \mathcal{C}^*$ $(P \lhd {}^*\mathcal{C})$ is a two-sided ideal, then the fully invariant right (left) $\mathcal{C}$-coideal $B := \mathrm{ann}_{\mathcal{C}}(P) \subseteq \mathcal{C}$ is a $\mathcal{C}$-bicoideal.

5. If ${}_A\mathcal{C}$ is locally projective and $I \lhd_r {}^*\mathcal{C}$ is a right ideal, then the left ${}^*\mathcal{C}$-submodule $I^{\perp(\mathcal{C})} \subseteq \mathcal{C}$ is a right $\mathcal{C}$-coideal.

   If $\mathcal{C}_A$ is locally projective and $I \lhd_l \mathcal{C}^*$ is a left ideal, then the right $\mathcal{C}^*$-submodule $I^{\perp(\mathcal{C})} \subseteq \mathcal{C}$ is a left $\mathcal{C}$-coideal.

6. If $K \subseteq \mathcal{C}$ is a (fully invariant) right $\mathcal{C}$-coideal, then $K^{\perp(\mathcal{C}^*)} = \mathrm{ann}_{\mathcal{C}^*}(K) \simeq \mathrm{An}_{\mathrm{E}^{\mathcal{C}}_{\mathcal{C}}}(K)$; in particular $K^{\perp(\mathcal{C}^*)} \subseteq \mathcal{C}^*$ is a right (two-sided) ideal.

   If $K \subseteq \mathcal{C}$ is a (fully invariant) left $\mathcal{C}$-coideal, then $K^{\perp({}^*\mathcal{C})} = \mathrm{ann}_{{}^*\mathcal{C}}(K) \simeq \mathrm{An}_{{}^{\mathcal{C}}_{\mathcal{C}}\mathrm{E}}(K)$; in particular $K^{\perp({}^*\mathcal{C})} \subseteq {}^*\mathcal{C}$ is a left (two-sided) ideal.

7. If A_A is an injective cogenerator and ${}_A\mathcal{C}$ is flat, then for every right ideal $I \lhd_r \mathcal{C}^*$ we have $\mathrm{ann}_{\mathcal{C}}(I) = I^{\perp(\mathcal{C})}$: Write $I = \bigcup_{\lambda \in \Lambda} I_\lambda$, where $I_\lambda \lhd_r \mathcal{C}^*$ is a finitely generated right ideal for each $\lambda \in \Lambda$. If $\mathrm{ann}_{\mathcal{C}}(I_{\lambda_0}) \subsetneqq I_{\lambda_0}^{\perp(\mathcal{C})}$ for some $\lambda_0 \in \Lambda$, then $\mathrm{Hom}_A(\mathcal{C}/\mathrm{ann}_{\mathcal{C}}(I_{\lambda_0}), A) \nsubseteq \mathrm{Hom}_A(\mathcal{C}/I_{\lambda_0}^{\perp(\mathcal{C})}, A)$ (since A_A is a cogenerator). Since A_A is injective, $\mathcal{C}$ is injective in $\mathbb{M}^{\mathcal{C}}$ by Lemma 1.2.3 and it follows by 1.3.1 (3-b) and the remarks above that $I_{\lambda_0} = \mathrm{ann}_{\mathcal{C}^*}(\mathrm{ann}_{\mathcal{C}}(I_{\lambda_0})) = (\mathrm{ann}_{\mathcal{C}}(I_{\lambda_0}))^{\perp(\mathcal{C}^*)} \subsetneqq I_{\lambda_0}^{\perp(\mathcal{C})\perp(\mathcal{C}^*)}$ (a contradiction). So $\mathrm{ann}_{\mathcal{C}}(I_\lambda) = I_\lambda^{\perp(\mathcal{C})}$ for each $\lambda \in \Lambda$ and we get

$$\mathrm{ann}_{\mathcal{C}}(I) = \bigcap_{\lambda \in \Lambda} \mathrm{ann}_{\mathcal{C}}(I_\lambda) = \bigcap_{\lambda \in \Lambda} I_\lambda^{\perp(\mathcal{C})} = (\bigcup_{\lambda \in \Lambda} I_\lambda)^{\perp(\mathcal{C})} = I^{\perp(\mathcal{C})}.\blacksquare$$

1.4 Bicomodules

To the end of this section, $\mathcal{C}$ is a non-zero A-coring and $\mathcal{D}$ is a non-zero B-coring with ${}_A\mathcal{C}, \mathcal{D}_B$ flat. Moreover, M is a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule.

Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. Then M is a $({}^*\mathcal{C}, \mathcal{D}^*)$-bimodule with actions

$$f \rightharpoonup m := \sum m_{<0>} f(m_{<1>}) \text{ and } m \leftharpoonup g := \sum g(m_{<-1>}) m_{<0>},$$

for all $f \in {}^*\mathcal{C}$, $g \in \mathcal{D}^*$, $m \in M$. Moreover, the set ${}^{\mathcal{D}}\mathrm{E}^{\mathcal{C}}_M := {}^{\mathcal{D}}\mathrm{End}^{\mathcal{C}}(M)^{op}$ of $(\mathcal{D}, \mathcal{C})$-bicolinear endomorphisms of M is a ring with multiplication the *opposite* composition of maps, so that M is canonically a $({}^*\mathcal{C} \otimes_R \mathcal{D}^{*op}, {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$-bimodule. A $(\mathcal{D}, \mathcal{C})$-subbicomodule $L \subseteq M$ is called *fully invariant,* iff it is a right ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$-submodule as well. We call $M \in {}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ *duo* (*quasi-duo*), iff every (simple) $(\mathcal{D}, \mathcal{C})$-subbicomodule of M is fully invariant. If ${}_A\mathcal{C}$ and $\mathcal{D}_B$ are locally projective, then ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}} \simeq {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{(\mathcal{D}^*)^{op}}\mathbb{M}_{({}^*\mathcal{C})^{op}}) = {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{{}^*\mathcal{C}}\mathbb{M}_{\mathcal{D}^*})$ (the category of $(\mathcal{D}, \mathcal{C})$-*birational* $({}^*\mathcal{C}, \mathcal{D}^*)$-*bimodules*, e.g. [Abu2003, Theorem 2.17.]).

Notation. Let M be a $(\mathcal{D}, \mathcal{C})$-bicomodule. With $\mathcal{L}(M)$ (resp. $\mathcal{L}_{f.i.}(M)$) we denote the lattice of (fully invariant) $(\mathcal{D}, \mathcal{C})$-subbicomodules of M and with $\mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ (resp. $\mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$) the lattice of right (two-sided) ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$. With $\mathcal{I}_r^{f.g.}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \subseteq \mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ (resp. $\mathcal{L}^{f.g.}(M) \subseteq \mathcal{L}(M)$) we denote the subclass of finitely generated right ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ (the subclass of $(\mathcal{D}, \mathcal{C})$-subbicomodules of M which are finitely generated as (B, A)-bimodules). For $\varnothing \neq K \subseteq M$ and $\varnothing \neq I \subseteq {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ we set

$$\mathrm{An}(K) := \{f \in {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}} \mid f(K) = 0\} \text{ and } \mathrm{Ke}(I) := \{m \in M \mid f(m) = 0 \ \forall \ f \in I\}.$$

In what follows we introduce some notions for an object in ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$:

Definition 1.4.1. We say that a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule M is

self-injective, iff for every $(\mathcal{D}, \mathcal{C})$-subbicomodule $K \subseteq M$, every $f \in {}^{\mathcal{D}}\mathrm{Hom}^{\mathcal{C}}(K, M)$ extends to some $(\mathcal{D}, \mathcal{C})$-bicolinear endomorphism $\widetilde{f} \in {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$;

self-cogenerator, iff M cogenerates M/K in ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ $\forall$ $(\mathcal{D}, \mathcal{C})$-subbicomodule $K \subseteq M$;

intrinsically injective, iff $\mathrm{AnKe}(I) = I$ for every finitely generated right ideal $I \lhd_r {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$.

simple, iff M has no non-trivial $(\mathcal{D}, \mathcal{C})$-subbicomodules;

subdirectly irreducible, iff M contains a unique simple $(\mathcal{D}, \mathcal{C})$-subbicomodule that is contained in every non-zero $(\mathcal{D}, \mathcal{C})$-subbicomodule of M (equivalently, iff $\bigcap_{0 \neq K \in \mathcal{L}(M)} \neq 0$).

semisimple, iff $M = \mathrm{Corad}(M)$, where $\mathrm{Corad}(M) := \sum\{K \subseteq M \mid K$ is a simple $(\mathcal{D}, \mathcal{C})$-subbicomodule$\}$ (:= 0, if M has no simple $(\mathcal{D}, \mathcal{C})$-subbicomodules).

Notation. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. We denote with $\mathcal{S}(M)$ ($\mathcal{S}_{f.i.}(M)$) the class of simple $(\mathcal{D}, \mathcal{C})$-subbicomodules of M (non-zero fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodules of M with no non-trivial fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodules). Moreover, we denote with $\mathrm{Max}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ ($\mathrm{Max}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$) the class of maximal right (two-sided) ideals of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$. The Jacobson radical (prime radical) of ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is denoted by $\mathrm{Jac}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}})$ $(\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}))$.

1.4.2. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. We M has *Property* $\mathbf{S}$ ($\mathbf{S}_{f.i.}$, iff $\mathcal{S}(L) \neq \varnothing$ ($\mathcal{S}_{f.i.}(L) \neq \varnothing$) for every (fully invariant) non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq L \subseteq M$. Notice that if M has $\mathbf{S}$, then M is subdirectly irreducible if and only if $L_1 \cap L_2 \neq 0$ for any two non-zero $(\mathcal{D},\mathcal{C})$-subbicomodules $0 \neq L_1, L_2 \subseteq M$.

Lemma 1.4.3. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. If $B \otimes_R A^{op}$ is left perfect and ${}_A\mathcal{C}$, $\mathcal{D}_B$ are locally projective, then*

1. *every finite subset of M is contained in a $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ that is finitely generated as a (B,A)-bimodule.*
2. *every non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq L \subseteq M$ has a simple $(\mathcal{D},\mathcal{C})$-subbicomodule, so that M has Property $\mathbf{S}$. If moreover, M is quasi-duo, then M has Property $\mathbf{S}_{f.i.}$.*
3. $\mathrm{Corad}(M) \subseteq^e M$ *(an essential $(\mathcal{D},\mathcal{C})$-subbicomodule).*

Proof. 1. It's enough to show the assertion for a single element $m \in M$. Let $\varrho_M^{\mathcal{C}}(m) = \sum_{i=1}^{n} m_i \otimes_A c_i$ and $\varrho_M^{\mathcal{D}}(m_i) = \sum_{j=1}^{k_i} d_{i,j} \otimes_B m_{ij}$ for each $i = 1, ..., n$. Since ${}_A\mathcal{C}$, $\mathcal{D}_B$ are locally projective, the $({}^*\mathcal{C}, \mathcal{D}^*)$-subbimodule $L := {}^*\mathcal{C} \rightharpoonup m \leftharpoonup \mathcal{D}^* \subseteq M$ is by [Abu2003, Theorem 2.17.] a $(\mathcal{D},\mathcal{C})$-subbicomodule. Moreover, $\{m_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq k_i\}$ generates ${}_BL_A$, since

$$f \rightharpoonup m \leftharpoonup g = [\sum_{i=1}^{n} m_i f(c_i)] \leftharpoonup g = \sum_{i=1}^{n} \sum_{j=1}^{k_i} g(d_{i,j}) m_{i,j} f(c_i)$$

for all $f \in {}^*\mathcal{C}$ and $g \in \mathcal{D}^*$.

2. Suppose $0 \neq L \subseteq M$ is a $(\mathcal{D},\mathcal{C})$-subbicomodule with no simple $(\mathcal{D},\mathcal{C})$-subbicomodules. By "1", L contains a non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq L_1 \subsetneqq M$ that is finitely generated as a (B,A)-bimodule. Since L contains no simple $(\mathcal{D},\mathcal{C})$-subbicomodules, for every $n \in \mathbb{N}$ we can pick (by induction) a non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq L_{n+1} \subsetneqq L_n$ that is finitely generated as a $B \otimes_R A^{op}$-module. In this way we obtain an

infinite chain $L_1 \supsetneqq L_2 \supsetneqq ... \supsetneqq L_n \supsetneqq L_{n+1} \supsetneqq ....$ of finitely generated $B \otimes_R A^{op}$-submodules of L (a contradiction to the assumption that $B \otimes_R A^{op}$ is left perfect, see [Fai1976, Theorem 22.29]). Consequently, L should contain at least one simple $(\mathcal{D}, \mathcal{C})$-subbicomodule. Hence M has property **S**. The last statement is obvious.

3. For every non-zero $(\mathcal{D}, \mathcal{C})$-subbicomodule $0 \neq L \subseteq M$, we have by "1" $L \cap \mathrm{Corad}(M) = \mathrm{Corad}(L) \neq 0$, hence $\mathrm{Corad}(M) \subseteq^e M$.■

Given a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule M, we have the following annihilator conditions. The proofs are similar to the corresponding results in [Wis1991, 28.1.], hence omitted:

1.4.4. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule and consider the order-reversing mappings

$$\mathrm{An}(-) : \mathcal{L}(M) \to \mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \text{ and } \mathrm{Ke}(-) : \mathcal{I}_r({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \to \mathcal{L}(M). \tag{1.3}$$

1. $\mathrm{An}(-)$ and $\mathrm{Ke}(-)$ restrict to order-reversing mappings

$$\mathrm{An}(-) : \mathcal{L}_{f.i.}(M) \to \mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \text{ and } \mathrm{Ke}(-) : \mathcal{I}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) \to \mathcal{L}_{f.i.}(M). \tag{1.4}$$

2. For a $(\mathcal{D}, \mathcal{C})$-subbicomodule $K \subseteq M$: $\mathrm{Ke}(\mathrm{An}(K)) = K$ if and only if M/K is M-cogenerated. So, if M is self-cogenerator, then the map $\mathrm{An}(-)$ in (1.3) and its restriction in (1.4) are injective.

3. If M is self-injective, then

   (a) $\mathrm{An}(\bigcap_{i=1}^{n} K_i) = \sum_{i=1}^{n} \mathrm{An}(K_i)$ for any $(\mathcal{D}, \mathcal{C})$-subbicomodules $K_1, ..., K_n \subseteq M$ (i.e. $\mathrm{An}(-)$ in (1.3) and its restriction in (1.4) are lattice antimorphisms).

   (b) M is intrinsically injective.

Remarks 1.4.5. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is *right-duo* (i.e. every right ideal is a two-sided ideal), then M is duo. On the other hand, if M is intrinsically injective and M is duo, then ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right-duo. If M is self-injective and duo, then every fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodule of M is also duo.

Chapter 2

(Co)Prime Comodules

In this chapter we introduce and study several (co)primeness properties of comodules of corings and investigate their relation with other (co)primeness conditions in the literature.

2.1 E-prime (E-semiprime) Comodules

In this section we study and characterize non-zero comodules, for which the ring of colinear endomorphisms is prime (respectively semiprime, domain, reduced). Throughout, we assume $\mathcal{C}$ is a non-zero A-coring with ${}_A\mathcal{C}$ flat, M is a non-zero right $\mathcal{C}$-comodule and $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op}$ is the ring of $\mathcal{C}$-colinear endomorphisms of M with the opposite composition of maps. We remark that analogous results to those obtained in this section can be obtained for left $\mathcal{C}$-comodules, by symmetry.

Definition 2.1.1. We define a fully invariant non-zero $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$ to be

E-*prime in* M, iff $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is prime;
E-*semiprime in* M, iff $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is semiprime;
completely E-*prime in* M, iff $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is completely prime;
completely E-*semiprime in* M, iff $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is completely semiprime.

Definition 2.1.2. We call the right $\mathcal{C}$-comodule M E-*prime* (respectively E-*semiprime*, *completely* E-*prime*, *completely* E-*semiprime*), provided M is E-prime in M (respectively E-semiprime in M, completely E-prime in M, completely E-semiprime in M), equivalently iff R-algebra $\mathrm{E}_M^{\mathcal{C}}$ is prime (respectively semiprime, domain, reduced).

Notation. For the right $\mathcal{C}$-comodule M we denote with $\mathrm{EP}(M)$ (resp. $\mathrm{ESP}(M)$, $\mathrm{CEP}(M)$, $\mathrm{CESP}(M)$) the class of fully invariant $\mathcal{C}$-subcomodules of M whose annihilator in $\mathrm{E}_M^{\mathcal{C}}$ is prime (resp. semiprime, completely prime, completely semiprime).

Example 2.1.3. Let $P \lhd \mathrm{E}_M^{\mathcal{C}}$ be a proper two-sided ideal with $P = \mathrm{AnKe}(P)$ (e.g. M intrinsically injective and $P_{\mathrm{E}_M^{\mathcal{C}}}$ finitely generated) and consider the fully invariant $\mathcal{C}$-subcomodule $0 \neq K := \mathrm{Ke}(P) \subseteq M$. Assume $P \lhd \mathrm{E}_M^{\mathcal{C}}$ to be prime (respectively semiprime, completely prime, completely semiprime). Then $K \in \mathrm{EP}(M)$ (respectively $K \in \mathrm{ESP}(M)$, $K \in \mathrm{CEP}(M)$, $K \in \mathrm{CESP}(M)$). If moreover M is self-injective, then we have isomorphisms of R-algebras

$$\mathrm{E}_K^{\mathcal{C}} \simeq \mathrm{E}_M^{\mathcal{C}}/\mathrm{An}(K) = \mathrm{E}_M^{\mathcal{C}}/\mathrm{AnKe}(P) = \mathrm{E}_M^{\mathcal{C}}/P,$$

hence K is E-prime (respectively E-semiprime, completely E-prime, completely E-semiprime).

For the right $\mathcal{C}$-comodule M we have

$$\mathrm{CEP}(M) \subseteq \mathrm{EP}(M) \subseteq \mathrm{ESP}(M) \text{ and } \mathrm{CEP}(M) \subseteq \mathrm{CESP}(M) \subseteq \mathrm{ESP}(M). \tag{2.1}$$

Remark 2.1.4. The idea of Example 2.1.3 can be used to construct *counterexamples*, which show that the inclusions in (2.1) are *in general* strict.

The E-prime coradical

Definition 2.1.5. We define the E*-prime coradical* of the right $\mathcal{C}$-comodule M as

$$\mathrm{EPcorad}(M) = \sum_{K \in \mathrm{EP}(M)} K.$$

Proposition 2.1.6. *Let M be intrinsically injective. If $\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then*

$$\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{EPcorad}(M)). \tag{2.2}$$

If moreover M is self-cogenerator, then

$$\mathrm{EPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}})). \tag{2.3}$$

Proof. If $K \in \mathrm{EP}(M)$, then $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is a prime ideal (by definition). On the otherhand, if $P \lhd \mathrm{E}_M^{\mathcal{C}}$ is a prime ideal then $P = \mathrm{AnKe}(P)$ (since $P_{\mathrm{E}_M^{\mathcal{C}}}$ is finitely generated and M is intrinsically injective) and so $K := \mathrm{Ke}(P) \in \mathrm{EP}(M)$. It follows then that

$$\begin{array}{rclcl} \mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}}) & = & \bigcap_{P \in \mathrm{Spec}(\mathrm{E}_M^{\mathcal{C}})} P & = & \bigcap_{P \in \mathrm{Spec}(\mathrm{E}_M^{\mathcal{C}})} \mathrm{AnKe}(P) \\ & = & \bigcap_{K \in \mathrm{EP}(M)} \mathrm{AnKeAn}(K) & = & \bigcap_{K \in \mathrm{EP}(M)} \mathrm{An}(K) \\ & = & \mathrm{An}(\sum_{K \in \mathrm{EP}(M)} K) & = & \mathrm{An}(\mathrm{EPcorad}(M)). \end{array}$$

If moreover M is self-cogenerator, then

$$\mathrm{EPcorad}(M) = \mathrm{KeAn}(\mathrm{EPcorad}(M)) = \mathrm{Ke}(\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}})).\blacksquare$$

Corollary 2.1.7. *Let M be intrinsically injective self-cogenerator. If $\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then*

$$M = \mathrm{EPcorad}(M) \Leftrightarrow M \textit{ is } \mathrm{E}\textit{-semiprime.}$$

Proof. Under the assumptions and Proposition 2.1.6 we have:

$$M = \mathrm{EPcorad}(M) \Rightarrow \mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{EPcorad}(M)) = \mathrm{An}(M) = 0,$$

i.e. $\mathrm{E}_M^{\mathcal{C}}$ is semiprime; on the otherhand

$$\mathrm{E}_M^{\mathcal{C}} \text{ semiprime} \Rightarrow \mathrm{EPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}})) = \mathrm{Ke}(0) = M.\blacksquare$$

Remark 2.1.8. Let ${}_A\mathcal{C}$ be locally projective and M be right $\mathcal{C}$-comodule. A sufficient condition for $\mathrm{E}_M^{\mathcal{C}}$ to be right Noetherian, so that the results of Proposition 2.1.6 and Corollary 2.1.7 follow, is that M is Artinian and self-injective (see 1.2.9 (1)). We recall here also that in case A_A is Artinian, every finitely generated right $\mathcal{C}$-comodule has finite length by [Abu2003, Corollary 2.25 (4)].

Proposition 2.1.9. *Let $\theta : L \to M$ be an isomorphism of right $\mathcal{C}$-comodules. Then we have bijections*

$$\begin{array}{lcllcl} \mathrm{EP}(L) & \leftrightarrow & \mathrm{EP}(M), & \mathrm{ESP}(L) & \leftrightarrow & \mathrm{ESP}(M), \\ \mathrm{CEP}(L) & \leftrightarrow & \mathrm{CEP}(M), & \mathrm{CESP}(L) & \leftrightarrow & \mathrm{CESP}(M). \end{array} \tag{2.4}$$

In particular

$$\theta(\mathrm{EPcorad}(L)) = \mathrm{EPcorad}(M). \tag{2.5}$$

If moreover L, M are self-injective, then we have bijections between the class of E*-prime* (*respectively* E*-semiprime, completely* E*-prime, completely* E*-semiprime*) *$\mathcal{C}$-subcomodules of L and the class of* E*-prime* (*respectively* E*-semiprime, completely* E*-prime, completely* E*-semiprime*) *$\mathcal{C}$-subcomodules of M.*

Proof. Sine θ is an isomorphism in $\mathbb{M}^{\mathcal{C}}$, we have an isomorphism of R-algebras

$$\widetilde{\theta} : \mathrm{E}^{\mathcal{C}}_M \to \mathrm{E}^{\mathcal{C}}_L,\ f \mapsto [\theta^{-1} \circ f \circ \theta].$$

The result follows then from the fact that for every fully invariant $\mathcal{C}$-subcomodule $0 \neq H \subseteq L$ (respectively $0 \neq K \subseteq M$), $\widetilde{\theta}$ induces an isomorphism of R-algebras

$$\mathrm{E}^{\mathcal{C}}_M/\mathrm{An}(\theta(H)) \simeq \mathrm{E}^{\mathcal{C}}_L/\mathrm{An}(H) \text{ (respectively } \mathrm{E}^{\mathcal{C}}_L/\mathrm{An}(\theta^{-1}(K)) \simeq \mathrm{E}^{\mathcal{C}}_M/\mathrm{An}(K)).\blacksquare$$

Remark 2.1.10. Let L be a non-zero right $\mathcal{C}$-comodule and $\theta : L \longrightarrow M$ be a $\mathcal{C}$-colinear map. If θ is not bijective, then it is NOT evident that we have the correspondences (2.4).

Despite Remark 2.1.10 we have

Proposition 2.1.11. *Let M be self-injective and $0 \neq L \subseteq M$ be a fully invariant non-zero $\mathcal{C}$-subcomodule. Then*

$$\begin{array}{lclclcl} \mathcal{C}_{f.i.}(L) \cap \mathrm{EP}(M) & = & \mathrm{EP}(L) & ; & \mathcal{C}_{f.i.}(L) \cap \mathrm{CEP}(M) & = & \mathrm{CEP}(L) \\ \mathcal{C}_{f.i.}(L) \cap \mathrm{ESP}(M) & = & \mathrm{ESP}(L) & ; & \mathcal{C}_{f.i.}(L) \cap \mathrm{CESP}(M) & = & \mathrm{CESP}(L). \end{array}$$

Proof. Assume M to be self-injective (so that L is also self-injective). Let $0 \neq K \subseteq L$ be an arbitrary non-zero fully invariant $\mathcal{C}$-subcomodule (so that $K \subseteq M$ is also fully invariant). The result follows then directly from the definitions and the canonical isomorphisms of R-algebras

$$\mathrm{E}^{\mathcal{C}}_M/\mathrm{An}_{\mathrm{E}^{\mathcal{C}}_M}(K) \simeq \mathrm{E}^{\mathcal{C}}_K \simeq \mathrm{E}^{\mathcal{C}}_L/\mathrm{An}_{\mathrm{E}^{\mathcal{C}}_L}(K).\blacksquare$$

Corollary 2.1.12. *Let M be self-injective and $0 \neq L \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. Then $L \in \mathrm{EP}(M)$* (*respectively $L \in \mathrm{ESP}(M)$, $L \in \mathrm{CEP}(M)$, $L \in \mathrm{CESP}(M)$*) *if and only if L is* E*-prime* (*respectively* E*-semiprime, completely* E*-prime, completely* E*-semiprime*).

Sufficient and necessary conditions

Given a fully invariant non-zero $\mathcal{C}$-subcomodule $K \subseteq M$, we give sufficient and necessary conditions for $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ to be prime (respectively semiprime, completely prime, completely semiprime). These generalize the conditions given in [XLZ1992] for the dual algebras of a coalgebra over a base field to be prime (respectively semiprime, domain).

Proposition 2.1.13. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. A sufficient condition for K to be in $\mathrm{EP}(M)$ is that $K = Kf\mathrm{E}_M^{\mathcal{C}}$ $\forall f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$, where the later is also necessary in case M is self-cogenerator (or M is self-injective and K is coretractable).*

Proof. Let $I, J \lhd \mathrm{E}_M^{\mathcal{C}}$ with $IJ \subseteq \mathrm{An}(K)$. Suppose $I \nsubseteq \mathrm{An}(K)$ and pick some $f \in I \backslash \mathrm{An}(K)$. By assumption $K = Kf\mathrm{E}_M^{\mathcal{C}}$ and it follows then that $KJ = (Kf\mathrm{E}_M^{\mathcal{C}})J \subseteq K(IJ) = 0$, i.e. $J \subseteq \mathrm{An}(K)$.

On the otherhand, assume M is self-cogenerator *(*or M is self-injective and K is coretractable*)*. Suppose there exists some $f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$, such that $H := Kf\mathrm{E}_M^{\mathcal{C}} \subsetneqq K \neq 0$. Then obviously $(\mathrm{E}_M^{\mathcal{C}} f \mathrm{E}_M^{\mathcal{C}})\mathrm{An}(H) \subseteq \mathrm{An}(K)$, whereas our assumptions and Remarks 1.3.2 (3) & (4) imply that $\mathrm{E}_M^{\mathcal{C}} f \mathrm{E}_M^{\mathcal{C}} \nsubseteq \mathrm{An}(K)$ and $\mathrm{An}(H) \nsubseteq \mathrm{An}(K)$ (i.e. $\mathrm{An}(K)$ is not prime).■

Proposition 2.1.14. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. A sufficient condition for K to be in $\mathrm{ESP}(M)$ is that $Kf = Kf\mathrm{E}_M^{\mathcal{C}} f$ $\forall f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$, where the later is also necessary in case M is self-cogenerator.*

Proof. Let $I^2 \subseteq \mathrm{An}(K)$ for some $I \lhd \mathrm{E}_M^{\mathcal{C}}$. Suppose $I \nsubseteq \mathrm{An}(K)$ and pick some $f \in I \backslash \mathrm{An}(K)$. Then $0 \neq Kf \neq Kf\mathrm{E}_M^{\mathcal{C}} f \subseteq KI^2 = 0$, a contradiction. So $I \subseteq \mathrm{An}(K)$.

On the otherhand, assume that M is self-cogenerator. Suppose there exists some $f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$ with $Kf\mathrm{E}_M^{\mathcal{C}} f \subsetneqq Kf \neq 0$. By assumptions and Remark 1.3.2 (3), there exists some $g \in \mathrm{An}(Kf\mathrm{E}_M^{\mathcal{C}} f) \backslash \mathrm{An}(Kf)$ and it follows then that $J := \mathrm{E}_M^{\mathcal{C}}(fg)\mathrm{E}_M^{\mathcal{C}} \nsubseteq \mathrm{An}(K)$ while $J^2 \subseteq \mathrm{An}(K)$ (i.e. $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is not semiprime).■

Proposition 2.1.15. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. A sufficient condition for K to be in $\mathrm{CEP}(M)$ is that $K = Kf$ $\forall f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$, where the later is also necessary in case M is self-cogenerator (or M is self-injective and K is coretractable).*

Proof. 1. Let $fg \in \mathrm{An}(K)$ for some $f, g \in \mathrm{E}_M^{\mathcal{C}}$ and suppose $f \notin \mathrm{An}(K)$. The assumption $K = Kf$ implies then that $Kg = (Kf)g = K(fg) = 0$, i.e. $g \in \mathrm{An}(K)$.

On the otherhand, assume M is self-cogenerator (or M is self-injective and K is coretractable). Suppose $Kf \subsetneqq K \neq 0$ for some $f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$. By assumptions and Remarks 1.3.2 (3) & (4) there exists some $g \in \mathrm{An}(Kf) \backslash \mathrm{An}(K)$ with $fg \in \mathrm{An}(K)$ (i.e. $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is not completely prime).■

Proposition 2.1.16. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. A sufficient condition for K to be in* $\mathrm{CESP}(M)$ *is that $Kf = Kf^2$ for every $f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$, where the later is also necessary in case M is self-cogenerator.*

Proof. Let $f \in \mathrm{E}_M^{\mathcal{C}}$ be such that $f^2 \in \mathrm{An}(K)$. The assumption $K = Kf$ implies then that $Kf = Kf^2 = 0$, i.e. $f \in \mathrm{An}(K)$. On the otherhand, assume M is self-cogenerator. Suppose that $Kf^2 \subsetneqq Kf \neq 0$ for some $f \in \mathrm{E}_M^{\mathcal{C}} \backslash \mathrm{An}(K)$. By assumptions and Remark 1.3.2 (3), there exists some $g \in \mathrm{An}(Kf^2) \backslash \mathrm{An}(Kf)$. Set

$$h := \begin{cases} fgf, & \text{in case } fgf \notin \mathrm{An}(K); \\ fg, & \text{otherwise.} \end{cases}$$

So $h^2 \in \mathrm{An}(K)$ while $h \notin \mathrm{An}(K)$ (i.e. $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is not completely semiprime).■

The proof of the following result can be obtained directly from the proofs of the previous four propositions by replacing K with M.

Theorem 2.1.17. *1. M is (completely)* E*-prime, if $M = Mf\mathrm{E}_M^{\mathcal{C}}$ ($M = Mf$) for every $0 \neq f \in \mathrm{E}_M^{\mathcal{C}}$. If M is coretractable, then M is (completely)* E*-prime if and only if $M = Mf\mathrm{E}_M^{\mathcal{C}}$ ($M = Mf$) for every $0 \neq f \in \mathrm{E}_M^{\mathcal{C}}$.*

2. M is (completely) E*-semiprime, if $Mf = Mf\mathrm{E}_M^{\mathcal{C}}f$ ($Mf = Mf^2$) for every $0 \neq f \in \mathrm{E}_M^{\mathcal{C}}$. If M is self-cogenerator, then M is (completely)* E*-semiprime if and only if $Mf = Mf\mathrm{E}_M^{\mathcal{C}}f$ ($Mf = Mf^2$) for every $0 \neq f \in \mathrm{E}_M^{\mathcal{C}}$.*

E-Prime versus simple

In what follows we show that E-prime comodules generalize simple comodules.

Theorem 2.1.18. *A sufficient condition for* $\mathrm{E}_M^{\mathcal{C}}$ *to be right simple (a division ring) is that* M *is simple, where the later is also necessary in case* M *is self-cogenerator.*

Proof. If M is simple, then $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op}$ is a Division ring by Schur's Lemma.

On the otherhand, assume M to be self-cogenerator. Let $K \subseteq M$ be a $\mathcal{C}$-subcomodule and consider the right ideal $\mathrm{An}(K) \lhd_r \mathrm{E}_M^{\mathcal{C}}$. If $\mathrm{E}_M^{\mathcal{C}}$ is right simple, then $\mathrm{An}(K) = (0_{\mathrm{E}_M^{\mathcal{C}}})$ so that $K = \mathrm{KeAn}(K) = \mathrm{Ke}(0_{\mathrm{E}_M^{\mathcal{C}}}) = M$; or $\mathrm{An}(K) = \mathrm{E}_M^{\mathcal{C}}$ so that $K = \mathrm{KeAn}(K) = \mathrm{Ke}(\mathrm{E}_M^{\mathcal{C}}) = (0_M)$. Consequently M is simple.■

Theorem 2.1.19. *A sufficient condition for* $\mathrm{E}_M^{\mathcal{C}}$ *to be simple, in case* M *is intrinsically injective and* $\mathrm{E}_M^{\mathcal{C}}$ *is right Noetherian, is to assume that* M *has no non-trivial fully invariant* $\mathcal{C}$*-subcomodules, where the later is also necessary if* M *is self-cogenerator.*

Proof. The proof is similar to that of Theorem 2.1.18 replacing right ideals of $\mathrm{E}_M^{\mathcal{C}}$ by two-sided ideals and arbitrary $\mathcal{C}$-subcomodules of M with fully invariant ones.■

Notation. Consider the non-zero right $\mathcal{C}$-comodule M. With $\mathcal{S}(M)$ ($\mathcal{S}_{f.i.}(M)$) we denote the class of simple $\mathcal{C}$-subcomodules of M (non-zero fully invariant $\mathcal{C}$-subcomodules of M with no non-trivial fully invariant $\mathcal{C}$-subcomodules).

Corollary 2.1.20. *Let* M *be self-injective self-cogenerator and* $0 \neq K \subseteq M$ *be a fully invariant non-zero* $\mathcal{C}$*-subcomodule. Then*

1. $K \in \mathcal{S}(M) \Leftrightarrow \mathrm{An}(K) \in \mathrm{Max}_r(\mathrm{E}_M^{\mathcal{C}})$;
2. *If* $\mathrm{E}_M^{\mathcal{C}}$ *is right Noetherian, then* $K \in \mathcal{S}_{f.i.}(M) \Leftrightarrow \mathrm{An}(K) \in \mathrm{Max}(\mathrm{E}_M^{\mathcal{C}})$.

Proof. Recall that, since M is self-injective self-cogenerator and $K \subseteq M$ is fully invariant, K is also self-injective self-cogenerator. The result follows then from Theorems 2.1.18 and 2.1.19 applied to the R-algebra $\mathrm{E}_K^{\mathcal{C}} \simeq \mathrm{E}_M^{\mathcal{C}}/\mathrm{An}(K)$.■

Lemma 2.1.21. *Let M be intrinsically injective self-cogenerator and assume $\mathrm{E}_M^{\mathcal{C}}$ to be right Noetherian. Then the order reversing mappings (1.2) give a bijection*

$$\mathcal{S}(M) \longleftrightarrow \mathrm{Max}_r(\mathrm{E}_M^{\mathcal{C}}) \text{ and } \mathcal{S}_{f.i.}(M) \longleftrightarrow \mathrm{Max}(\mathrm{E}_M^{\mathcal{C}}). \tag{2.6}$$

Proof. Let $K \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$) and consider the proper right ideal $\mathrm{An}(K) \subsetneqq \mathrm{E}_M^{\mathcal{C}}$. If $\mathrm{An}(K) \subseteq I \subseteq \mathrm{E}_M^{\mathcal{C}}$, for some right (two-sided) ideal $I \subseteq \mathrm{E}_M^{\mathcal{C}}$, then $\mathrm{Ke}(I) \subseteq \mathrm{KeAn}(K) = K$ and it follows from the assumption $K \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$) that $\mathrm{Ke}(I) = 0$ so that $I = \mathrm{AnKe}(I) = \mathrm{E}_M^{\mathcal{C}}$; or $\mathrm{Ke}(I) = K$ so that $I = \mathrm{AnKe}(I) = \mathrm{An}(K)$. This means that $\mathrm{An}(K) \in \mathrm{Max}_r(\mathrm{E}_M^{\mathcal{C}})$ ($\mathrm{An}(K) \in \mathrm{Max}(\mathrm{E}_M^{\mathcal{C}})$).

On the otherhand, let $I \in \mathrm{Max}_r(\mathrm{E}_M^{\mathcal{C}})$ ($I \in \mathrm{Max}(\mathrm{E}_M^{\mathcal{C}})$) and consider the non-zero $\mathcal{C}$-subcomodule $0 \neq \mathrm{Ke}(I) \subseteq M$. If $K \subseteq \mathrm{Ke}(I)$ for some (fully invariant) $\mathcal{C}$-subcomodule $K \subseteq M$, then $I \subseteq \mathrm{AnKe}(I) \subseteq \mathrm{An}(K) \subseteq \mathrm{E}_M^{\mathcal{C}}$ and it follows by the maximality of I that $\mathrm{An}(K) = \mathrm{E}_M^{\mathcal{C}}$ so that $K = \mathrm{KeAn}(K) = 0$; or $\mathrm{An}(K) = I$ so that $K = \mathrm{KeAn}(K) = \mathrm{Ke}(I)$. Consequently $\mathrm{Ke}(I) \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$). Since M is intrinsically injective self-cogenerator, $\mathrm{Ke}(-)$ and $\mathrm{An}(-)$ are injective by 1.3.1 and we are done.■

Lemma 2.1.22. *Let A be left perfect and ${}_A\mathcal{C}$ be locally projective.*

1. *The non-zero right $\mathcal{C}$-comodule contains a simple $\mathcal{C}$-subcomodule.*

2. $\mathrm{Soc}(M) \vartriangleleft_e M$ *(an essential $\mathcal{C}$-subcomodule).*

Proof. Let ${}_AA$ be perfect and ${}_A\mathcal{C}$ be locally projective.

1. By [Abu2003, Corollary 2.25] M satisfies the descending chain condition on finitely generated non-zero $\mathcal{C}$-subcomodules, which turn out to be finitely generated right A-modules, hence M contains a non-zero simple $\mathcal{C}$-subcomodule.

2. Let M be a non-zero right $\mathcal{C}$-comodule. For every $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$ we have $K \cap \mathrm{Soc}(M) = \mathrm{Soc}(K) \neq 0$, by (1).■

Proposition 2.1.23. *We have*

$$\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{Soc}(M)) \text{ and } \mathrm{Soc}(M) = \mathrm{Ke}(\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}})), \tag{2.7}$$

if any of the following conditions holds:

1. *M is intrinsically injective self-cogenerator with $\mathrm{E}_M^{\mathcal{C}}$ right Noetherian;*

2. *${}_A\mathcal{C}$ is locally projective and M is Artinian self-injective self cogenerator;*

3. *A is left perfect, ${}_A\mathcal{C}$ is locally projective and M is self-injective self-cogenerator.*

Proof. 1. By Lemma 2.1.21 we have

$$\begin{aligned}\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}) &= \bigcap\{Q \mid Q \lhd_r \mathrm{E}_M^{\mathcal{C}} \text{ is a maximal right ideal}\}\\ &= \bigcap\{\mathrm{AnKe}(Q) \mid Q \lhd_r \mathrm{E}_M^{\mathcal{C}} \text{ is a maximal right ideal}\}\\ &= \bigcap\{\mathrm{AnKe}(\mathrm{An}(K)) \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\}\\ &= \bigcap\{\mathrm{An}(K) \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\}\\ &= \mathrm{An}(\textstyle\sum\{K \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\})\\ &= \mathrm{An}(\mathrm{Soc}(M)).\end{aligned}$$

Since M is self-cogenerator, we have $\mathrm{Soc}(M) = \mathrm{KeAn}(\mathrm{Soc}(M)) = \mathrm{Ke}(\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}))$.

2. Since M is Artinian and self-injective in $\mathbb{M}^{\mathcal{C}} = \sigma[{}_{^*\mathcal{C}}\mathcal{C}]$, we conclude that $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op} = \mathrm{End}({}_{^*\mathcal{C}}M)$ is right Noetherian by Proposition 1.2.9 (2). The result follows then by (1).

3. Since A is left perfect and ${}_A\mathcal{C}$ is locally projective, $\mathrm{Soc}(M) \lhd_e M$ is an essential $\mathcal{C}$-subcomodule by Lemma 2.1.22 (2) and it follows then, since M is self-injective, that

$$\begin{aligned}\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}) &= \mathrm{Jac}(\mathrm{End}({}_{^*\mathcal{C}}M)^{op}) &&= \mathrm{Hom}_{^*\mathcal{C}}(M/\mathrm{Soc}(M), M)^1\\ &= \mathrm{Hom}^{\mathcal{C}}(M/\mathrm{Soc}(M), M) &&\simeq \mathrm{An}(\mathrm{Soc}(M)).\end{aligned}$$

Since M is self-cogenerator, we have moreover

$$\mathrm{Soc}(M) = \mathrm{KeAn}(\mathrm{Soc}(M)) = \mathrm{Ke}(\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}})).\blacksquare$$

Corollary 2.1.24. *If any of the three conditions in Proposition 2.1.23 holds, then we have*

$$M \text{ is semisimple} \Leftrightarrow \mathrm{E}_M^{\mathcal{C}} \text{ is semiprimitive.}$$

[1] by [Wis1991, 22.1 (5)]

Proof. By assumptions and Proposition 2.1.23 we have $\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{Soc}(M))$ and $\mathrm{Soc}(M) = \mathrm{Ke}(\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}))$. Hence,

$$M \text{ semisimple} \Rightarrow \mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{Soc}(M)) = \mathrm{An}(M) = 0,$$

i.e. $\mathrm{E}_M^{\mathcal{C}}$ is semiprimitive; on the otherhand $\mathrm{E}_M^{\mathcal{C}}$ semiprimitive implies

$$\mathrm{Soc}(M) = \mathrm{Ke}(\mathrm{Jac}(\mathrm{E}_M^{\mathcal{C}})) = \mathrm{Ke}(0) = M,$$

i.e. M is semisimple.■

E-Prime versus subdirectly irreducible

In what follows we clarify the relation between E-prime and subdirectly irreducible comodules.

Remark 2.1.25. Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant $\mathcal{C}$-subcomodules of M and consider the fully invariant $\mathcal{C}$-subcomodule $K := \sum_{\lambda\in\Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \mathrm{EP}(M)$ ($K_\lambda \in \mathrm{CEP}(M)$) for every $\lambda \in \Lambda$, then $\mathrm{An}(K) = \bigcap_{\lambda\in\Lambda} \mathrm{An}(K_\lambda)$ is an intersection of (completely) prime ideals, hence a (completely) semiprime ideal, i.e. $K \in \mathrm{ESP}(M)$ ($K \in \mathrm{CESP}(M)$). If M is self-injective, then we conclude that an arbitrary sum of (completely) E-prime $\mathcal{C}$-subcomodules of M is in general (completely) E-semiprime.

Despite Remark 2.1.25 we have the following result (which is most interesting in case $K = M$):

Proposition 2.1.26. *Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant $\mathcal{C}$-subcomodules of M, such that for any $\gamma, \delta \in \Lambda$ either $K_\gamma \subseteq K_\delta$ or $K_\delta \subseteq K_\gamma$, and consider the fully invariant $\mathcal{C}$-subcomodule $K := \sum_{\lambda\in\Lambda} K_\lambda = \bigcup_{\lambda\in\Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \mathrm{EP}(M)$ ($K_\lambda \in \mathrm{CEP}(M)$) for every $\lambda \in \Lambda$, then $K \in \mathrm{EP}(M)$ ($K \in \mathrm{CEP}(M)$).*

Proof. Let $I, J \lhd \mathrm{E}_M^{\mathcal{C}}$ be such that $IJ \subseteq \mathrm{An}(K) = \bigcap_{\lambda\in\Lambda} \mathrm{An}(K_\lambda)$ and suppose $I \nsubseteq \mathrm{An}(K)$. Pick some $\lambda_0 \in \Lambda$ with $I \nsubseteq \mathrm{An}(K_{\lambda_0})$. By assumption $\mathrm{An}(K_{\lambda_0}) \lhd \mathrm{E}_M^{\mathcal{C}}$ is prime and $IJ \subseteq \mathrm{An}(K_{\lambda_0})$, so $J \subseteq \mathrm{An}(K_{\lambda_0})$. We **claim** that $J \subseteq \bigcap_{\lambda\in\Lambda} \mathrm{An}(K_\lambda)$: Let $\lambda \in \Lambda$ be arbitrary. If $K_\lambda \subseteq K_{\lambda_0}$, then $J \subseteq \mathrm{An}(K_{\lambda_0}) \subseteq \mathrm{An}(K_\lambda)$. On the other hand, if $K_{\lambda_0} \subseteq K_\lambda$ and $J \nsubseteq \mathrm{An}(K_\lambda)$, then the primeness of $\mathrm{An}(K_\lambda)$ implies that $I \subseteq \mathrm{An}(K_\lambda) \subseteq \mathrm{An}(K_{\lambda_0})$, a contradiction. So $J \subseteq \bigcap_{\lambda\in\Lambda} \mathrm{An}(K_\lambda) = \mathrm{An}(K)$. Consequently $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is a prime ideal, i.e. $K \in \mathrm{EP}(M)$.

In case $\mathrm{An}(K_\lambda) \lhd \mathrm{E}_M^{\mathcal{C}}$ is completely prime for every $\lambda \in \Lambda$, then replacing ideals in the argument above with elements yields that $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is a completely prime ideal, i.e. $K \in \mathrm{CEP}(M)$.■

Remark 2.1.27. If M is self-injective and the subcomodule K_λ in Proposition 2.1.26 are *(*completely*)* E-prime, then K is *(*completely*)* E-prime (recall that we have in this case an isomorphism of algebras $\mathrm{E}_M^{\mathcal{C}}/\mathrm{Ann}(K_\lambda) \simeq \mathrm{E}_{K_\lambda}^{\mathcal{C}}$).

Proposition 2.1.28. *Let M be self-cogenerator and $K \in \mathrm{EP}(M)$. Then K admits no decomposition as an internal direct sum of non-trivial fully invariant $\mathcal{C}$-subcomodules.*

Proof. Let $K \subseteq M$ be a fully invariant $\mathcal{C}$-subcomodule with $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ a prime ideal and suppose $K = K_{\lambda_0} \oplus \sum_{\lambda\neq\lambda_0} K_\lambda$ to be a decomposition of K as an internal direct sum of non-trivial fully invariant $\mathcal{C}$-subcomodules. Consider the two-sided ideals $I := \mathrm{An}(K_{\lambda_0})$, $J := \mathrm{An}(\sum_{\lambda\neq\lambda_0} K_\lambda)$ of $\mathrm{E}_M^{\mathcal{C}}$, so that $IJ \subseteq \mathrm{An}(K)$. If $J \subseteq \mathrm{An}(K)$, then $K_{\lambda_0} \subseteq K = \mathrm{KeAn}(K) \subseteq \mathrm{Ke}(J) = \sum_{\lambda\neq\lambda_0} K_\lambda$ (a contradiction). Since $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is prime, $I \subseteq \mathrm{An}(K)$ and we conclude that $K = \mathrm{KeAn}(K) \subseteq \mathrm{Ke}(I) = \mathrm{KeAn}(K_{\lambda_0}) = K_{\lambda_0}$ (a contradiction).■

The following result clarifies, under suitable conditions, the relation between E-prime and subdirectly irreducible comodules.

Theorem 2.1.29. *Assume ${}_A\mathcal{C}$ to be locally projective, M to be self-injective self-cogenerator and* $\mathrm{End}^{\mathcal{C}}(M)$ *to be commutative. If M is* E*-prime, then M is subdirectly irreducible.*

Proof. If $\mathrm{End}^{\mathcal{C}}(M = \mathrm{End}({}_{^*\mathcal{C}}M)$ is commutative, then under the assumptions on M, [Wis1991, 48.16] yield that M is a direct sum of subdirectly irreducible fully invariant $\mathcal{C}$-subcomodules. The results follows then by Proposition 2.1.28.■

2.2 Fully Coprime (fully cosemiprime) comodules

As before, $\mathcal{C}$ is a non-zero A-coring with ${}_A\mathcal{C}$ flat, M is a non-zero right $\mathcal{C}$-comodule and $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op}$ is the ring of $\mathcal{C}$-colinear endomorphisms of M with the opposite composition of maps.

2.2.1. For R-submodules $X, Y \subseteq M$, set

$$(X :_M^{\mathcal{C}} Y) := \bigcap\{f^{-1}(Y)|\ f \in \mathrm{End}^{\mathcal{C}}(M) \text{ and } f(X) = 0\}.$$

If $Y \subseteq M$ is a right $\mathcal{C}$-subcomodule, then $f^{-1}(Y) \subseteq M$ is a $\mathcal{C}$-subcomodule for each $f \in \mathrm{E}_M^{\mathcal{C}}$, being the kernel of the $\mathcal{C}$-colinear map $\pi_Y \circ f : M \longrightarrow M/Y$, and it follows then that $(X :_M^{\mathcal{C}} Y) \subseteq M$ is a right $\mathcal{C}$-subcomodule, being the intersection of right $\mathcal{C}$-subcomodules of M. If $X \subseteq M$ is fully invariant, i.e. $f(X) \subseteq X$ for every $f \in \mathrm{E}_M^{\mathcal{C}}$, then $(X :_M^{\mathcal{C}} Y) \subseteq M$ is clearly fully invariant. If $X, Y \subseteq M$ are right $\mathcal{C}$-subcomodules, then the right $\mathcal{C}$-subcomodule $(X :_M^{\mathcal{C}} Y)$ is called the *internal coproduct* of X and Y in the category $\mathbb{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules. If $\mathcal{C}_A$ is flat, then the internal coproduct of $\mathcal{C}$-subcomodules of left $\mathcal{C}$-comodules can be defined analogously.

Remark 2.2.2. The *internal coproduct* of submodules of a given module over a ring was first introduced by Bican et. al. [BJKN1980] to present the notion of *coprime modules*. The definition was modified in [RRW2005], where arbitrary submodules are replaced by the fully invariant ones. To avoid any possible confusion, we refer to coprime modules in the sense of [RRW2005] as *fully coprime modules* and transfer that terminology to *fully coprime comodules*.

Definition 2.2.3. A fully invariant $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$ will be called

fully M-coprime, iff for any two fully invariant $\mathcal{C}$-subcomodules X, $Y \subseteq M$ with $K \subseteq (X :_M^{\mathcal{C}} Y)$, we have $K \subseteq X$ or $K \subseteq Y$;

fully M-cosemiprime, iff for any fully invariant $\mathcal{C}$-subcomodule $X \subseteq M$ with $K \subseteq (X :_M^{\mathcal{C}} X)$, we have $K \subseteq X$.

We call M *fully coprime* (*fully cosemiprime*), iff M is fully M-coprime (fully M-cosemiprime).

The fully coprime coradical

Definition 2.2.4. We define the *fully coprime spectrum* of M as

$$\mathrm{CPSpec}(M) := \{K \mid 0 \neq K \subseteq M \text{ is a fully } M\text{-coprime } \mathcal{C}\text{-subcomodule}\}.$$

We define the *fully coprime coradical* of M as

$$\mathrm{CPcorad}(M) = \sum_{K \in \mathrm{CPSpec}(M)} K.$$

Moreover, we set

$$\mathrm{CSP}(M) := \{K \mid 0 \neq K \subseteq M \text{ is an fully } M\text{-cosemiprime } \mathcal{C}\text{-subcomodule}\}.$$

The fully coprime spectra (fully coprime coradicals) of comodules are invariant under isomorphisms of comodules:

Proposition 2.2.5. *Let $\theta : L \to M$ be an isomorphism of $\mathcal{C}$-comodules. Then we have bijections*

$$\mathrm{CPSpec}(L) \longleftrightarrow \mathrm{CPSpec}(M) \text{ and } \mathrm{CSP}(L) \longleftrightarrow \mathrm{CSP}(M).$$

In particular

$$\theta(\mathrm{CPcorad}(L)) = \mathrm{CPcorad}(M). \tag{2.8}$$

Proof. Let $\theta : L \to M$ be an isomorphism of right $\mathcal{C}$-comodules. Let $0 \neq H \subseteq L$ be a fully invariant $\mathcal{C}$-subcomodule that is fully L-coprime and consider the fully invariant $\mathcal{C}$-subcomodule $0 \neq \theta(H) \subseteq M$. Let $X, Y \subseteq M$ be two fully invariant $\mathcal{C}$-subcomodules with $\theta(H) \subseteq (X :^{\mathcal{C}}_{M} Y)$. Then $\theta^{-1}(X)$, $\theta^{-1}(Y) \subseteq L$ are two fully invariant $\mathcal{C}$-subcomodules and $H \subseteq (\theta^{-1}(X) :^{\mathcal{C}}_{L} \theta^{-1}(Y))$. By assumption H is fully L-coprime and we conclude that $H \subseteq \theta^{-1}(X)$ so that $\theta(H) \subseteq X$; or $H \subseteq \theta^{-1}(Y)$ so that $\theta(H) \subseteq Y$. Consequently $\theta(H)$ is fully M-coprime. Analogously one can show that for any fully invariant fully M-coprime $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$, the fully invariant $\mathcal{C}$-subcomodule $0 \neq \theta^{-1}(K) \subseteq L$ is fully L-coprime.

Repeating the proof above with $Y = X$, one can prove that for any fully L-cosemiprime (fully M-cosemiprime) fully invariant $\mathcal{C}$-subcomodule $0 \neq H \subseteq L$ (resp. $0 \neq K \subseteq M$), the fully invariant $\mathcal{C}$-subcomodule $0 \neq \theta(H) \subseteq M$ $(0 \neq \theta^{-1}(K) \subseteq L)$ is fully M-cosemiprime (fully L-cosemiprime).■

Remark 2.2.6. Let L be a non-zero right $\mathcal{C}$-comodules and $\theta : L \to M$ be a $\mathcal{C}$-colinear map. If θ is not bijective, then it is NOT evident that for $K \in \mathrm{CPSpec}(L)$ (respectively $K \in \mathrm{CSP}(L)$) we have $\theta(K) \subseteq \mathrm{CPSpec}(M)$ (respectively $\theta(K) \in \mathrm{CSP}(M)$).

Despite Remark 2.2.6 we have

Proposition 2.2.7. *Let $0 \neq L \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. Then we have*

$$\mathcal{M}_{f.i.}(L) \cap \mathrm{CPSpec}(M) \subseteq \mathrm{CPSpec}(L) \text{ and } \mathcal{M}_{f.i.}(L) \cap \mathrm{CSP}(M) \subseteq \mathrm{CSP}(L), \tag{2.9}$$

with equality in case M is self injective.

Proof. Let $0 \neq H \subseteq L$ be a fully invariant $\mathcal{C}$-subcomodule and assume H to be fully M-coprime (fully M-cosemiprime). Suppose $H \subseteq (X :^{\mathcal{C}}_{L} Y)$ for two (equal) fully invariant $\mathcal{C}$-subcomodules $X,\ Y \subseteq L$. Since $L \subseteq M$ is a fully invariant $\mathcal{C}$-subcomodule, it follows that X, Y are also fully invariant $\mathcal{C}$-subcomodules of M and moreover $(X :^{\mathcal{C}}_{L} Y) \subseteq (X :^{\mathcal{C}}_{M} Y)$. By assumption H is fully M-coprime (fully M-cosemiprime), and so the inclusions $H \subseteq (X :^{\mathcal{C}}_{L} Y) \subseteq (X :^{\mathcal{C}}_{M} Y)$ imply $H \subseteq X$ or $H \subseteq Y$. Consequently H is fully L-coprime (fully L-cosemiprime). Hence the inclusions in (2.9) hold.

Assume now that M is self-injective. Let $0 \neq H \subseteq L$ to be an fully L-coprime (fully L-cosemiprime) $\mathcal{C}$-subcomodule. Suppose $X, Y \subseteq M$ are two (equal) fully invariant $\mathcal{C}$-subcomodules with $H \subseteq (X :^{\mathcal{C}}_{M} Y)$ and consider the fully invariant $\mathcal{C}$-subcomodules $X \cap L,\ Y \cap L \subseteq L$. Since M is self-injective, the embedding $\iota : L/X \cap L \hookrightarrow M/X$ induces a surjective set map

$$\Phi : \mathrm{Hom}^{\mathcal{C}}(M/X, M) \to \mathrm{Hom}^{\mathcal{C}}(L/X \cap L, M),\ f \mapsto f_{|_{L/X\cap L}}.$$

Since $L \subseteq M$ is fully invariant, Φ induces a surjective set map

$$\Psi : \mathrm{An}_{\mathrm{E}^{\mathcal{C}}_{M}}(X) \to \mathrm{An}_{\mathrm{E}^{\mathcal{C}}_{L}}(X \cap L),\ g \mapsto g_{|_{L}}, \tag{2.10}$$

which implies that $H \subseteq (X \cap L :^{\mathcal{C}}_{L} Y \cap L)$. By assumption H is fully L-coprime, hence $H \subseteq X \cap L$ so that $H \subseteq X$; or $H \subseteq Y \cap L$ so that $H \subseteq Y$. Hence H is fully M-coprime (fully M-cosemiprime). Consequently the inclusions in (2.9) become equality.■

Remark 2.2.8. Let $0 \neq L \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. By Proposition 2.2.7, a sufficient condition for L to be fully coprime *(*fully cosemiprime*)* is that L is fully M-coprime *(*fully M-cosemiprime*)*, where the later is also necessary in case M is self-injective.

Lemma 2.2.9. *Let $X, Y \subseteq M$ be any R-submodules. Then*

$$(X :^{\mathcal{C}}_M Y) \subseteq \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y)), \tag{2.11}$$

with equality in case M is self-cogenerator and $Y \subseteq M$ is a $\mathcal{C}$-subcomodule.

Proof. Let $m \in (X :^{\mathcal{C}}_M Y)$ be arbitrary. Then for all $f \in \mathrm{An}(X)$ we have $f(m) = y$ for some $y \in Y$ and so for each $g \in \mathrm{An}(Y)$ we get

$$(f \circ^{op} g)(m) = (g \circ f)(m) = g(f(m)) = g(y) = 0,$$

i.e. $(X :^{\mathcal{C}}_M Y) \subseteq \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y))$.

Assume now that M is self-cogenerator and that $Y \subseteq M$ is a $\mathcal{C}$-subcomodule (so that $\mathrm{KeAn}(Y) = Y$ by 1.3.1 (2)). If $m \in \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y))$ and $f \in \mathrm{An}(X)$ are arbitrary, then by our choice

$$g(f(m)) = (f \circ^{op} g)(m) = 0 \text{ for all } g \in \mathrm{An}(Y),$$

so $f(m) \in \mathrm{KeAn}(Y) = Y$, i.e. $m \in (X :^{\mathcal{C}}_M Y)$. Hence, $(X :^{\mathcal{C}}_M Y) = \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y))$.■

Proposition 2.2.10. *Let M be self-cogenerator. Then*

$$\mathrm{EP}(M) \subseteq \mathrm{CPSpec}(M) \text{ and } \mathrm{ESP}(M) \subseteq \mathrm{CSP}(M)$$

with equality, if M is intrinsically injective self-cogenerator, whence

$$\mathrm{EPcorad}(M) = \mathrm{CPcorad}(M).$$

Proof. Assume M to be self-cogenerator. Let $0 \neq K \subseteq M$ be a fully invariant $\mathcal{C}$-subcomodule that is E-prime (E-semiprime) in M, and suppose $X, Y \subseteq M$ are two (equal) fully invariant $\mathcal{C}$-subcomodules with $K \subseteq (X :^{\mathcal{C}}_M Y)$. Then we have by Lemma 2.2.9 (1)

$$\mathrm{An}(X) \circ^{op} \mathrm{An}(Y) \subseteq \mathrm{AnKe}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y)) \subseteq \mathrm{An}(X :^{\mathcal{C}}_M Y) \subseteq \mathrm{An}(K).$$

By assumption $\mathrm{An}(K) \lhd \mathrm{E}^{\mathcal{C}}_M$ is prime (semiprime), hence $\mathrm{An}(X) \subseteq \mathrm{An}(K)$, so that $K = \mathrm{KeAn}(K) \subseteq \mathrm{KeAn}(X) = X$; or $\mathrm{An}(Y) \subseteq \mathrm{An}(K)$ so that $K = \mathrm{KeAn}(K) \subseteq \mathrm{KeAn}(Y) = Y$. Consequently K is fully M-coprime (fully M-cosemiprime).

Assume now that M is intrinsically injective self-cogenerator. Let $0 \neq K \subseteq M$ be an fully M-coprime (fully M-cosemiprime) $\mathcal{C}$-subcomodule and consider the proper two-sided ideal $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$. Suppose $I, J \lhd \mathrm{E}_M^{\mathcal{C}}$ are two (equal) ideals with $I \circ^{op} J \subseteq \mathrm{An}(K)$ and $I_{\mathrm{E}_M^{\mathcal{C}}}$, $J_{\mathrm{E}_M^{\mathcal{C}}}$ are *finitely generated.* Consider the fully invariant $\mathcal{C}$-subcomodules $X := \mathrm{Ke}(I)$, $Y := \mathrm{Ke}(J)$ of M. Since M is self-cogenerator, it follows by Lemma 2.2.9 that

$$K = \mathrm{KeAn}(K) \subseteq \mathrm{Ke}(I \circ^{op} J) = \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y)) = (X :_M^{\mathcal{C}} Y).$$

Since K is fully M-coprime (fully M-cosemiprime), we conclude that $K \subseteq X$ so that $I = \mathrm{AnKe}(I) = \mathrm{An}(X) \subseteq \mathrm{An}(K)$; or $K \subseteq Y$ so that $J = \mathrm{AnKe}(J) = \mathrm{An}(Y) \subseteq \mathrm{An}(K)$. Consequently $\mathrm{An}(K) \lhd \mathrm{E}_M^{\mathcal{C}}$ is prime (semiprime), i.e. K is E-prime (E-semiprime) in M.■

Remark 2.2.11. It follows from Proposition 2.2.10 that a sufficient condition for M to be fully coprime (fully cosemiprime), in case M is self-cogenerator, is that M is E-prime (E-semiprime), where the later is also necessary in case M is intrinsically injective self-cogenerator.

As a direct consequence of Propositions 2.1.6, 2.2.10 we have

Proposition 2.2.12. *Let M be intrinsically injective self-cogenerator and $\mathrm{E}_M^{\mathcal{C}}$ be right Noetherian. Then*

$$\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{CPcorad}(M)) \text{ and } \mathrm{CPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}})). \quad (2.12)$$

Using Proposition 2.2.12, a similar proof to that of Corollary 2.1.7 yields:

Corollary 2.2.13. *Let M be intrinsically injective self-cogenerator and $\mathrm{E}_M^{\mathcal{C}}$ be right Noetherian. Then*

$$M \text{ is fully cosemiprime} \Leftrightarrow M = \mathrm{CPcorad}(M).$$

Corollary 2.2.14. *Let ${}_A\mathcal{C}$ be locally projective and M be self injective self-cogenerator. If M is Artinian (e.g. A is right Artinian and M is finitely generated), then*

1. $\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{CPcorad}(M))$ *and* $\mathrm{CPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}(\mathrm{E}_M^{\mathcal{C}})$.
2. *M is fully cosemiprime* $\Leftrightarrow M = \mathrm{CPcorad}(M)$.

Comodules with rings of colinear endomorphisms right Artinian

Under the *strong assumption* $\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op}$ is right Artinian, several primeness and coprimeness properties of the non-zero right $\mathcal{C}$-comodule M coincide and become, in case ${}_A\mathcal{C}$ locally projective, equivalent to M being simple as a $({}^*\mathcal{C}, \mathrm{E}_M^{\mathcal{C}})$-bimodule. This follows from the fact that right Artinian prime rings are simple.

If M has no non-trivial fully invariant $\mathcal{C}$-subcomodules, then it is obviously fully coprime. The following result gives a partial converse:

Theorem 2.2.15. *Let M be intrinsically injective self-cogenerator and assume $\mathrm{E}_M^{\mathcal{C}}$ to be right Artinian. Then the following are equivalent:*

1. *M is* E*-prime (i.e. $\mathrm{E}_M^{\mathcal{C}}$ is a prime ring);*
2. *$\mathrm{E}_M^{\mathcal{C}}$ is simple;*
3. *M has no non-trivial fully invariant $\mathcal{C}$-subcomodules;*
4. *M is fully coprime.*

Proof. Let M be intrinsically injective self-cogenerator and assume $\mathrm{E}_M^{\mathcal{C}}$ to be right Artinian.

(1) $\Rightarrow$ (2) : Right Artinian prime rings are simple (e.g. [Wis1991, 4.5 (2)]).

(2) $\Rightarrow$ (3) : Since M is self-cogenerator, this follows by Theorem 2.1.19.

(3) $\Rightarrow$ (4) : Trivial.

(4) $\Rightarrow$ (1) : Since M be intrinsically injective self-cogenerator, this follows by 2.2.11.■

Proposition 2.2.16. *Let ${}_A\mathcal{C}$ be locally projective and M be self-injective self-cogenerator. If any of the following additional conditions is satisfied, then M is fully coprime if and only if M is simple as a $({}^*\mathcal{C}, \mathrm{E}_M^{\mathcal{C}})$-bimodule:*

1. *M has finite length;* or
2. *A is right Artinian and M_A is finitely generated;* or
3. *M is Artinian and self-projective.*

Proof. By Theorem 2.2.15, it suffices to show that $\mathrm{E}_M^{\mathcal{C}} = \mathrm{End}({}_{^*\mathcal{C}}M)^{op}$ is right Artinian under each of the additional conditions.

1. By assumption M is self-injective and Artinian (semi-injective and Noetherian) and it follows then by Proposition 1.2.9 that $\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian (semiprimary). Applying Hopkins Theorem (e.g. [Wis1991, 31.4]), we conclude that $\mathrm{E}_M^{\mathcal{C}}$ is right Artinian.

2. If A is right Artinian and ${}_A\mathcal{C}$ is locally projective, then every finitely generated right $\mathcal{C}$-comodule has finite length by [Abu2003, Corollary 2.25].

3. Since M is Artinian, self-injective and self-projective, $\mathrm{E}_M^{\mathcal{C}}$ is right Artinian by Proposition 1.2.9 (2).■

Fully coprimeness versus irreducibility

In what follows we clarify, under suitable conditions, the relation between fully coprime and subdirectly irreducible comodules:

Proposition 2.2.17. *Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant $\mathcal{C}$-subcomodules of M, such that for any $\gamma, \delta \in \Lambda$ either $K_\gamma \subseteq K_\delta$ or $K_\delta \subseteq K_\gamma$, and consider the fully invariant $\mathcal{C}$-subcomodule $K := \sum_{\lambda\in\Lambda} K_\lambda = \bigcup_{\lambda\in\Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \mathrm{CPSpec}(M)$ for all $\lambda \in \Lambda$, then $K \in \mathrm{CPSpec}(M)$.*

Proof. Let $X, Y \subseteq M$ be any fully invariant $\mathcal{C}$-subcomodules with $K \subseteq (X :_M^{\mathcal{C}} Y)$ and suppose $K \nsubseteq X$. We **claim** that $K \subseteq Y$.

Since $K \nsubseteq X$, there exists some $\lambda_0 \in \Lambda$ with $K_{\lambda_0} \nsubseteq X$. Since $K_{\lambda_0} \subseteq (X :_M^{\mathcal{C}} Y)$, it follows from the assumption $K_{\lambda_0} \in \mathrm{CPSpec}(M)$ that $K_{\lambda_0} \subseteq Y$. Let $\lambda \in \Lambda$ be arbitrary. If $K_\lambda \subseteq K_{\lambda_0}$, then $K_\lambda \subseteq Y$. If otherwise $K_{\lambda_0} \subseteq K_\lambda$, then the inclusion $K_\lambda \subseteq (X :_M^{\mathcal{C}} Y)$ implies $K_\lambda \subseteq Y$ (since $K_\lambda \subseteq X$ would imply $K_{\lambda_0} \subseteq X$, a contradiction). So $K := \bigcup_{\lambda\in\Lambda} K_\lambda \subseteq Y$.■

Corollary 2.2.18. *Let $M = \sum_{\lambda\in\Lambda} M_\lambda$, where $\{M_\lambda\}_\Lambda$ is a family of non-zero fully invariant $\mathcal{C}$-subcomodules of M such that for any $\gamma, \delta \in \Lambda$ either $M_\gamma \subseteq M_\delta$ or $M_\delta \subseteq M_\gamma$. If $M_\lambda \in \mathrm{CPSpec}(M)$ for each $\lambda \in \Lambda$, then M is fully coprime.*

Proposition 2.2.19. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant $\mathcal{C}$-subcomodule. If $K \in \mathrm{CPSpec}(M)$, then K has no decomposition as an internal direct sum of non-trivial fully invariant $\mathcal{C}$-subcomodules.*

Proof. Let $K \in \mathrm{CPSpec}(M)$ and suppose $K := K_{\lambda_0} \oplus \sum_{\lambda \neq \lambda_0} K_\lambda$, an internal direct sum of non-trivial fully invariant $\mathcal{C}$-subcomodules. Then $K \subseteq (K_{\lambda_0} :^{\mathcal{C}}_M \sum_{\lambda \neq \lambda_0} K_\lambda)$ and it follows that $K \subseteq K_{\lambda_0}$ or $K \subseteq \sum_{\lambda \neq \lambda_0} K_\lambda$ (contradiction).■

Corollary 2.2.20. *If M is fully coprime, then M has no decomposition as an internal direct sum of non-trivial fully invariant $\mathcal{C}$-subcomodules.*

As a direct consequence of Corollary 2.2.20 we get a restatement of Theorem 2.1.29:

Theorem 2.2.21. *Let ${}_A\mathcal{C}$ locally projective, M self-injective self-cogenerator in $\mathbb{M}^{\mathcal{C}}$ with $\mathrm{End}^{\mathcal{C}}(M)$ commutative. If M is fully coprime, then M is subdirectly irreducible.*

2.3 Prime and Endo-prime comodules

Every right $\mathcal{C}$-comodule M can be considered as a $({}^*\mathcal{C}, \mathrm{E}^{\mathcal{C}}_M)$-bimodule in the canonical way. Given a non-zero right $\mathcal{C}$-comodule M, we consider in this section several primeness conditions of M as a left ${}^*\mathcal{C}$-module as well as a right $\mathrm{E}^{\mathcal{C}}_M$-module. In particular, we clarify the relations between these primeness properties and the ring structure of ${}^*\mathcal{C}$ and $\mathrm{E}^{\mathcal{C}}_M$.

Prime comodules

Given an A-coring $\mathcal{C}$, M. Ferrero and V. Rodrigues studied in [FR2005] prime and semiprime right $\mathcal{C}$-comodules considered as rational left ${}^*\mathcal{C}$-modules in the canonical way.

Definition 2.3.1. Let ${}_A\mathcal{C}$ ($\mathcal{C}_A$) be locally projective. A non-zero right (left) $\mathcal{C}$-comodule M is said to be *prime* (resp. *semiprime, strongly prime, strongly semiprime*), provided the left (right) module ${}_{^*\mathcal{C}}M$ ($M_{\mathcal{C}^*}$) is prime (resp. semiprime, strongly prime, strongly semiprime).

Lemma 2.3.2. ([FR2005, Proposition 3.2., Lemma 3.9.]) *Let ${}_A\mathcal{C}$ be locally projective and M be a non-zero right $\mathcal{C}$-comodule.*

1. *If A is left perfect and M is prime, then ${}^*\mathcal{C}/\mathrm{ann}_{{}^*\mathcal{C}}(M)$ is simple Artinian.*

2. *If A is right Artinian and ${}^*\mathcal{C}m$ is semiprime for some $0 \neq m \in M$, then ${}^*\mathcal{C}/\mathrm{ann}_{{}^*\mathcal{C}}({}^*\mathcal{C}m)$ is left Artinian.*

Combining Lemma 2.3.2 (1) with Proposition 1.1.7 one obtains

Proposition 2.3.3. ([FR2005, Theorem 3.3., Corollary 3.5.]) *Let A be left perfect, ${}_A\mathcal{C}$ be locally projective and M be a non-zero right $\mathcal{C}$-comodule. Then the following are equivalent:*

1. *M is prime;*
2. *${}^*\mathcal{C}/\mathrm{ann}_{{}^*\mathcal{C}}(M)$ is simple Artinian;*
3. *M is strongly prime;*
4. *$M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, a direct sum of isomorphic simple $\mathcal{C}$-subcomodules;*
5. *$M = \sum_{\lambda\in\Lambda} M_\lambda$, a sum of isomorphic simple $\mathcal{C}$-subcomodules of M;*
6. *M is generated by each of its non-zero $\mathcal{C}$-subcomodules of M;*
7. *M has no non-trivial fully invariant $\mathcal{C}$-subcomodules;*
8. *For any pretorsion class $\mathcal{T}$ in $\sigma[M]^{\mathcal{C}}$, $\mathcal{T}(M) = 0$ or $\mathcal{T}(M) = M$ (where $\sigma[M]^{\mathcal{C}} \subseteq \mathbb{M}^{\mathcal{C}}$ is the subcategory of M-subgenerated right $\mathcal{C}$-comodules).*

Combining Lemma 2.3.2 (2) with Proposition 1.1.8 one obtains

Proposition 2.3.4. ([FR2005, Theorem 3.10., Corollary 3.11.]) *Let A be right Artinian, ${}_A\mathcal{C}$ be locally projective and M be a non-zero right $\mathcal{C}$-comodule. Then the following are equivalent:*

1. *M is semiprime;*
2. *M is semisimple;*
3. *M is strongly semiprime;*
4. *$M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, a direct sum of prime $\mathcal{C}$-subcomodules;*

5. $M = \sum_{\lambda \in \Lambda} M_\lambda$, *a sum of prime $\mathcal{C}$-subcomodules;*

6. *Any semiprime $\mathcal{C}$-subcomodule of M is a direct summand.*

Endo-prime (endo-semiprime) comodules

In what follows we consider non-zero right $\mathcal{C}$-comodules M that are prime (semiprime) as right $\mathrm{E}_M^{\mathcal{C}}$-modules.

Remark 2.3.5. If M is a non-zero right $\mathcal{C}$-comodule and $K \subseteq M$ is a *fully invariant* $\mathcal{C}$-subcomodule, then K and M/K are right $\mathrm{E}_M^{\mathcal{C}}$-modules in the canonical way and

$$\begin{array}{lllll} \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(K) & := & \{f \in \mathrm{E}_M^{\mathcal{C}} : Kf = 0\} & = & \{f \in \mathrm{E}_M^{\mathcal{C}} : f(K) = 0\} \\ & = & \mathrm{An}(K); & & \\ \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/K) & := & \{f \in \mathrm{E}_M^{\mathcal{C}} : Mf \subseteq K\} & = & \{f \in \mathrm{E}_M^{\mathcal{C}} : f(M) \subseteq K\} \\ & = & \mathrm{Hom}^{\mathcal{C}}(M, K). & & \end{array}$$

The following definition is inspired by [HV2005], in which A. Haghany and R. Vedadi studied modules that are prime over their endomorphisms rings (called *endo-prime modules*).

Definition 2.3.6. We call a non-zero right $\mathcal{C}$-comodule M :
endo-prime, iff $\mathit{ann}_{\mathrm{E}_M^{\mathcal{C}}}(K) = 0$ for all non-zero fully invariant $\mathcal{C}$-subcomodules $0 \neq K \subseteq M$;
endo-coprime, iff $\mathit{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/K) = 0$ for all proper fully invariant $\mathcal{C}$-subcomodules $K \subsetneqq M$;
endo-diprime, iff $\mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(K) = 0$ or $\mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/K) = 0$ for every non-trivial fully invariant $\mathcal{C}$-subcomodule $0 \neq K \subsetneqq M$.
The corresponding notions for a non-zero left $\mathcal{C}$-comodule M can be defined analogously.

Remarks 2.3.7. Let M be a non-zero right $\mathcal{C}$-comodule.

1. If M is retractable (i.e. $\mathrm{Hom}^{\mathcal{C}}(M, K) \neq 0$ for every right $\mathcal{C}$-subcomodule $0 \neq K \subseteq M$), then M is endo-diprime if and only if M is endo-prime.

2. If M is coretractable (i.e. $\mathrm{Hom}^{\mathcal{C}}(M/K, M) \neq 0$ for every $\mathcal{C}$-subcomodule $K \subsetneqq M$), then M is endo-diprime if and only if M is endo-coprime.

Lemma 2.3.8. *Let M be a non-zero right $\mathcal{C}$-comodule.*

1. *Let ${}_A\mathcal{C}$ be locally projective. Then M is endo-prime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime.*

   If M is retractable, then M is endo-diprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime.

2. *M is endo-coprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is coprime.*

   If M is coretractable, then M is endo-diprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is coprime.

3. *M is endo-diprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is diprime.*

   If moreover, $M_{\mathrm{E}_M^{\mathcal{C}}}$ satisfies condition (*) (condition (**)), then M is endo-diprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime ($M_{\mathrm{E}_M^{\mathcal{C}}}$ is coprime) if and only if M is endo-prime (endo-coprime).

Proof. 1. Since ${}_A\mathcal{C}$ is locally projective, $\mathbb{M}^{\mathcal{C}} = \sigma[{}_{^*\mathcal{C}}\mathcal{C}]$ and

$$\mathrm{E}_M^{\mathcal{C}} := \mathrm{End}^{\mathcal{C}}(M)^{op} = \mathrm{End}({}_{^*\mathcal{C}}M)^{op}.$$

Assume M to be endo-prime and let $0 \neq N \subseteq M$ be an $\mathrm{E}_M^{\mathcal{C}}$-submodule. Then $\mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(N) = \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}({}^*\mathcal{C}N) = 0$, where the first equality is obvious and the second follows from the assumption that M is endo-prime (notice that $0 \neq {}^*\mathcal{C}N \subseteq M$ is a fully invariant $\mathcal{C}$-subcomodule by Proposition 1.2.7 (1)). So $M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime. The other implication is obvious.

If moreover M is retractable, then M is endo-diprime if and only if M is endo-prime and we are done.

2. Assume M to be endo-coprime and let $N \subsetneqq M$ be an arbitrary right $\mathrm{E}_M^{\mathcal{C}}$-submodule. Then $I := \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/N) \subseteq \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/MI) = 0$, where the inclusion is obvious and the equality follows from the assumption that M is endo-coprime (notice that $MI \subsetneqq M$ is a fully invariant $\mathcal{C}$-subcomodule). So $M_{\mathrm{E}_M^{\mathcal{C}}}$ is coprime. The other implication is obvious.

   If M is coretractable, then M is endo-diprime if and only if M is endo-coprime and we are done.

3. Assume M to be endo-diprime and let $0 \neq N \subsetneqq M$ be an arbitrary right $\mathrm{E}_M^{\mathcal{C}}$-submodule. If $I := \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(M/N) \neq 0$, then $\mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(N) \subseteq \mathrm{ann}_{\mathrm{E}_M^{\mathcal{C}}}(MI) = 0$, where the equality follows from the assumption that M is endo-diprime (notice that $0 \neq MI \subsetneqq M$ is a fully invariant $\mathcal{C}$-subcomodule). So $M_{\mathrm{E}_M^{\mathcal{C}}}$ is diprime. The other implication is obvious.

   If moreover, $M_{\mathrm{E}_M^{\mathcal{C}}}$ satisfies condition (*) (condition (**)), then $M_{\mathrm{E}_M^{\mathcal{C}}}$ is diprime if and only if $M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime (coprime) and we are done.■

The following result is a combinations of Proposition 1.1.5 and Lemma 2.3.8:

Theorem 2.3.9. *Let M be a non-zero right $\mathcal{C}$-comodule.*

1. *The following are equivalent:*

   (a) *M is* E*-prime (i.e. $\mathrm{E}_M^{\mathcal{C}}$ is prime);*

   (b) *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is diprime;*

   (c) *M is endo-diprime.*

2. *If M is retractable, then (a)-(c) are equivalent to:*

   (d) *M is endo-prime.*

   If ${}_A\mathcal{C}$ is locally projective and M is retractable, then (a)-(d) above are equivalent to:

   (e) *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime.*

3. *If M is coretractable, then (a)-(c) above are equivalent to:*

   (d′) *M is endo-coprime.*

   (e′) *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is coprime.*

4. *If $M_{\mathrm{E}_M^{\mathcal{C}}}$ satisfies condition (*) (condition (**)), then M is* E*-prime if and only if M is endo-prime (M is endo-coprime).*

Theorem 2.3.10. *Let M be a non-zero right $\mathcal{C}$-comodule with $\mathrm{E}_M^{\mathcal{C}}$ right Artinian. Then the following are equivalent:*

1. *M is* E*-prime (i.e. $\mathrm{E}_M^{\mathcal{C}}$ is prime);*

2. *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is diprime;*

3. *M is endo-diprime.*

4. *M is endo-prime.*

5. *$\mathrm{E}_M^{\mathcal{C}}$ is simple;*

6. *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is prime;*

7. *$M_{\mathrm{E}_M^{\mathcal{C}}}$ is strongly prime;*

8. *$M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, a direct sum of isomorphic simple right $\mathrm{E}_M^{\mathcal{C}}$-submodules of M;*

9. *$M = \sum_{\lambda\in\Lambda} M_\lambda$, a sum of isomorphic simple $\mathrm{E}_M^{\mathcal{C}}$-submodules of M;*

10. *M is generated by each of its non-zero $\mathrm{E}_M^{\mathcal{C}}$-submodules.*

11. *M has no non-trivial fully invariant $\mathrm{E}_M^{\mathcal{C}}$-submodules.*

12. *For any pretorsion class $\mathcal{T}$ in $\sigma[M_{\mathrm{E}_M^{\mathcal{C}}}]$, $\mathcal{T}(M) = 0$ or $\mathcal{T}(M) = M$.*

Proof. The first three statements are equivalent by Theorem 2.3.9 (1).

(1) $\Leftrightarrow$ (5) For right Artinian rings, the equivalence between primeness and simplicity is folklore (e.g. [Wis1991, 4.4]).

(6)-(12) are equivalent to (5) by Proposition 1.1.7 (notice that $M_{\mathrm{E}_M^{\mathcal{C}}}$ is faithful).

The remaining implications (4) $\Rightarrow$ (3) and (6) $\Rightarrow$ (4) are trivial.■

Chapter 3

(Co)Prime Corings

Throughout this chapter $(\mathcal{C}, \Delta, \varepsilon)$ is a non-zero coring. We consider in what follows several coprimeness (cosemiprimeness) and primeness (semiprimeness) properties of $\mathcal{C}$, considered as an object in the category $\mathbb{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules, denoted by $\mathcal{C}^r$, as well as an object in the category ${}^{\mathcal{C}}\mathbb{M}$ of left $\mathcal{C}$-comodules, denoted by $\mathcal{C}^l$. In particular, we clarify the relation between these properties and the simplicity (semisimplicity) of $\mathcal{C}$.

Several results in this section can be obtained directly from the corresponding ones in the previous chapter. However, we state many of these in the case A is a QF ring, as in this case $\mathcal{C}$ is an injective cogenerator in both the categories of right and left $\mathcal{C}$-comodules by Lemma 1.2.3.

3.1 Sufficient and necessary conditions

The following result gives sufficient and necessary conditions for the dual rings of the non-zero coring $\mathcal{C}$ to be prime (respectively semiprime, domain, reduced) generalizing results of [XLZ1992] for coalgebras over base fields. Several results follow directly from previous sections recalling the isomorphism of rings $\mathcal{C}^* \simeq \mathrm{End}^{\mathcal{C}}(\mathcal{C})^{op}$. Analogous statements can be formulated for ${}^*\mathcal{C} \simeq {}^{\mathcal{C}}\mathrm{End}(\mathcal{C})$.

Theorem 2.1.17 yields directly.

Theorem 3.1.1. *Let ${}_A\mathcal{C}$ be flat.*

1. *$\mathcal{C}^*$ is prime (domain), if $\mathcal{C} = \mathcal{C}f\mathcal{C}^*$ ($\mathcal{C} = \mathcal{C}f$) for all $0 \neq f \in \mathcal{C}^*$. If $\mathcal{C}$ is coretractable in $\mathbb{M}^{\mathcal{C}}$, then $\mathcal{C}^*$ is prime (domain) if and only if $\mathcal{C} = \mathcal{C}f\mathcal{C}^*$ ($\mathcal{C} = \mathcal{C}f$) $\forall\ 0 \neq f \in \mathcal{C}^*$.*

2. *$\mathcal{C}^*$ is semiprime (reduced), if $\mathcal{C}f = \mathcal{C}f\mathcal{C}^*f$ ($\mathcal{C}f = \mathcal{C}f^2$) for all $0 \neq f \in \mathcal{C}^*$. If $\mathcal{C}$ is self-cogenerator in $\mathbb{M}^{\mathcal{C}}$, then $\mathcal{C}^*$ is semiprime (reduced) if and only if $\mathcal{C}f = \mathcal{C}f\mathcal{C}^*f$ ($\mathcal{C}f = \mathcal{C}f^2$) $\forall\ 0 \neq f \in \mathcal{C}^*$.*

Proposition 3.1.2. *Let ${}_A\mathcal{C}$ and $\mathcal{C}_A$ be flat.*

1. *Let $\mathcal{C}$ be coretractable in $\mathbb{M}^{\mathcal{C}}$ and $\mathcal{C}_{\mathcal{C}^*}$ satisfy condition (**). If $\mathcal{C}^*$ is prime (domain), then ${}^*\mathcal{C}$ is prime (domain).*

2. *Let $\mathcal{C}$ be coretractable in $\mathbb{M}^{\mathcal{C}}$, ${}^{\mathcal{C}}\mathbb{M}$ and $\mathcal{C}_{\mathcal{C}^*,{}^*\mathcal{C}}\mathcal{C}$ satisfy condition (**). Then $\mathcal{C}^*$ is prime (domain) if and only if ${}^*\mathcal{C}$ is prime (domain).*

3. *Let $\mathcal{C}$ be coretractable in $\mathbb{M}^{\mathcal{C}}$ and ${}_A\mathcal{C}$ be locally projective. If $\mathcal{C}^*$ is prime, then ${}^*\mathcal{C}$ is prime.*

4. *Let $\mathcal{C}$ be coretractable in $\mathbb{M}^{\mathcal{C}}$, ${}^{\mathcal{C}}\mathbb{M}$ and ${}_A\mathcal{C}$, $\mathcal{C}_A$ be locally projective. Then $\mathcal{C}^*$ is prime if and only if ${}^*\mathcal{C}$ is prime.*

Proof. 1. Let $\mathcal{C}^*$ be prime (domain). If ${}^*\mathcal{C}$ were not prime (not a domain), then there exists by an analogous statement of Theorem 3.1.1 some $0 \neq f \in {}^*\mathcal{C}$ with ${}^*\mathcal{C}f\mathcal{C} \subsetneqq \mathcal{C}$ ($f\mathcal{C} \subsetneqq \mathcal{C}$). By assumption $\mathcal{C}_{\mathcal{C}^*}$ satisfies condition (**) and so there exists some $0 \neq h \in \mathcal{C}^*$ such that $({}^*\mathcal{C}f\mathcal{C})h = 0$ ($(f\mathcal{C})h = 0$). But this implies $\mathcal{C} \neq \mathcal{C}h\mathcal{C}^*$ ($\mathcal{C} \neq \mathcal{C}h$): otherwise $f\mathcal{C} = f(\mathcal{C}h\mathcal{C}^*) = ((f\mathcal{C})h)\mathcal{C}^* = 0$ ($f\mathcal{C} = f(\mathcal{C}h) = (f\mathcal{C})h = 0$), which implies $f = 0$, a contradiction. Since $\mathcal{C}$ is coretractable in $\mathbb{M}^{\mathcal{C}}$, Theorem 3.1.1 (1) implies that $\mathcal{C}^*$ is not prime (not a domain), which contradicts our assumptions.

2. Follows from (1) by symmetry.

3. The proof is similar to that of (1) recalling that, in case ${}_A\mathcal{C}$ locally projective, for any $f \in {}^*\mathcal{C}$, the left ${}^*\mathcal{C}$-submodule ${}^*\mathcal{C}f\mathcal{C} \subseteq \mathcal{C}$ is a right $\mathcal{C}$-subcomodule.

4. Follows from (3) by symmetry.■

E-Prime versus simple

In what follows we show that E-prime corings generalize simple corings. The results are obtained by direct application of the corresponding results in the Section 3.

As a direct consequence of Theorems 2.1.18 and 2.1.19 we get

Theorem 3.1.3. *Let A be a QF ring and assume ${}_A\mathcal{C}$ to be (locally) projective.*

1. *$\mathcal{C}^r$ is simple if and only if $\mathcal{C}^*$ is right simple.*

2. *If $\mathcal{C}^*$ is simple, then $\mathcal{C}$ is simple (as a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule).*

3. *Let $\mathcal{C}^*$ be right Noetherian. Then $\mathcal{C}^*$ is simple if and only if $\mathcal{C}$ is simple (as a $({}^*\mathcal{C}, \mathcal{C}^*)$-bimodule).*

Corollary 3.1.4. *Let A be a QF ring, ${}_A\mathcal{C}$, $\mathcal{C}_A$ be locally projective, ${}^*\mathcal{C}$ be left Noetherian and $\mathcal{C}^*$ be right Noetherian. Then*

$$\mathcal{C}^* \text{ is simple} \Leftrightarrow \mathcal{C} \text{ is simple (as a } ({}^*\mathcal{C}, \mathcal{C}^*)\text{-bimodule)} \Leftrightarrow {}^*\mathcal{C} \text{ is simple.}$$

Proposition 3.1.5. *Let A be a QF ring. If ${}_A\mathcal{C}$ is (locally) projective, then we have*

$$\mathrm{Jac}(\mathcal{C}^*) = \mathrm{ann}_{\mathcal{C}^*}(\mathrm{Soc}(\mathcal{C}^r)) = \mathrm{Soc}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)} \text{ and } \mathrm{Soc}(\mathcal{C}^r) = \mathrm{Jac}(\mathcal{C}^*)^{\perp(\mathcal{C})}.$$

Proof. The result in (1) follows from Proposition 2.1.23 (3) recalling the isomorphisms of R-algebras $\mathcal{C}^* \simeq \mathrm{End}^{\mathcal{C}}(\mathcal{C})^{op}$ and Remarks 1.3.5 (6) & (7).■

Corollary 3.1.6. *Let A be a QF ring.*

1. *If ${}_A\mathcal{C}$ is (locally) projective, then $\mathcal{C}$ is right semisimple if and only if $\mathcal{C}^*$ is semiprimitive.*

2. *If ${}_A\mathcal{C}$ and $\mathcal{C}_A$ are (locally) projective, then $\mathcal{C}^*$ is semiprimitive if and only if ${}^*\mathcal{C}$ is semiprimitive.*

The wedge product

The *wedge product* of subspaces of a given coalgebra C over a base field was already defined and investigated in [Swe1969, Section 9]. In [NT2001], the wedge product of subcoalgebras was used to define *fully coprime coalgebras.*

Definition 3.1.7. We define the *wedge product* of a right A-submodule $K \subseteq \mathcal{C}$ and a left A-submodule $L \subseteq \mathcal{C}$ as

$$K \wedge L := \Delta^{-1}(\mathrm{Im}(K \otimes_A \mathcal{C}) + \mathrm{Im}(\mathcal{C} \otimes_A L)) = \mathrm{Ker}((\pi_K \otimes \pi_L) \circ \Delta : \mathcal{C} \to \mathcal{C}/K \otimes_A \mathcal{C}/L).$$

Remark 3.1.8. ([Swe1969, Proposition 9.0.0.]) Let C be a coalgebra over a base field and $K, L \subseteq C$ be any subspaces. Then $K \wedge L = (K^{\perp(C^*)} * L^{\perp(C^*)})^{\perp(C)}$. If moreover K is a left C-coideal and L is a right C-coideal, then $K \wedge L \subseteq C$ is a subcoalgebra.

Lemma 3.1.9. ([Abu2003, Corollary 2.9.]) *Let $K, L \subseteq \mathcal{C}$ be A-subbimodules.*

1. *Consider the canonical A-bilinear map*

$$\kappa_l : K^{\perp({}^*\mathcal{C})} \otimes_A L^{\perp({}^*\mathcal{C})} \to {}^*(\mathcal{C} \otimes_A \mathcal{C}),\ [f \otimes_A g \mapsto (c \otimes_A c') = g(cf(c'))].$$

If A is right Noetherian, $\mathcal{C}_A$ is flat and $L^{\perp({}^*\mathcal{C})\perp} \subseteq \mathcal{C}$ is pure as a right A-module, then

$$(\kappa_l(K^{\perp({}^*\mathcal{C})} \otimes_A L^{\perp({}^*\mathcal{C})}))^{\perp(\mathcal{C}\otimes_A\mathcal{C})} = L^{\perp({}^*\mathcal{C})\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A K^{\perp({}^*\mathcal{C})\perp}. \quad (3.1)$$

2. *Consider the canonical A-bilinear map*

$$\kappa_r : L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)} \to (\mathcal{C} \otimes_A \mathcal{C})^*,\ [g \otimes_A f \mapsto (c' \otimes_A c) = g(f(c')c)].$$

If A is left Noetherian, ${}_A\mathcal{C}$ is flat and $L^{\perp(\mathcal{C}^*)\perp} \subseteq \mathcal{C}$ is pure as a left A-module, then

$$(\kappa_r(L \otimes_A K))^{\perp(\mathcal{C}\otimes_A\mathcal{C})} = K^{\perp(\mathcal{C}^*)\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L^{\perp(\mathcal{C}^*)\perp}. \quad (3.2)$$

Definition 3.1.10. For R-submodules $K, L \subseteq \mathcal{C}$ we set

$$\begin{aligned}(K :_{\mathcal{C}^r} L) &:= \bigcap\{f^{-1}(Y) \mid f \in \mathrm{End}^{\mathcal{C}}(\mathcal{C})^{op} \text{ and } f(K) = 0\} \\ &= \bigcap\{c \in \mathcal{C} \mid c \leftharpoonup f \in L \text{ for all } f \in \mathrm{ann}_{\mathcal{C}^*}(K)\}.\end{aligned}$$

and

$$\begin{aligned}(K :_{\mathcal{C}^l} L) &:= \bigcap\{f^{-1}(L) \mid f \in {}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) \text{ and } f(K) = 0\} \\ &= \bigcap\{c \in \mathcal{C} \mid f \rightharpoonup c \in L \text{ for all } f \in \mathrm{ann}_{{}^*\mathcal{C}}(K)\}.\end{aligned}$$

If $K, L \subseteq \mathcal{C}$ are right (left) $\mathcal{C}$-coideals, then we call $(K :_{\mathcal{C}^r} L)$ ($(K :_{\mathcal{C}^l} L)$) the *internal coproduct* of X and Y in $\mathbb{M}^{\mathcal{C}}$ (in ${}^{\mathcal{C}}\mathbb{M}$).

Lemma 3.1.11. *Let $K, L \subseteq \mathcal{C}$ be $\mathcal{C}$-bicoideals.*

1. *If ${}_A\mathcal{C}$ is flat and $\mathcal{C}$ is self-cogenerator in $\mathbb{M}^{\mathcal{C}}$, then*

$$(K :_{\mathcal{C}^r} L) = \mathrm{ann}_{\mathcal{C}}(\mathrm{ann}_{\mathcal{C}^*}(K) *^r \mathrm{ann}_{\mathcal{C}^*}(L)).$$

2. *If $\mathcal{C}_A$ is flat and $\mathcal{C}$ is self-cogenerator in ${}^{\mathcal{C}}\mathbb{M}$, then*

$$(K :_{\mathcal{C}^l} L) = \mathrm{ann}_{\mathcal{C}}(\mathrm{ann}_{^*\mathcal{C}}(K) *^l \mathrm{ann}_{^*\mathcal{C}}(L)).$$

Proof. The proof of (1) is analogous to that of Lemma 2.2.9, while (2) follows by symmetry.■

The following result clarifies the relation between the *wedge product* and the *internal coproduct* of right (left) $\mathcal{C}$-coideals under suitable purity conditions:

Proposition 3.1.12. *Let A be a QF ring, $(\mathcal{C}, \Delta, \varepsilon)$ be an A-coring and $K, L \subseteq \mathcal{C}$ be A-subbimodules.*

1. *Let ${}_A\mathcal{C}$ be flat and K, L be right $\mathcal{C}$-coideals. If ${}_AL \subseteq {}_A\mathcal{C}$ is pure, then $(K :_{\mathcal{C}^r} L) = K \wedge L$.*
2. *Let $\mathcal{C}_A$ be flat and K, L be left $\mathcal{C}$-coideals. If $K_A \subseteq \mathcal{C}_A$ is pure, then $(K :_{\mathcal{C}^l} L) = K \wedge L$.*
3. *Let ${}_A\mathcal{C}, \mathcal{C}_A$ be flat and $K, L \subseteq \mathcal{C}$ be $\mathcal{C}$-bicoideals. If ${}_AK \subseteq {}_A\mathcal{C}$ and $L_A \subseteq \mathcal{C}_A$ are pure, then*

$$(K :_{\mathcal{C}^r} L) = K \wedge L = (K :_{\mathcal{C}^l} L). \tag{3.3}$$

Proof. 1. Assume ${}_A\mathcal{C}$ to be flat and consider the map

$$\kappa_r : L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)} \longrightarrow (\mathcal{C} \otimes_A \mathcal{C})^*, \ [g \otimes_A f \mapsto (c' \otimes_A c) = g(f(c')c)].$$

Then we have

$$\begin{array}{rcll} (K :_{\mathcal{C}^r} L) & = & (ann_{\mathcal{C}^*}(K) *^r ann_{\mathcal{C}^*}(L))^{\perp} & (3.1.11) \\ & = & (K^{\perp(\mathcal{C}^*)} *^r L^{\perp(\mathcal{C}^*)})^{\perp} & (X, Y \subseteq \mathcal{C} \text{ right coideals}) \\ & = & ((\Delta^* \circ \kappa_r)(L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)}))^{\perp} & \\ & = & \Delta^{-1}((\kappa_r(L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)}))^{\perp}) & ([\text{Abu2006, 1.10 (3-c)}]) \\ & = & \Delta^{-1}(K^{\perp(\mathcal{C}^*)\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L^{\perp(\mathcal{C}^*)\perp}) & ([\text{Abu2006, 2.9}]) \\ & = & \Delta^{-1}(K \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L) & A \text{ is cogenerator} \\ & = & K \wedge L. & \end{array}$$

2. This follows from (1) by symmetry.

3. This is a combination of (1) and (2).■

3.2 Fully coprime (fully cosemiprime) corings

In addition to the notions of *right* (*left*) *fully coprime* and *right* (*left*) *fully cosemiprime bicoideals*, considered as right (left) comodules in the canonical way, we present the notion of a *fully coprime* (*fully cosemiprime*) *bicoideal.*

Definition 3.2.1. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a non-zero A-coring and assume ${}_A\mathcal{C}, \mathcal{C}_A$ to be flat. Let $0 \neq B \subseteq \mathcal{C}$ be a $\mathcal{C}$-bicomodule and consider the right $\mathcal{C}$-comodule B^r and the left $\mathcal{C}$-comodule B^l. We call B :

fully $\mathcal{C}$-coprime (*fully $\mathcal{C}$-cosemiprime*), iff both B^r and B^l are fully $\mathcal{C}$-coprime (fully $\mathcal{C}$-cosemiprime);

fully coprime (*fully cosemiprime*), iff both B^r and B^l are fully coprime (fully cosemiprime).

The fully coprime coradical

The *prime spectra* and the associated *prime radicals* for rings play an important role in the study of structure of rings. Dually, we define the *fully coprime spectra* and the *fully coprime coradicals* for corings.

Definition 3.2.2. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a non-zero ring and assume ${}_A\mathcal{C}$ to be flat. We define the *fully coprime spectrum* of $\mathcal{C}^r$ as

$$\mathrm{CPSpec}(\mathcal{C}^r) := \{0 \neq B \in \mathcal{B}(C) \mid B^r \subseteq \mathcal{C}^r \text{ is a fully } \mathcal{C}\text{-coprime}\}$$

and the *fully coprime coradical* of $\mathcal{C}^r$ as

$$\mathrm{CPcorad}(\mathcal{C}^r) := \sum_{B \in \mathrm{CPSpec}(\mathcal{C}^r)} B.$$

Moreover, we set

$$\mathrm{CSP}(\mathcal{C}^r) := \{0 \neq B \in \mathcal{B}(C) \mid B^r \subseteq \mathcal{C}^r \text{ is a fully } \mathcal{C}\text{-cosemiprime}\}.$$

In case $\mathcal{C}_A$ is flat, one defines analogously $\mathrm{CPSpec}(\mathcal{C}^l)$, $\mathrm{CPcorad}(\mathcal{C}^l)$ and $\mathrm{CSP}(\mathcal{C}^l)$.

As a direct consequence of Remark 2.2.11 we get:

Theorem 3.2.3. *Let A be a QF ring and ${}_A\mathcal{C}$ be flat. Then $\mathcal{C}^*$ is prime* (*semiprime*) *if and only if $\mathcal{C}^r$ is fully coprime* (*fully cosemiprime*).

The following result shows that fully coprime spectrum (fully coprime coradical) of corings is invariant under isomorphisms of corings. The proof is analogous to that of Proposition 2.2.5.

Proposition 3.2.4. *Let $\theta : \mathcal{C} \to \mathcal{D}$ be an isomorphism of A-corings and assume ${}_A\mathcal{C}$, ${}_A\mathcal{D}$ to be flat. Then we have bijections*

$$\mathrm{CPSpec}(\mathcal{C}^r) \longleftrightarrow \mathrm{CPSpec}(\mathcal{D}^r) \text{ and } \mathrm{CSP}(\mathcal{C}^r) \longleftrightarrow \mathrm{CSP}(\mathcal{D}^r).$$

In particular, $\theta(\mathrm{CPcorad}(\mathcal{C}^r)) = \mathrm{CPcorad}(\mathcal{D}^r)$.

Remark 3.2.5. If $\theta : \mathcal{C} \to \mathcal{D}$ is a morphism of A-corings, then it is NOT evident that θ maps fully $\mathcal{C}$-coprime (fully $\mathcal{C}$-cosemiprime) $\mathcal{C}$-bicoideals into fully $\mathcal{D}$-coprime (fully $\mathcal{D}$-cosemiprime) $\mathcal{D}$-bicoideals, contrary to what was mentioned in [NT2001, Theorem 2.4(i)].

The following example, given by Chen Hui-Xiang in his review of [NT2001] (Zbl 1012.16041), shows moreover that a homomorphic image of a fully coprime coalgebra need not be fully coprime:

Example 3.2.6. Let $A := M_n(F)$ be the algebra of all $n \times n$ matrices over a field F, $B := T_n(F)$ be the subalgebra of upper-triangular $n \times n$ matrices over F where $n > 1$. Consider the dual coalgebras A^*, B^*. The embedding of F-algebras $\iota : B \hookrightarrow A$ induces a surjective map of F-coalgebras $A^* \xrightarrow{\iota^*} B^* \longrightarrow 0$. However, A is prime while B is not, i.e. A^* is a fully coprime F-coalgebra, while B^* is not (see Theorem 3.2.3).

As a direct consequence of Proposition 2.2.12 we have

Proposition 3.2.7. *Let A be a QF ring. If ${}_A\mathcal{C}$ is flat and $\mathcal{C}^*$ is right Noetherian, then*

$$\mathrm{Prad}(\mathcal{C}^*) = \mathrm{CPcorad}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)} \text{ and } \mathrm{CPcorad}(\mathcal{C}^r) = \mathrm{Prad}(\mathcal{C}^*)^{\perp(\mathcal{C})}.$$

Making use of Proposition 3.2.7, a similar proof to that of Corollary 2.1.7 yields:

Corollary 3.2.8. *Let A be a QF ring. If ${}_A\mathcal{C}$ is flat and $\mathcal{C}^*$ is Noetherian, then*

$$\mathcal{C}^r \text{ is fully cosemiprime} \Leftrightarrow \mathcal{C} = \mathrm{CPcorad}(\mathcal{C}^r).$$

Corollary 3.2.9. *Let A be a QF ring. If ${}_A\mathcal{C}$ is (locally) projective and $\mathcal{C}^r$ is Artinian (e.g. $\mathcal{C}_A$ is finitely generated), then*

1. $\mathrm{Prad}(\mathcal{C}^*) = \mathrm{CPcorad}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)}$ *and* $\mathrm{CPcorad}(\mathcal{C}^r) = \mathrm{Prad}(\mathcal{C}^*)^{\perp(\mathcal{C})}$.

2. *$\mathcal{C}^r$ is fully cosemiprime if and only if* $\mathcal{C} = \mathrm{CPcorad}(\mathcal{C}^r)$.

Corings with Artinian dual rings

For corings over QF ground rings several primeness and coprimeness properties become equivalent. As a direct consequence Theorems 2.2.15, 3.2.3 and [FR2005, Theorem 2.9, Corollary 2.10] we get the following characterizations of fully coprime locally projective corings over QF ground rings:

Theorem 3.2.10. *Let A be a QF ring and ${}_A\mathcal{C}$, $\mathcal{C}_A$ be projective and assume $\mathcal{C}^*$ is right Artinian and ${}^*\mathcal{C}$ is left Artinian. Then the following statements are equivalent:*

1. *$\mathcal{C}^*$ (or ${}^*\mathcal{C}$) is prime;*
2. *$\mathcal{C}_{\mathcal{C}^*}$ (or ${}_{^*\mathcal{C}}\mathcal{C}$) is diprime;*
3. *$\mathcal{C}_{\mathcal{C}^*}$ (or ${}_{^*\mathcal{C}}\mathcal{C}$) is prime;*
4. *$\mathcal{C}^*$ (or ${}^*\mathcal{C}$) is simple Artinian;*
5. *$\mathcal{C}_{\mathcal{C}^*}$ (or ${}_{^*\mathcal{C}}\mathcal{C}$) is strongly prime;*
6. *$\mathcal{C}^r$ (or $\mathcal{C}^l$) is fully coprime;*
7. *$\mathcal{C}$ has non-trivial fully invariant right (left) $\mathcal{C}$-coideals;*
8. *$\mathcal{C}$ is simple.*

As a direct consequence of Theorem 2.2.21 we get

Theorem 3.2.11. *Let C be a locally projective cocommutative R-coalgebra and assume C to be self-injective self-cogenerator in $\mathbb{M}^C$. If C is fully coprime, then C is subdirectly irreducible.*

3.3 Examples and Counterexamples

In what follows we give some examples of *fully coprime* corings (coalgebras) over arbitrary (commutative) ground rings. An important class of fully coprime path coalgebras over fields is considered by Prof. Jara et. al. in [JMNR]. For other examples of fully coprime coalgebras over fields, the reader is referred to [NT2001].

We begin with a counterexample to a conjecture in [NT2001], communicated to the author by Ch. Lomp, which shows that the converse of Theorem 3.2.11 is not true in general:

Counterexample 3.3.1. Let C be a $\mathbb{C}$-vector space spanned by g and an infinite family of elements $\{x_\lambda\}_\Lambda$ where Λ is a non-empty set. Define a coalgebra structure on C by

$$\begin{array}{ll}\Delta(g) = g \otimes g, & \varepsilon(g) = 1; \\ \Delta(x_\lambda) = g \otimes x_\lambda + x_\lambda \otimes g, & \varepsilon(x_\lambda) = 0.\end{array} \tag{3.4}$$

Then C is a cocommutative coalgebra with unique simple (1-dimensional) subcoalgebra $C_0 = \mathbb{C}g$. Let $V(\Lambda)$ be the $\mathbb{C}$-vector space of families $\{b_\lambda\}_\Lambda$, where $b_\lambda \in \mathbb{C}$ and consider the trivial extension

$$\mathbb{C} \ltimes V(\Lambda) = \left\{ \begin{pmatrix} a & w \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \text{ and } w \in V(\Lambda) \right\}, \tag{3.5}$$

which is a ring under the ordinary matrix multiplication and addition. Then there exists a ring isomorphism

$$C^* \simeq \mathbb{C} \ltimes V(\Lambda),\ f \mapsto \begin{pmatrix} f(g) & (f(x_\lambda))_\Lambda \\ 0 & f(g) \end{pmatrix} \text{ for all } f \in C^*. \tag{3.6}$$

Since

$$\mathrm{Jac}(C^*) \simeq \mathrm{Jac}(\mathbb{C} \ltimes V(\Lambda)) = \begin{pmatrix} 0 & V(\Lambda) \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

we have $(\mathrm{Jac}(C^*))^2 = 0$, which means that C^* is not semiprime. So C is an infinite dimensional subdirectly irreducible cocommutative coalgebra, which is not fully coprime (even not fully cosemiprime).

3.3.2. (**The comatrix coring** [E-GT2003]) Let A, B be R-algebras, Q a (B, A)-bimodule and assume Q_A to be finitely generated projective with dual basis $\{(e_i, \pi_i)\}_{i=1}^n \subset Q \times Q^*$. By [E-GT2003], $\mathcal{C} := Q^* \otimes Q$ is an A-coring (called the *comatrix coring*) with coproduct and counit given by

$$\Delta_{\mathcal{C}}(f \otimes_B q) := \sum_{i=1}^n (f \otimes_B e_i) \otimes_A (\pi_i \otimes_B q) \text{ and } \varepsilon_{\mathcal{C}}(f \otimes_B q) := f(q).$$

Notice that we have R-algebra isomorphisms

$$\mathcal{C}^* := \mathrm{Hom}_{-A}(Q^* \otimes_B Q, A) \simeq \mathrm{Hom}_{-B}(Q^*, \mathrm{Hom}_{-A}(Q, A)) = \mathrm{End}_{-B}(Q^*);$$

and

$${}^*\mathcal{C} := \mathrm{Hom}_{A-}(Q^* \otimes_B Q, A) \simeq \mathrm{Hom}_{B-}(Q, \mathrm{Hom}_{A-}(Q^*, A))^{op} \simeq \mathrm{End}_{B-}(Q)^{op}.$$

Example 3.3.3. Consider the (A,A)-bimodule $Q = A^n$ and the corresponding comatrix A-coring $\mathcal{C} := Q^* \otimes_A Q$ (called also the *matrix coalgebra* in case $A = R$, a commutative ring). Then we have isomorphisms of rings

$$\begin{array}{rcl} \mathcal{C}^* & \simeq & \mathrm{End}_{-A}((A^n)^*) \quad \simeq \quad \mathrm{End}_{-A}((A^*)^n) \\ & \simeq & \mathbb{M}_n(\mathrm{End}_{-A}(A^*)) \quad \simeq \quad \mathbb{M}_n(\mathrm{End}_{-A}(A)) \\ & \simeq & \mathbb{M}_n(A), \end{array}$$

and

$$^*\mathcal{C} \simeq \mathrm{End}_{A-}(A^n)^{op} \simeq \mathbb{M}_n(\mathrm{End}_{A-}(A))^{op} \simeq \mathbb{M}_n(A^{op})^{op}.$$

Let A be prime. Then $\mathcal{C}^* \simeq \mathbb{M}_n(A)$ and $^*\mathcal{C} \simeq \mathbb{M}_n(A^{op})^{op}$ are prime (e.g. [AF1974, Proposition 13.2]). If moreover A_A ($_AA$) is a cogenerator, then $\mathcal{C}^r$ ($\mathcal{C}^l$) is self-cogenerator and it follows by Remark 2.2.11 that $\mathcal{C}^r$ is fully coprime ($\mathcal{C}^l$ is fully coprime).

Example 3.3.4. Let $A \to B$ be a ring homomorphism and assume B_A to finitely generated and projective. Then the A-comatrix coring $\mathcal{C} := B^* \otimes_B B \simeq B^*$, is called the *dual A-coring* of the A-ring B as its coring structure can also be obtained from the the A-ring structure of B (see [Swe1975, 3.7.]). If B is a prime ring, then $^*\mathcal{C} := {}^*(B^*) \simeq B$ is prime. If moreover, B_A^* is flat and self-cogenerator, then it follows by analogy to Remark 2.2.11 that $^l\mathcal{C}$ is fully coprime.

Example 3.3.5. Let R be Noetherian, A a non-zero R-algebra for which the *finite dual* $A^\circ \subset R^A$ is a pure submodule (e.g. R is a Dedekind domain) and assume $_RA^\circ$ to be a self-cogenerator. By [AG-TW2000], A° is an R-coalgebra. If the R-algebra $A^{\circ *}$ is prime, then A° is a fully coprime R-coalgebra. If A is a reflexive R-algebra (i.e. $A \simeq A^{\circ *}$ canonically), then A is prime if and only if A° is fully coprime.

Example 3.3.6. Let A be a prime R-algebra and assume $_RA$ to be finitely generated projective. Then $C := A^*$ is an R-coalgebra (with no assumption on the commutative ground ring R) and $C^* := A^{**} \simeq A$. If $_RA^*$ is self-cogenerator (e.g. $_RR$ is a cogenerator), then A is a prime R-algebra if and only if C is a fully coprime R-coalgebra.

Example 3.3.7. Let R be a integral domain and $C := R[x]$ be the R-coalgebra with coproduct and counit defined on the generators by

$$\Delta(x^n) := \sum_{j=0}^{n} x^j \otimes_R x^{n-j} \text{ and } \varepsilon(x^n) := \delta_{n,0} \text{ for all } n \geq 0.$$

Then $C^* \simeq R[[x]]$, the power series ring, is an integral domain. If moreover, ${}_RC$ is self-cogenerator (e.g. ${}_RR$ is a cogenerator), then C is fully coprime.

Chapter 4

Zariski-Topologies for Corings and Bicomodules

All rings and their modules in this paper are assumed to be unital. For a ring T, we denote with $Z(T)$ the *center* of T and with T^{op} the opposite ring of T. We denote by $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ a non-zero A-coring with ${}_A\mathcal{C}$ flat and by $\mathcal{D} = (\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ a non-zero B-coring with $\mathcal{D}_B$ flat, so that the categories ${}^{\mathcal{D}}\mathbb{M}^{\mathcal{C}}$ of $(\mathcal{D}, \mathcal{C})$-bicomodules, $\mathbb{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules and ${}^{\mathcal{D}}\mathbb{M}$ of left $\mathcal{D}$-comodules are Grothendieck.

4.1 Fully coprime (fully cosemiprime) bicomodules

4.1.1. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. For any R-submodules $X, Y \subseteq M$ we set

$$\begin{aligned}(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) &:= \bigcap \{f^{-1}(Y) \mid f \in \mathrm{An}_{{}^{\mathcal{D}}\mathrm{E}^{\mathcal{C}}_M}(X)\} \\ &= \bigcap_{f \in \mathrm{An}(X)} \{\mathrm{Ker}(\pi_Y \circ f : M \to M/Y)\}.\end{aligned}$$

If $Y \subseteq M$ is a $(\mathcal{D}, \mathcal{C})$-subbicomodule (and $f(X) \subseteq X$ for all $f \in {}^{\mathcal{D}}\mathrm{E}^{\mathcal{C}}_M$), then $(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) \subseteq M$ is a (fully invariant) $(\mathcal{D}, \mathcal{C})$-subbicomodule. If $X, Y \subseteq M$ are $(\mathcal{D}, \mathcal{C})$-subbicomodules, then we call $(X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y) \subseteq M$ the *internal coproduct* of X and Y in M.

Lemma 4.1.2. *Let $X, Y \subseteq M$ be any R-submodules. Then*

$$(X :^{\mathcal{C}}_{M} Y) \subseteq \mathrm{Ke}(\mathrm{An}(X) \circ^{op} \mathrm{An}(Y)), \tag{4.1}$$

with equality in case M is self-cogenerator and $Y \subseteq M$ is a $(\mathcal{D}, \mathcal{C})$-subbicomodule.

Definition 4.1.3. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. We call a non-zero fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodule $0 \neq K \subseteq M$:

fully M-coprime, iff for any fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodules $X, Y \subseteq M$ with $K \subseteq (X \overset{(\mathcal{D},\mathcal{C})}{:_M} Y)$, we have $K \subseteq X$ or $K \subseteq Y$;

fully M-cosemiprime, iff for any fully invariant $(\mathcal{D}, \mathcal{C})$-subbicomodule $X \subseteq M$ with $K \subseteq (X \overset{(\mathcal{D},\mathcal{C})}{:_M} X)$, we have $K \subseteq X$;

In particular, we call M *fully coprime* (*fully cosemiprime*), iff M is fully M-coprime (fully M-cosemiprime).

The fully coprime coradical

The *prime spectra* and the associated *prime radicals* for rings play an important role in the study of structure of rings. Dually, we define the *fully coprime spectra* and the *fully coprime coradicals* for bicomodules.

Definition 4.1.4. Let M be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule. We define the *fully coprime spectrum* of M as

$$\mathrm{CPSpec}(M) := \{0 \neq K \mid K \subseteq M \text{ is a fully } M\text{-coprime } (\mathcal{D}, \mathcal{C})\text{-subbicomodule}\}$$

and the *fully coprime coradical* of M as $\mathrm{CPcorad}(M) := \sum\limits_{K \in \mathrm{CPSpec}(M)} K$ ($:= 0$, in case $\mathrm{CPSpec}(M) = \varnothing$). Moreover, we set

$$\mathrm{CSP}(M) := \{K \mid K \subseteq M \text{ is a fully } M\text{-cosemiprime } (\mathcal{D}, \mathcal{C})\text{-subbicomodule}\}.$$

Remark 4.1.5. We should mention here that the definition of fully coprime (bi)comodules we present is motivated by the modified version of the definition of coprime modules (in the sense of Bican et. al. [BJKN1980]) as presented in [RRW2005]. (Fully) coprime coalgebras over base fields were introduced first in [NT2001] and considered in [JMNR] using the *wedge product* of subcoalgebras.

4.1.6. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and $L \subseteq M$ a fully invariant non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule. Then L is called E-*prime* (E-*semiprime*), iff $\mathrm{An}(K) \lhd {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is prime (semiprime). With $\mathrm{EP}(M)$ ($\mathrm{ESP}(M)$) we denote the class of E-prime (E-semiprime) $(\mathcal{D},\mathcal{C})$-subbicomodules of M.

The results of [Abu2006] on comodules can be reformulated (with slight modifications of the proofs) for bicomodules. We state only two of them that are needed in the sequel.

Proposition 4.1.7. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. If M is self-cogenerator, then* $\mathrm{EP}(M) \subseteq \mathrm{CPSpec}(M)$ *and* $\mathrm{ESP}(M) \subseteq \mathrm{CSP}(M)$, *with equality if M is intrinsically injective. If moreover ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then*

$$\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}) = \mathrm{An}(\mathrm{CPcorad}(M)) \text{ and } \mathrm{CPcorad}(M) = \mathrm{Ke}(\mathrm{Prad}({}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}));$$

in particular, M is fully cosemiprime if and only if $M = \mathrm{CPcorad}(M)$.

Proposition 4.1.8. *Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule and $0 \neq L \subseteq M$ a fully invariant $(\mathcal{D},\mathcal{C})$-subbicomodule. If M is self-injective, then*

$$\mathrm{CPSpec}(L) = \mathcal{M}_{f.i.}(L) \cap \mathrm{CPSpec}(M) \text{ and } \mathrm{CSP}(L) = \mathcal{M}_{f.i.}(L) \cap \mathrm{CSP}(M); \tag{4.2}$$

hence $\mathrm{CPcorad}(L) := L \cap \mathrm{CPcorad}(M)$.

Remark 4.1.9. Let M be a non-zero $(\mathcal{D},\mathcal{C})$-bicomodule. Then every $L \in \mathcal{S}_{f.i.}(M)$ is trivially a fully coprime $(\mathcal{D},\mathcal{C})$-bicomodule. If M is self-injective, then $\mathcal{S}_{f.i.}(M) \subseteq \mathrm{CPSpec}(M)$ by Proposition 4.1.8; hence if M has Property $\mathbf{S}_{f.i.}$, then every *fully invariant* non-zero $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ contains a fully M-coprime $(\mathcal{D},\mathcal{C})$-subbicomodule $K \subseteq L$ (in particular, $\varnothing \neq \mathrm{CPSpec}(L) \subseteq \mathrm{CPSpec}(M) \neq \varnothing$).

4.2 Top Bicomodules

In what follows we introduce *top (bi)comodules*, which can be considered (in some sense) as dual to *top (bi)modules,* [Lu1999], [MMS1997], [Zha1999]. We define a Zariski topology on the fully coprime spectrum of such (bi)comodules in a way dual to that of defining the classical Zariski topology on the prime spectrum of (commutative) rings. A reference for the topological terminology and other results we use could be any standard book in general topology (notice that in our case, a *compact space* is not necessarily Hausdorff; such spaces are called *quasi-compact*topological space by some authors, e.g. [Bou1966, I.9.1.]).

Notation. For every $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ set

$$\mathcal{V}_L := \{K \in \mathrm{CPSpec}(M) \mid K \subseteq L\},\ \mathcal{X}_L := \{K \in \mathrm{CPSpec}(M) \mid K \nsubseteq L\}.$$

Moreover, we set

$$\begin{array}{lllllll} \xi(M) & := & \{\mathcal{V}_L \mid L \in \mathcal{L}(M)\}; & \xi_{f.i.}(M) & := & \{\mathcal{V}_L \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \tau_M & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(M)\}; & \tau_M^{f.i.} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}_{f.i.}(M)\}. \\ \mathbf{Z}_M & := & (\mathrm{CPSpec}(M), \tau_M); & \mathbf{Z}_M^{f.i.} & := & (\mathrm{CPSpec}(M), \tau_M^{f.i.}). \end{array}$$

Lemma 4.2.1. *1.* $\mathcal{X}_M = \emptyset$ *and* $\mathcal{X}_{\{0\}} = \mathrm{CPSpec}(M)$.

2. If $\{L_\lambda\}_\Lambda \subseteq \mathcal{L}(M)$, *then* $\mathcal{X}_{\sum\limits_\Lambda L_\lambda} \subseteq \bigcap\limits_\Lambda \mathcal{X}_{L_\lambda} \subseteq \bigcup\limits_\Lambda \mathcal{X}_{L_\lambda} = \mathcal{X}_{\bigcap\limits_\Lambda L_\lambda}$.

3. For any $L_1, L_2 \in \mathcal{L}_{f.i.}(M)$, *we have* $\mathcal{X}_{L_1+L_2} = \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$.

Proof. Notice that "1" and "2" and the inclusion $\mathcal{X}_{L_1+L_2} \subseteq \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$ in (3) are obvious. If $K \in \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$, and $K \notin \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$, then $K \subseteq L_1$ or $K \in L_2$ since K is fully M-coprime, hence $K \notin \mathcal{X}_{L_1}$ or $K \notin \mathcal{X}_{L_2}$ (a contradiction, hence $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} \subseteq \mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)}$). Since $L_2 \subseteq M$ is a fully invariant, we have $L_1 + L_2 \subseteq (L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)$, hence $\mathcal{X}_{(L_1 \overset{(\mathcal{D},\mathcal{C})}{:_M} L_2)} \subseteq \mathcal{X}_{L_1+L_2}$ and we are done.■

Remark 4.2.2. Let $L_1, L_2 \subseteq M$ be *arbitrary* $(\mathcal{D},\mathcal{C})$-subbicomodules. If $L_1, L_2 \subseteq M$ are not fully invariant, then it is not evident that there exists a $(\mathcal{D},\mathcal{C})$-subbicomodule $L \subseteq M$ such that $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_L$. So, for an arbitrary $(\mathcal{D},\mathcal{C})$-bicomodule M, the set $\xi(M)$ is not necessarily closed under finite unions.

The remark above motivates the following

Definition 4.2.3. We call M a *top bicomodule*, iff $\xi(M)$ is closed under finite unions.

As a direct consequence of Lemma 4.2.1 we get

Theorem 4.2.4. $\mathbf{Z}_M^{f.i.} := (\mathrm{CPSpec}(M), \tau_M^{f.i.})$ *is a topological space (which we call the Zariski-topology for* M*). In particular, if* M *is duo, then* M *is a top* $(\mathcal{D}, \mathcal{C})$*-bicomodule (i.e.* $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$ *is a topological space).*

To the end of this section,

M is *duo*, *self-injective* and *has Property* **S**,

so that $\varnothing \neq \mathcal{S}(L) = \mathcal{S}_{f.i.}(L) \subseteq \mathrm{CPSpec}(M)$ for every non-zero $(\mathcal{D}, \mathcal{C})$-subbicomodule $0 \neq L \subseteq M$ (by Remark 4.1.9), and hence a top $(\mathcal{D}, \mathcal{C})$-bicomodule.

Remarks 4.2.5. Consider the Zariski topology $\mathbf{Z}_M := (\mathrm{CPSpec}(M), \tau_M)$.

1. $\mathbf{Z}_M$ is a T_0 (Kolmogorov) space.

2. $\mathcal{B} := \{\mathcal{X}_L \mid L \in \mathcal{L}^{f.g.}(M)\}$ is a basis of open sets for the Zariski topology $\mathbf{Z}_M$: any $K \in \mathrm{CPSpec}(M)$ is contained in some $\mathcal{X}_L$ for some $L \in \mathcal{L}^{f.g.}(M)$ (e.g. $L = 0$); and if $L_1, L_2 \in \mathcal{L}^{f.g.}(M)$ and $K \in \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$, then setting $L := L_1 + L_2 \in \mathcal{L}^{f.g.}(M)$, we have $K \in \mathcal{X}_L \subseteq \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$.

3. Let $L \subseteq M$ be a $(\mathcal{D}, \mathcal{C})$-subbicomodule.

   (a) L is simple if and only if L is fully M-coprime and $\mathcal{V}_L = \{L\}$.

   (b) Assume $L \in \mathrm{CPSpec}(M)$. Then $\overline{\{L\}} = \mathcal{V}_L$; in particular, L is simple if and only if $\{L\}$ is closed in $\mathbf{Z}_M$.

   (c) $\mathcal{X}_L = \mathrm{CPSpec}(M)$ if and only if $L = 0$.

   (d) If $\mathcal{X}_L = \emptyset$, then $\mathrm{Corad}(M) \subseteq L$.

4. Let $0 \neq L \overset{\theta}{\hookrightarrow} M$ be a non-zero $(\mathcal{D}, \mathcal{C})$-bicomodule and consider the embedding $\mathrm{CPSpec}(L) \overset{\widetilde{\theta}}{\hookrightarrow} \mathrm{CPSpec}(M)$ (compare Proposition 4.1.8). Since $\theta^{-1}(\mathcal{V}_N) = \mathcal{V}_{N \cap L}$ for every $N \in \mathcal{L}(M)$, the induced map $\boldsymbol{\theta} : \mathbf{Z}_L \to \mathbf{Z}_M$, $K \mapsto \theta(K)$ is continuous.

5. Let $M \overset{\theta}{\simeq} N$ be an isomorphism of non-zero $(\mathcal{D}, \mathcal{C})$-bicomodules. Then we have bijections $\mathrm{CPSpec}(M) \longleftrightarrow \mathrm{CPSpec}(N)$ and $\mathrm{CSP}(M) \longleftrightarrow \mathrm{CSP}(N)$; in particular, $\theta(\mathrm{CPcorad}(M)) = \mathrm{CPcorad}(N)$. Moreover, $\mathbf{Z}_M \approx \mathbf{Z}_N$ are *homeomorphic spaces.*

Theorem 4.2.6. *The following are equivalent:*

1. $\mathrm{CPSpec}(M) = \mathcal{S}(M)$;
2. $\mathbf{Z}_M$ *is discrete;*
3. $\mathbf{Z}_M$ *is a* T_2 *(Hausdorff space);*
4. $\mathbf{Z}_M$ *is a* T_1 *(Fréchet space).*

Proof. (1) $\Rightarrow$ (2). For every $K \in \mathrm{CPSpec}(M) = \mathcal{S}(M)$, we have $\{K\} = \mathcal{X}_{\mathcal{Y}_K}$ whence open, where $\mathcal{Y}_K := \sum\{L \in \mathrm{CPSpec}(M) \mid K \nsubseteq L\}$.

(2) $\Rightarrow$ (3) & (3) $\Rightarrow$ (4) : Every discrete topological space is T_2 and every T_2 space is T_1.

(4) $\Rightarrow$ (1) Let $\mathbf{Z}_M$ be T_1 and suppose $K \in \mathrm{CPSpec}(M)\backslash\mathcal{S}(M)$, so that $\{K\} = \mathcal{V}_L$ for some $L \in \mathcal{L}(M)$. Since K is not simple, there exists by assumptions and Remark 4.1.9 $K_1 \in \mathcal{S}(K) \subseteq \mathrm{CPSpec}(M)$ with $K_1 \subsetneqq K$, i.e. $\{K_1, K\} \subsetneqq \mathcal{V}_L = \{K\}$, a contradiction. Consequently, $\mathrm{CPSpec}(M) = \mathcal{S}(M)$.■

Proposition 4.2.7. *Let M be self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ be Noetherian with every prime ideal maximal (e.g. a biregular ring*[1]*).*

1. $\mathcal{S}(M) = \mathrm{CPSpec}(M)$ *(so M is subdirectly irreducible* $\Leftrightarrow$ $|\mathrm{CPSpec}(M)| = 1$*).*
2. *If* $L \subseteq M$ *is a* $(\mathcal{D}, \mathcal{C})$*-subbicomodule, then* $\mathcal{X}_L = \varnothing$ *if and only if* $\mathrm{Corad}(M) \subseteq L$.

Proof.

1. Notice that $\mathcal{S}(M) \subseteq \mathrm{CPSpec}(M)$ by Remark 4.1.9. If $K \in \mathrm{CPSpec}(M)$, then $\mathrm{An}(K) \lhd {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is prime by Proposition 2.2.10, whence maximal by assumption and it follows then that $K = \mathrm{Ke}(\mathrm{An}(K))$ is simple (if $0 \neq K_1 \subsetneqq K$, for some $K_1 \in \mathcal{L}(M)$, then $\mathrm{An}(K) \subsetneqq \mathrm{An}(K_1) \subsetneqq {}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ since $\mathrm{Ke}(-)$ is injective, a contradiction).
2. If $L \subseteq M$ is a $(\mathcal{D}, \mathcal{C})$-subbicomodule, then it follows from "1" that $\mathcal{X}_L = \varnothing$ if and only if $\mathrm{Corad}(M) = \mathrm{CPcorad}(M) \subseteq L$.■

[1] a ring in which every two-sided ideal is generated by a central idempotent (see [Wis1991, 3.18(6,7)]).

Remark 4.2.8. Proposition 4.2.7 corrects [NT2001, Lemma 2.6.], which is absurd since it assumes C^* PID, while C is not (fully) coprime (but C^* domain implies C is (fully) coprime!!).

Theorem 4.2.9. *If $|\mathcal{S}(M)|$ is countable (finite), then $\mathbf{Z}_M$ is Lindelöf (compact). The converse holds, if $\mathcal{S}(M) = \mathrm{CPSpec}(M)$.*

Proof. Assume $\mathcal{S}(M) = \{S_{\lambda_k}\}_{k\geq 1}$ is countable (finite). Let $\{\mathcal{X}_{L_\alpha}\}_{\alpha\in I}$ be an open cover of CPSpec(M) (i.e. $\mathrm{CPSpec}(M) \subseteq \bigcup\limits_{\alpha\in I} \mathcal{X}_{L_\alpha}$). Since $\mathcal{S}(M) \subseteq$ CPSpec(M) we can pick for each $k \geq 1$, some $\alpha_k \in I$ such that $S_{\lambda_k} \nsubseteq L_{\alpha_k}$. If $\bigcap\limits_{k\geq 1} L_{\alpha_k} \neq 0$, then it contains by Property **S** a simple $(\mathcal{D}, \mathcal{C})$-subbicomodule $0 \neq S \subseteq \bigcap\limits_{k\geq 1} L_{\alpha_k}$, (a contradiction, since $S = S_{\lambda_k} \nsubseteq L_{\alpha_k}$ for some $k \geq 1$). Hence $\bigcap\limits_{k\geq 1} L_{\alpha_k} = 0$ and we conclude that $\mathrm{CPSpec}(M) = \mathcal{X}_{\bigcap\limits_{k\geq 1} L_{\alpha_k}} = \bigcup\limits_{k\geq 1} \mathcal{X}_{L_{\alpha_k}}$ (i.e. $\{\mathcal{X}_{L_{\alpha_k}} \mid k \geq 1\} \subseteq \{\mathcal{X}_{L_\alpha}\}_{\alpha\in I}$ is a countable (finite) subcover). Notice that if $\mathcal{S}(M) = \mathrm{CPSpec}(M)$, then $\mathbf{Z}_M$ is discrete by Theorem 4.2.6 and so $\mathbf{Z}_M$ is Lindelöf (compact) if and only if CPSpec(M) is countable (finite).■

Definition 4.2.10. A collection $\mathcal{G}$ of subsets of a topological space $\mathbf{X}$ is *locally finite*, iff every point of $\mathbf{X}$ has a neighbourhood that intersects only finitely many elements of $\mathcal{G}$.

Proposition 4.2.11. *Let $\mathcal{K} = \{K_\lambda\}_\Lambda \subseteq \mathcal{S}(M)$ be a non-empty family of simple $(\mathcal{D}, \mathcal{C})$-subbicomodules. If $|\mathcal{S}(L)| < \infty$ for every $L \in \mathrm{CPSpec}(M)$, then $\mathcal{K}$ is locally finite.*

Proof. Let $L \in \mathrm{CPSpec}(M)$ and set $F := \sum\{K \in \mathcal{K} \mid K \nsubseteq L\}$. Since $|\mathcal{S}(L)| < \infty$, there exists a finite number of simple $(\mathcal{D}, \mathcal{C})$-subbicomodules $\{S_{\lambda_1}, .., S_{\lambda_n}\} = \mathcal{K} \cap \mathcal{V}_L$. If $L \subseteq F$, then $0 \neq L \subseteq \sum\limits_{i=1}^{n} S_{\lambda_i} \subseteq (S_{\lambda_1} :_M^{(\mathcal{D},\mathcal{C})} \sum\limits_{i=2}^{n} S_{\lambda_i})$ and it follows by induction that $0 \neq L \subsetneqq S_{\lambda_i}$ for some $1 \leq i \leq n$ (a contradiction, since S_{λ_i} is simple), whence $L \in \mathcal{X}_F$. It is clear then that $\mathcal{K} \cap \mathcal{X}_F = \{K_{\lambda_1}, .., K_{\lambda_n}\}$ and we are done.■

Definition 4.2.12. ([Bou1966], [Bou1998]) A topological space $\mathbf{X}$ is said to be *irreducible* (*connected*), iff $\mathbf{X}$ is not the (disjoint) union of two proper closed subsets; equivalently, iff the intersection of any two non-empty open subsets is non-empty (the only subsets of $\mathbf{X}$ that are open and closed are

$\varnothing$ and $\mathbf{X}$). A maximal irreducible subspace of $\mathbf{X}$ is called an *irreducible component*.

Proposition 4.2.13. CPSpec(M) *is irreducible if and only if* CPcorad(M) *is fully M-coprime.*

Proof. Let CPSpec(M) be irreducible. By Remark 4.1.9, CPcorad$(M) \neq 0$. Suppose that CPcorad(M) is not fully M-coprime, so that there exist $(\mathcal{D},\mathcal{C})$-subbicomodules $X, Y \subseteq M$ with CPcorad$(M) \subseteq (X :_M^{(\mathcal{D},\mathcal{C})} Y)$ but CPcorad$(M) \nsubseteq X$ and CPcorad$(M) \nsubseteq Y$. It follows then that CPSpec$(M) = \mathcal{V}_{(X:_M^{(\mathcal{D},\mathcal{C})}Y)} = \mathcal{V}_X \cup \mathcal{V}_Y$ a union of proper closed subsets, a contradiction. Consequently, CPcorad(M) is fully M-coprime.

On the other hand, assume CPcorad$(M) \in$ CPSpec(M) and suppose that CPSpec$(M) = \mathcal{V}_{L_1} \cup \mathcal{V}_{L_2} = \mathcal{V}_{(L_1:_M^{(\mathcal{D},\mathcal{C})}L_2)}$ for some $(\mathcal{D},\mathcal{C})$-subbicomodules $L_1, L_2 \subseteq M$. It follows then that CPcorad$(M) \subseteq L_1$, so that $\mathcal{V}_{L_1} =$ CPSpec(M); or CPcorad$(M) \subseteq L_2$, so that $\mathcal{V}_{L_2} =$ CPSpec(M). Consequently CPSpec(M) is not the union of two *proper* closed subsets, i.e. it is irreducible.■

Lemma 4.2.14.

1. *M is subdirectly irreducible if and only if the intersection of any two non-empty closed subsets of* CPSpec(M) *is non-empty.*
2. *If M is subdirectly irreducible, then* CPSpec(M) *is connected. If* CPSpec(M) *is connected and* CPSpec$(M) = \mathcal{S}(M)$, *then M is subdirectly irreducible.*

Proof.

1. Assume M is subdirectly irreducible with unique simple $(\mathcal{D},\mathcal{C})$-subbicomodule $0 \neq S \subseteq M$. If $\mathcal{V}_{L_1}, \mathcal{V}_{L_2} \subseteq$ CPSpec(M) are any two non-empty closed subsets, then $L_1 \neq 0 \neq L_2$ and so $\mathcal{V}_{L_1} \cap \mathcal{V}_{L_2} = \mathcal{V}_{L_1 \cap L_2} \neq \varnothing$, since $S \subseteq L_1 \cap L_2 \neq 0$. On the other hand, assume that the intersection of any two non-empty closed subsets of CPSpec(M) is non-empty. Let $0 \neq L_1, L_2 \subseteq M$ be any non-zero $(\mathcal{D},\mathcal{C})$-subbicomodules, so that $\mathcal{V}_{L_1} \neq \varnothing \neq \mathcal{V}_{L_2}$. By assumption $\mathcal{V}_{L_1 \cap L_2} = \mathcal{V}_{L_1} \cap \mathcal{V}_{L_2} \neq \varnothing$, hence $L_1 \cap L_2 \neq 0$ and it follows by 1.4.2 that M is subdirectly irreducible.
2. If M is subdirectly irreducible, then CPSpec(M) is connected by "1". On the other hand, if CPSpec$(M) = \mathcal{S}(M)$, then $\mathbf{Z}_M$ is discrete by Theorem 4.2.6 and so M is subdirectly irreducible (since a discrete topological space is connected if and only if it has only one point).■

Proposition 4.2.15. *1. If $K \in \mathrm{CPSpec}(M)$, then $\mathcal{V}_K \subseteq \mathrm{CPSpec}(M)$ is irreducible.*

2. If $\mathcal{V}_L$ is an irreducible component of $\mathbf{Z}_M$, then L is a maximal fully M-coprime $(\mathcal{D},\mathcal{C})$-subbicomodule.

Proof. 1. Let $K \in \mathrm{CPSpec}(M)$ and suppose $\mathcal{V}_K = A \cup B = (\mathcal{V}_K \cap \mathcal{V}_X) \cup (\mathcal{V}_K \cap \mathcal{V}_Y)$ for two $(\mathcal{D},\mathcal{C})$-subbicomodules $X, Y \subseteq M$ (so that $A, B \subseteq \mathcal{V}_K$ are closed subsets w.r.t. the relative topology on $\mathcal{V}_K \hookrightarrow \mathrm{CPSpec}(M)$). It follows then that $\mathcal{V}_K = (\mathcal{V}_{K\cap X}) \cup (\mathcal{V}_{K\cap Y}) = \mathcal{V}_{(K\cap X :_M^{(\mathcal{D},\mathcal{C})} K\cap Y)}$ and so $K \subseteq (K \cap X :_M^{(\mathcal{D},\mathcal{C})} K \cap Y)$, hence $K \subseteq X$ so that $\mathcal{V}_K = A$; or $K \subseteq Y$, so that $\mathcal{V}_K = B$. Consequently $\mathcal{V}_K$ is irreducible.

2. Assume $\mathcal{V}_L$ is an irreducible component of $\mathrm{CPSpec}(M)$ for some $0 \neq L \in \mathcal{L}(M)$. If $L \subseteq K$ for some $K \in \mathrm{CPSpec}(M)$, then $\mathcal{V}_L \subseteq \mathcal{V}_K$ and it follows then that $L = K$ (since $\mathcal{V}_K \subseteq \mathrm{CPSpec}(M)$ is irreducible by "1"). We conclude then that L is fully M-coprime and is moreover maximal in $\mathrm{CPSpec}(M)$.■

Lemma 4.2.16. *If $n \geq 2$ and $\mathcal{A} = \{K_1, ..., K_n\} \subseteq \mathrm{CPSpec}(M)$ is a connected subset, then for every $i \in \{1, ..., n\}$, there exists $j \in \{1, ..., n\} \backslash \{i\}$ such that $K_i \subseteq K_j$ or $K_j \subseteq K_i$.*

Proof. Without loss of generality, suppose $K_1 \nsubseteq K_j$ and $K_j \nsubseteq K_1$ for all $2 \leq j \leq n$ and set $F := \sum_{i=2}^{n} K_i$, $W_1 := \mathcal{A} \cap \mathcal{X}_{K_1} = \{K_2, ..., K_n$ and $W_2 := \mathcal{A} \cap \mathcal{X}_F = \{K_1\}$ (if $n = 2$, then clearly $W_2 = \{K_1\}$; if $n > 2$ and $K_1 \notin W_2$, then $K_1 \subseteq \sum_{i=2}^{n} K_i \subseteq (K_2 :_M^{(\mathcal{D},\mathcal{C})} \sum_{i=3}^{n} K_i)$ and it follows that $K_1 \subseteq \sum_{i=3}^{n} K_i$; by induction one shows that $K_1 \subseteq K_n$, a contradiction). So $\mathcal{A} = W_1 \cup W_2$, a disjoint union of proper non-empty open subsets (a contradiction).■

Notation. For $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$ set $\varphi(\mathcal{A}) := \sum_{K\in\mathcal{A}} K$ ($:= 0$, iff $\mathcal{A} = \varnothing$). Moreover, set

$$\begin{array}{lcl}\mathbf{CL}(\mathbf{Z}_M) & := & \{\mathcal{A} \subseteq \mathrm{CPSpec}(M) \mid \mathcal{A} = \overline{\mathcal{A}}\}; \\ \mathcal{E}(M) & := & \{L \in \mathcal{L}(M) \mid \mathrm{CPcorad}(L) = L\}.\end{array}$$

Lemma 4.2.17. *The closure of any subset $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$ is $\overline{\mathcal{A}} = \mathcal{V}_{\varphi(\mathcal{A})}$.*

Proof. Let $\mathcal{A} \subseteq \mathrm{CPSpec}(M)$. Since $\mathcal{A} \subseteq \mathcal{V}_{\varphi(\mathcal{A})}$ and $\mathcal{V}_{\varphi(\mathcal{A})}$ is a closed set, we have $\overline{\mathcal{A}} \subseteq \mathcal{V}_{\varphi(\mathcal{A})}$. On the other hand, suppose $H \in \mathcal{V}_{\varphi(\mathcal{A})} \backslash \mathcal{A}$ and let $\mathcal{X}_L$ be a neighbourhood of H, so that $H \nsubseteq L$. Then there exists $W \in \mathcal{A}$ with $W \nsubseteq L$ (otherwise $H \subseteq \varphi(\mathcal{A}) \subseteq L$, a contradiction), i.e. $W \in \mathcal{X}_L \cap (\mathcal{A} \backslash \{H\}) \neq \varnothing$ and so K is a cluster point of $\mathcal{A}$. Consequently, $\overline{\mathcal{A}} = \mathcal{V}_{\varphi(\mathcal{A})}$.■

Theorem 4.2.18. *We have a bijection*

$$\mathbf{CL}(\mathbf{Z}_M) \longleftrightarrow \mathcal{E}(M).$$

If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then there is a bijection

$$\mathbf{CL}(\mathbf{Z}_M) \backslash \{\varnothing\} \longleftrightarrow \mathrm{CSPSpec}(M).$$

Proof. For $L \in \mathcal{E}(M)$, set $\psi(L) := \mathcal{V}_L$. Then for $L \in \mathcal{E}(M)$ and $\mathcal{A} \in \mathbf{CL}(\mathbf{Z}_M)$ we have $\varphi(\psi(L)) = \varphi(\mathcal{V}_L) = L \cap \mathrm{CPcorad}(M) = \mathrm{CPcorad}(L) = L$ and $\psi(\varphi(\mathcal{A})) = \mathcal{V}_{\varphi(\mathcal{A})} = \overline{\mathcal{A}} = \mathcal{A}$. If M is self-cogenerator and ${}^{\mathcal{D}}\mathrm{E}_M^{\mathcal{C}}$ is right Noetherian, then $\mathrm{CSPSpec}(M) = \mathcal{E}(M) \backslash \{0\}$ by Proposition 4.1.7 and we are done.■

4.3 Applications and Examples

In this section we give some applications and examples. First of all we remark that taking $\mathcal{D} := R$ ($\mathcal{C} := R$), considered with the trivial coring structure, our results on the Zariski topology for bicomodules in the third section can be reformulated for Zariski topology on the fully coprime spectrum of right $\mathcal{C}$-comodules (left $\mathcal{D}$-comodules). However, our main application will be to the Zariski topology on the fully coprime spectrum of non-zero corings, considered as *duo bicomodules* in the canonical way.

Throughout this section, $\mathcal{C}$ is a non-zero A-coring with ${}_A\mathcal{C}$ and $\mathcal{C}_A$ flat.

4.3.1. The (A, A)-bimodule ${}^*\mathcal{C}^* := \mathrm{Hom}_{(A,A)}(\mathcal{C}, A) := {}^*\mathcal{C} \cap \mathcal{C}^*$ is an A^{op}-ring with multiplication $(f * g)(c) = \sum f(c_1)g(c_2)$ for all $f, g \in {}^*\mathcal{C}^*$ and unit $\varepsilon_{\mathcal{C}}$; hence every $(\mathcal{C}, \mathcal{C})$-bicomodule M is a $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bimodule and the *centralizer*

$$\mathbf{C}(M) := \{f \in {}^*\mathcal{C}^* \mid f \rightharpoonup m = m \leftharpoonup f \text{ for all } m \in M\}$$

is an R-algebra. If M is faithful as a left (right) ${}^*\mathcal{C}^*$-module, then $\mathbf{C}(M) \subseteq Z({}^*\mathcal{C}^*)$.

4.3.2. Considering $\mathcal{C}$ as a $(\mathcal{C},\mathcal{C})$-bicomodule in the natural way, $\mathcal{C}$ is a $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bimodule that is faithful as a left (right) ${}^*\mathcal{C}^*$-module, hence the centralizer

$$\mathbf{C}(\mathcal{C}) := \{f \in {}^*\mathcal{C}^* \mid f \rightharpoonup c = c \leftharpoonup f \text{ for every } c \in \mathcal{C}\}$$

embeds in the center of ${}^*\mathcal{C}^*$ as an R-subalgebra, i.e. $\mathbf{C}(\mathcal{C}) \hookrightarrow Z({}^*\mathcal{C}^*)$. If $ac = ca$ for all $a \in A$, then we have a morphism of R-algebras $\eta : Z(A) \to \mathbf{C}(\mathcal{C})$, $a \mapsto [\varepsilon_{\mathcal{C}}(a-) = \varepsilon_{\mathcal{C}}(-a)]$.

Remark 4.3.3. Notice that $\mathbf{C}(\mathcal{C}) \subseteq Z({}^*\mathcal{C}) \subseteq Z({}^*\mathcal{C}^*)$ and $\mathbf{C}(\mathcal{C}) \subseteq Z(\mathcal{C}^*) \subseteq Z({}^*\mathcal{C}^*)$ (compare [BW2003, 17.8. (4)]). If ${}_A\mathcal{C}$ ($\mathcal{C}_A$) is A-cogenerated, then it follows by [BW2003, 19.10 (3)] that $Z({}^*\mathcal{C}) = \mathbf{C}(\mathcal{C}) \subseteq Z(\mathcal{C}^*)$ ($Z(\mathcal{C}^*) = \mathbf{C}(\mathcal{C}) \subseteq Z({}^*\mathcal{C})$). If ${}_A\mathcal{C}_A$ is A-cogenerated, then $Z({}^*\mathcal{C}^*) \subseteq \mathbf{C}(\mathcal{C})$ (e.g. [BW2003, 19.10 (4)]), whence $Z({}^*\mathcal{C}) = Z({}^*\mathcal{C}^*) = Z(\mathcal{C}^*)$.

Lemma 4.3.4. *For every $(\mathcal{C},\mathcal{C})$-bicomodule M we have a morphism of R-algebras*

$$\phi_M : \mathbf{C}(M) \to {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(M)^{op},\ f \mapsto [m \mapsto f \rightharpoonup m = m \leftharpoonup f]. \tag{4.3}$$

Moreover, $\mathrm{Im}(\phi_M) \subseteq Z({}^{\mathcal{C}}\mathrm{E}_M^{\mathcal{C}})$.

Proof. First of all we prove that ϕ_M is well-defined: for $f \in \mathbf{C}(M)$ and $m \in M$ we have

$$\begin{aligned}
\sum(\phi_M(f)(m))_{<0>} \otimes_A (\phi_M(f)(m))_{<1>} &= \sum(m \leftharpoonup f)_{<0>} \otimes_A (m \leftharpoonup f)_{<1>} \\
&= \sum f(m_{<-1>})m_{<0><0>} \otimes_A m_{<0><1>} \\
&= \sum f(m_{<0><-1>})m_{<0><0>} \otimes_A m_{<1>} \\
&= \sum(m_{<0>} \leftharpoonup f) \otimes_A m_{<1>} \\
&= \sum \phi_M(f)(m_{<0>}) \otimes_A m_{<1>},
\end{aligned}$$

and

$$\begin{aligned}
\sum(\phi_M(f)(m))_{<-1>} \otimes_A (\phi_M(f)(m))_{<0>} &= \sum(f \rightharpoonup m)_{<-1>} \otimes_A (f \rightharpoonup m)_{<0>} \\
&= \sum m_{<0><-1>} \otimes_A m_{<0><0>} f(m_{<1>}) \\
&= \sum m_{<-1>} \otimes_A m_{<0><0>} f(m_{<0><1>}) \\
&= \sum m_{<-1>} \otimes_A (f \rightharpoonup m_{<0>}) \\
&= \sum m_{<-1>} \otimes_A \phi_M(f)(m_{<0>}),
\end{aligned}$$

i.e. $\phi_M(f) : M \to M$ is $(\mathcal{C},\mathcal{C})$-bicolinear. Obviously, $\phi_M(f * g) = \phi_M(f) \circ^{op} \phi_M(g)$ for all $f, g \in \mathbf{C}(M)$, i.e. ϕ_M is a morphism of R-algebras. Moreover, since every $g \in {}^{\mathcal{C}}\mathrm{E}_M^{\mathcal{C}}$ is $({}^*\mathcal{C}^*, {}^*\mathcal{C}^*)$-bilinear, we have $g(f \rightharpoonup m) = f \rightharpoonup g(m)$ for every $f \in {}^*\mathcal{C}^*$ and $m \in M$, i.e. $\mathrm{Im}(\phi_M) \subseteq Z({}^{\mathcal{C}}\mathrm{E}_M^{\mathcal{C}})$.■

Lemma 4.3.5. *We have an isomorphism of R-algebras* $\mathbf{C}(\mathcal{C}) \overset{\phi_{\mathcal{C}}}{\simeq} {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$, *with inverse* $\psi_{\mathcal{C}} : g \mapsto \varepsilon_{\mathcal{C}} \circ g$. *In particular,* $({}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C}), \circ)$ *is commutative and* $\mathcal{C} \in {}^{\mathcal{C}}\mathbb{M}^{\mathcal{C}}$ *is duo.*

Proof. First of all we prove that ψ is well-defined: for $g \in {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ and $c \in \mathcal{C}$ we have

$$\begin{array}{lclclcl}
\psi_{\mathcal{C}}(g) \rightharpoonup c & = & \sum c_1\psi(g)(c_2) & = & \sum c_1\varepsilon_{\mathcal{C}}(g(c_2)) & = & \sum g(c)_1\varepsilon_{\mathcal{C}}(g(c)_2) \\
& = & g(c) & = & \sum \varepsilon_{\mathcal{C}}(g(c)_1)g(c)_2 & = & \sum \varepsilon_{\mathcal{C}}(g(c_1))c_2 \\
& = & \sum \psi(g)(c_1)c_2 & = & c \leftharpoonup \psi(g), & &
\end{array}$$

i.e. $\psi_{\mathcal{C}}(g) \in \mathbf{C}(\mathcal{C})$. For any $f \in \mathbf{C}(\mathcal{C})$, $g \in {}^{\mathcal{C}}\mathrm{End}^{\mathcal{C}}(\mathcal{C})$ and $c \in \mathcal{C}$ we have $((\psi_{\mathcal{C}} \circ \phi_{\mathcal{C}})(f))(c) = \varepsilon_{\mathcal{C}}(\phi_{\mathcal{C}}(f)(c)) = \varepsilon_{\mathcal{C}}(f \rightharpoonup c) = f(c)$ and $((\phi_{\mathcal{C}} \circ \psi_{\mathcal{C}})(g))(c) = \sum c_1\psi_{\mathcal{C}}(g)(c_2) = \sum c_1\varepsilon_{\mathcal{C}}(g(c_2)) = \sum g(c)_1\varepsilon_{\mathcal{C}}(g(c)_2) = g(c)$.■

Zariski topologies for corings

Notation. With $\mathcal{B}(\mathcal{C})$ we denote the class of $\mathcal{C}$-bicoideals and with $\mathcal{L}(\mathcal{C}^r)$ (resp. $\mathcal{L}(\mathcal{C}^l)$) the class of right (left) $\mathcal{C}$-coideals. For a $\mathcal{C}$-bicoideal $K \in \mathcal{B}(\mathcal{C})$, K^r (K^l) indicates that we consider K as a right (left) $\mathcal{C}$-comodule, rather than a $(\mathcal{C}, \mathcal{C})$-bicomodule. We also set

$$\begin{array}{lcl}
\mathrm{CPSpec}(\mathcal{C}) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K \text{ is fully } \mathcal{C}\text{-coprime}\}; \\
\mathrm{CPSpec}(\mathcal{C}^r) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K^r \text{ is fully } \mathcal{C}^r\text{-coprime}\}; \\
\mathrm{CPSpec}(\mathcal{C}^l) & := & \{K \in \mathcal{B}(\mathcal{C}) \mid K^l \text{ is fully } \mathcal{C}^l\text{-coprime}\};
\end{array}$$

and

$$\begin{array}{lcl}
\tau_{\mathcal{C}} & := & \{\mathcal{X}_L \mid L \in \mathcal{B}(\mathcal{C})\}; \\
\tau_{\mathcal{C}^r} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(\mathcal{C}^r)\}; \\
\tau_{\mathcal{C}^l} & := & \{\mathcal{X}_L \mid L \in \mathcal{L}(\mathcal{C}^l)\}.
\end{array}$$

In what follows we announce only the main result on the Zariski topologies for corings, leaving to the interested reader the restatement of the other results of the third section.

Theorem 4.3.6.
1. $\mathbf{Z}_{\mathcal{C}} := (\mathrm{CPSpec}(\mathcal{C}), \tau_{\mathcal{C}})$ *is a topological space (which we call the Zariski topology for* $\mathcal{C}$*)..*
2. $\mathbf{Z}_{\mathcal{C}^r}^{f.i.} := (\mathrm{CPSpec}(C^r), \tau_{\mathcal{C}^r}^{f.i.})$ *and* $\mathbf{Z}_{\mathcal{C}^l}^{f.i.} := (\mathrm{CPSpec}(C^l), \tau_{\mathcal{C}^l}^{f.i.})$ *are topological spaces.*

Proposition 4.3.7. *Let $\theta : \mathcal{C} \to \mathcal{C}'$ be a morphism of non-zero A-corings with ${}_A\mathcal{C}$, ${}_A\mathcal{C}'$ flat, $\mathcal{C}^r$ intrinsically injective self-cogenerator and $\mathcal{C}'^r$ self-cogenerator.*

1. *If θ is injective and $\mathcal{C}'^r$ is self-injective, or if $\mathcal{C}^*$ is right-duo, then we have a map $\widetilde{\theta} : \mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$, $K \mapsto \theta(K)$ (and so $\theta(\mathrm{CPcorad}(\mathcal{C}^r)) \subseteq \mathrm{CPcorad}(\mathcal{C}'^r)$).*

2. *If $\mathcal{C}^r$, $\mathcal{C}'^r$ are duo, then the induced map $\boldsymbol{\theta} : \mathbf{Z}_{\mathcal{C}^r} \to \mathbf{Z}_{\mathcal{C}'^r}$ is continuous.*

3. *If every $K \in \mathrm{CPSpec}(\mathcal{C}^r)$ is inverse image of a $K' \in \mathrm{CPSpec}(\mathcal{C}'^r)$, then $\widetilde{\theta}$ is injective.*

4. *If θ is injective and $\mathcal{C}'^r$ is self-injective, then $\boldsymbol{\theta} : \mathbf{Z}^{f.i.}_{\mathcal{C}^r} \to \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ is continuous. If moreover, $\widetilde{\theta} : \mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$ is surjective, then $\boldsymbol{\theta}$ is open and closed.*

5. *If $\mathcal{C} \overset{\theta}{\simeq} \mathcal{C}'$, then $\mathbf{Z}^{f.i.}_{\mathcal{C}^r} \overset{\boldsymbol{\theta}}{\approx} \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ (homeomorphic spaces).*

Proof. First of all notice for every $K \in \mathcal{L}(\mathcal{C}^r)$ ($K \in \mathcal{B}(\mathcal{C})$), we have $\theta(K) \in \mathcal{L}(\mathcal{C}'^r)$ ($\theta(K) \in \mathcal{B}(\mathcal{C}')$) and for every $K' \in \mathcal{L}(\mathcal{C}'^r)$ ($K' \in \mathcal{B}(\mathcal{C}')$), $\theta^{-1}(K') \in \mathcal{L}(\mathcal{C}^r)$ ($\theta^{-1}(K') \in \mathcal{B}(\mathcal{C})$).

1. If θ is injective and $\mathcal{C}'^r$ is self-injective, then $\mathrm{CPSpec}(\mathcal{C}^r) = \mathcal{B}(\mathcal{C}) \cap \mathrm{CPSpec}(\mathcal{C}'^r)$ by [Abu2006, Proposition 4.7.]. Assume now that $\mathcal{C}^*$ is right-duo. Since θ is a morphism of A-corings, the canonical map $\theta^* : \mathcal{C}'^* \to \mathcal{C}^*$ is a morphism of A^{op}-rings. If $K \in \mathrm{CPSpec}(\mathcal{C}^r)$, then $\mathrm{ann}_{\mathcal{C}^*}(K) \lhd \mathcal{C}^*$ is a prime ideal by [Abu2006, Proposition 4.10.], whence completely prime since $\mathcal{C}^*$ is right-duo. It follows then that $\mathrm{ann}_{\mathcal{C}'^*}(\theta(K)) = \theta(K)^{\perp \mathcal{C}'^*} = (\theta^*)^{-1}(K^{\perp \mathcal{C}^*}) = (\theta^*)^{-1}(\mathrm{ann}_{\mathcal{C}^*}(K))$ is a prime ideal, whence $\theta(K) \in \mathrm{CPSpec}(\mathcal{C}'^r)$ by [Abu2006, Proposition 4.10.]. It is obvious then that $\theta(\mathrm{CPcorad}(\mathcal{C}^r)) \subseteq \mathrm{CPcorad}(\mathcal{C}'^r)$.

2. Since $\mathcal{C}^r \in \mathbb{M}^{\mathcal{C}}$, $\mathcal{C}'^r \in \mathbb{M}^{\mathcal{C}'}$ are duo, $\mathbf{Z}_{\mathcal{C}^r} := \mathbf{Z}^{f.i.}_{\mathcal{C}^r}$ and $\mathbf{Z}_{\mathcal{C}'^r} := \mathbf{Z}^{f.i.}_{\mathcal{C}'^r}$ are topological spaces. Since $\mathcal{C}^r$ is intrinsically injective, $\mathcal{C}^*$ is right-due and by "1" $\widetilde{\theta} : \mathrm{CPSpec}(\mathcal{C}^r) \to \mathrm{CPSpec}(\mathcal{C}'^r)$ is well-defined. For $L' \in \mathcal{L}(\mathcal{C}'^r)$, $\widetilde{\theta}^{-1}(\mathcal{X}_{L'}) = \mathcal{X}_{\theta^{-1}(L')}$, i.e. $\boldsymbol{\theta}$ is continuous.

3. Suppose $\widetilde{\theta}(K_1) = \widetilde{\theta}(K_2)$ for some $K_1, K_2 \in \mathrm{CPSpec}(\mathcal{C}^r)$ with $K_1 = \theta^{-1}(K_1')$, $K_2 = \theta^{-1}(K_2')$ where $K_1', K_2' \in \mathrm{CPSpec}(\mathcal{C}'^r)$. Then $K_1 = \theta^{-1}(K_1') = \theta^{-1}(\theta(\theta^{-1}(K_1'))) = \theta^{-1}(\theta(\theta^{-1}(K_1'))) = \theta^{-1}(\theta(\theta^{-1}(K_2'))) = \theta^{-1}(K_2') = K_2$.

4. By [Abu2006, Proposition 4.7.] $\mathrm{CPSpec}(\mathcal{C}^r) = \mathcal{B}(\mathcal{C}^r) \cap \mathrm{CPSpec}(\mathcal{C}'^r)$, hence for $L \in \mathcal{L}(\mathcal{C}^r)$ and $L' \in \mathcal{L}(\mathcal{C}'^r)$ we have $\boldsymbol{\theta}^{-1}(\mathcal{V}_{L'}) = \mathcal{V}_{\theta^{-1}(L')}$, $\boldsymbol{\theta}(\mathcal{V}_L) = \mathcal{V}_{\theta(L)}$ and $\boldsymbol{\theta}(\mathcal{X}_L) = \mathcal{X}_{\theta(L)}$.

5. Since θ is an isomorphism, $\widetilde{\theta}$ is bijective by [Abu2006, Proposition 4.5.]. In this case $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{-1}$ are obviously continuous (see "4").■

Example 4.3.8. ([NT2001, Example 1.1.]) Let k be a field and $C := k[X]$ be the cocommutative k-coalgebra with $\Delta(X^n) := X^n \otimes_k X^n$ and $\varepsilon(X^n) := 1$ for all $n \geq 0$. For each $n \geq 0$, set $C_n := kX^n$. Then $\mathrm{CPSpec}(C) = \mathcal{S}(C) = \{C_n \mid n \geq 0\}$. Notice that

1. $\mathbf{Z}_C$ is discrete by Theorem 4.2.6, hence $\mathbf{Z}_C$ is Lindelöf (but not compact) by Theorem 4.2.9.

2. $\mathrm{CPSpec}(C)$ is not connected: $\mathrm{CPSpec}(C) = \{C_n \mid n \geq 1\} \cup \{C_0\} = \mathcal{X}_{\{k\}} \cup \mathcal{X}_{<X,X^2,\ldots>}$ (notice that $\mathrm{CPSpec}(C)$ is not subdirectly irreducible, compare with Lemma 4.2.14.

Example 4.3.9. ([NT2001, Example 1.2.]) Let k be a field and $C := k[X]$ be the cocommutative k-coalgebra with $\Delta(X^n) := \sum_{j=1}^{n} X^j \otimes_k X^{n-j}$ and $\varepsilon(X^n) := \delta_{n,0}$ for all $n \geq 0$. For each $n \geq 0$ set $C_n := < 1, ..., X^n >$. For each $n \geq 1$, $C_n \subseteq (C_{n-1} :_C < kX^n >)$, hence not fully C-coprime and it follows that $\mathrm{CPSpec}(C) = \{k, C\}$ (since k is simple, whence fully C-coprime and $C^* \simeq k[[X]]$ is an integral domain, whence C is fully coprime). Notice that

1. C is subdirectly irreducible with unique simple subcoalgebra $C_0 = k$;

2. the converse of Remark 4.2.5 "3(d)" does not hold in general: $\mathrm{Corad}(C) = k \subseteq C_1$ while $\mathcal{X}_{C_1} = \{C\} \neq \varnothing$ (compare Proposition 4.2.7 "2").

3. $\mathrm{CPSpec}(C)$ is connected, although $\mathcal{S}(C) \subsetneqq \mathrm{CPSpec}(C)$ (see Lemma 4.2.14 "2").

4. $\mathbf{Z}_C$ is not T_1 by Theorem 4.2.6, since $C \in \mathrm{CPSpec}(C) \backslash \mathcal{S}(C)$: in fact, if $C \in \mathcal{X}_{L_1}$ and $C_0 \in \mathcal{X}_{L_2}$ for some C-subcoalgebras $L_1, L_2 \subseteq C$, then $L_2 = 0$ (since C is subdirectly irreducible with unique simple subcoalgebra C_0); hence $\mathcal{X}_{L_2} = \{C_0, C\} = \mathrm{CPSpec}(C)$ and $\mathcal{X}_{L_1} \cap \mathcal{X}_{L_2} = \mathcal{X}_{L_1} \neq \varnothing$.

Remark 4.3.10. As this paper extends results of [NT2001], several proofs and ideas are along the lines of the original ones. However, our results are much more general (as [NT2001] is restricted to coalgebras over fields). Moreover, we should warn the reader that in addition to the fact that several results in that paper are redundant or repeated, several other results are even *absurd*, e.g. Proposition 2.8., Corollary 2.4. and Theorem 2.4. (as noticed by Chen Hui-Xiang in his review; Zbl 1012.16041) in addition to [NT2001, Lemma 2.6.] as we clarified in Remark 4.2.8. We corrected the statement of some of these results (e.g. Proposition 4.2.7 corrects [NT2001, Lemma 2.6.]; while Proposition 4.3.7 suggests a correction of [NT2001, Theorem 2.4.] which does not hold in general as the counterexample [Abu2006, 5.20.] shows). Moreover, we improved some other results (e.g. applying Theorem 4.2.6 to coalgebras over base fields improves and puts together several scattered results of [NT2001]).

Future Research

The main results in this report were published in two papers [Abu2006] and [Abu2008]. Moreover, the investigations carried out during this project inspired several new approaches in research. In what follows I list some of the ideas suggested by such investigations[2]:

- The notions of endo-prime (endo-semiprime) comodules and the dual notions of endo-coprime (endo-cosemiprime) comodules for corings will be studied and investigated in [Abu-a] (where some unpublished partial results of Section 2.3 will be included).

- Different notions of primeness and coprimeness in this report can be investigated in categories of modules and bimodules over (commutative) rings. Some generalizations will be carried out in [Abu-b].

- Different Zariski-like topologies can be introduced and investigated for modules and bimodules over (commutative) rings. This will be done in [Abu-c].

- Most of the results in this paper can be transformed to investigate the notion of coprimeness in the sense of Annin [Ann2002] in categories of (bi)comodules and define a Zariski topology on the spectrum of *coprime sub(bi)comodules* of a given (bi)comodule.

- More generally, such (co)primeness notions can be developed in more general Abelian categories. These and other applications will be considered in forthcoming papers.

[2] Some of these were suggested by referees of the two published papers mentioned above. I do thank the anonymous referees for their useful suggestions.

Open Problems

One of the goals of this project was to shed more light on the (co)localization problem for corings and comodules. Although we were successful in at least introducing candidates of what plays the rule of a prime ideal in the classical localization of rings and modules (namely coprime bicoideals), we are more convinced than before that a well-established theory of *colocalization* for corings and comodules is **still far from being achieved**.

Given a coalgebra C (over a base filed), it's not clear how to choose a good "localizing set" S to define a new coalgebra $C_{[S]}$. In what follows we provide some references with different approaches to handle the (co)localization of coalgebras and comodules:

- One approach suggested by M. Takeuchi [Tak1985] was to work in the category of topological coalgebras. In [FS1998][3], M. Farinati and A. Solotar followed this approach and considered the localizing sets as multiplicatively closed subsets of the center of the topological dual algebra. Such localizations were used then to study cohomology theories for topological coalgebras.

- In [NT1996] C. Nastasescu and B. Torrecillas applied results on colocalizations for general Grothendieck categories to study colocalizing full subcategories of the category of right C-comodules of a given coalgebra C over a base field.

- Several papers addressing the colocalization of coalgebras appeared recently (e.g. [G-TNT2007] , [JMNR2006], [JMN2007], [Sim2007]). The methods applied in most of these papers depend heavily of the assumption that the ground ring is a base field.

- In [Wij2006], I. Wijayanti applied techniques of colocalization in Wisbauer categories of type $\sigma[M]$ to categories of comodules $\mathbb{M}^C \simeq \sigma[_{C^*}C]$ of a coalgebra C over a commutative ring R with $_RC$ locally projective. This approach will be followed in [Abu-d] to handle the colocalization in categories of comodules over corings.

[3] In his review # MR1655467 of this paper, M. Takeuchi notices there are several gaps in this paper and that several results could be incorrect.

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Index