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A Mass Function Based on Correlation Coefficient and Its Application

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Abstract A discrete distribution involving product of two gamma functions has been proposed. It arises naturally in connection with the distribution of sample variances and correlation coefficient based on a bivariate normal population. The first four raw moments of the distribution, corrected moments, coefficient of skewness and kurtosis have been derived. Some illustrations have been provided to show how the product moments of sample variances and correlation can be derived by exploiting the new distribution. These moments are important for correlation analysis, covariance analysis, intra-sire regression and inference in bivariate normal population.

AMS Mathematical Subject Classification: 60E05, 60E10

Key Words and Phrases: Wishart distribution, distribution of the correlation coefficient, product moments

1. Introduction

Fisher (1915) derived the distribution of sum of squares and mean-centered sum of products to study the distribution of correlation coefficient from a normal sample. Let $X_1, X_2, \cdots, X_N \ (N > 2)$ be a two-dimensional independent random vectors where $X_j = (X_{1j}, X_{2j})'$, $j = 1, 2, \cdots, N$ is distributed as bivariate normal distribution denoted by $N_2(\theta, \Sigma)$ with $\theta = (\theta_1, \theta_2)'$ and a 2×2 covariance matrix $\Sigma = (\sigma_{ik})$, $i = 1, 2;\ k = 1, 2$. The true correlation coefficient (ρ) between X_1 and X_2 is given by $\sigma_{12} = \rho\sigma_1\sigma_2$ where $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2$. The sample mean-centered sums of squares and sums of products are given by $a_{ii} = mS_i^2$, $m = N - 1, (i = 1, 2)$ and $a_{12} = mRS_1S_2$ respectively.

The quantity R is the sample product moment correlation coefficient. The distribution of a_{11}, a_{22} and a_{12} was derived by Fisher (1915) and may be called the bivariate Wishart distribution after Wishart (1928) who obtained the distribution of p-variate Wishart matrix as the joint distribution of sample variances and covariances from multivariate normal population. Obviously a_{11} / σ_{11} has a chi-square distribution with m degrees of freedom.

It may be mentioned that the joint distribution of $U = mS_1^2 / \sigma_1^2$ and $V = mS_2^2 / \sigma_2^2$ is called the bivariate chi-square distribution (Joarder, 2007).

In this paper we introduce a discrete distribution based on the density identity of the product moment correlation coefficient. The distribution is then used to derive product moments of sample variances and coefficient of correlation.

In what follows, for any nonnegative integer k, we will use the notation

$$k^{\{a\}} = k(k-1)\cdots(k-a+1).. \tag{1.1}$$

The duplication formula for gamma function is given by

$$\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{1.2}$$

Multiplying both sides by $2z$ we have

$$(2z)!\sqrt{\pi} = 2^{2z} z!\Gamma\left(z+\tfrac{1}{2}\right) \tag{1.3}$$

The hypergeometric function is defined by

$$\begin{aligned} &{}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)}{\Gamma(a_1)}\frac{\Gamma(a_2+k)}{\Gamma(a_2)}\cdots\frac{\Gamma(a_p+k)}{\Gamma(a_p)}\left(\frac{\Gamma(b_1+k)}{\Gamma(b_1)}\frac{\Gamma(b_2+k)}{\Gamma(b_2)}\cdots\frac{\Gamma(b_q+k)}{\Gamma(b_q)}\right)^{-1}\frac{z^k}{k!}. \end{aligned} \tag{1.4}$$

In particular,

$${}_1F_0(a;z) = (1-z)^{-a}. \tag{1.5}$$

Note that ${}_2F_1(a,b;c;z)$ can be transformed as

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \tag{1.6}$$

(Gradshtyen and Ryzhik, 1992, 1069).

In Section 2, we derive a mass function from the density function of the product moment correlation coefficient. In Section 3, we study some properties of the distribution. In section 4, we demonstrate how some of the product moments of the bivariate Wishart distribution, the bivariate chi-square distribution and the product moment correlation coefficient distribution can be derived through the new mass function. These product moments have found applications in statistical inference for parameters of bivariate normal distribution (Sunthornworasiri, Tienswan and Sinha, 2006), intra-sire regression (Prabhakaran, Mahajan and Uma, 1991) and radar systems (Lawson and Uhlenbeck, 1950).

2. The Probability Mass Function

The density function of the bivariate Wishart distribution was originally derived by Fisher (1915). The density function $h(r)$ of R can be obtained from the Wishart distribution (see e.g. Anderson 2203, 123). The density identity is based on the following equation:

$$\int_{-1}^{1} h(r)dr = 1$$

Theorem 2.1 Let $\gamma_{k,m} = \frac{1+(-1)^k}{2} \frac{2^k}{k!}\Gamma\left(\frac{k+m}{2}\right)\Gamma\left(\frac{k+1}{2}\right)$. Then the density identity based on the distribution of product is given by

$$\sum_{k=0}^{\infty} \rho^k \gamma_{k,m} = \frac{\sqrt{\pi}\ \Gamma(m/2)}{(1-\rho^2)^{m/2}}, \quad m > 2, \ -1 < \rho < 1. \tag{2.2}$$

Proof. The distribution of the product moment correlation is given by

$$h(r) = \frac{2^{m-2}(1-\rho^2)^{m/2}}{\pi\ \Gamma(m-1)}(1-r^2)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k!}\ \Gamma^2\left(\frac{m+k}{2}\right)$$

(Anderson, 1984, 113). Then the density identity is given by

$$\int_{-1}^{1}\frac{2^{m-2}(1-\rho^2)^{m/2}}{\pi\ \Gamma(m-1)}\ (1-r^2)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k!}\ \Gamma^2\left(\frac{m+k}{2}\right)dr = 1.$$

By integrating the density function with the substitution $r^2 = y$ we have

$$\frac{2^{m-2}(1-\rho^2)^{m/2}}{\pi\ \Gamma(m-1)}\Gamma\left(\frac{m-1}{2}\right)\sum_{k=0}^{\infty}\frac{1+(-1)^k}{2}\frac{(2\rho)^k}{k!}\ \Gamma\left(\frac{m+k}{2}\right)\Gamma\left(\frac{k+1}{2}\right) = 1$$

which can be rewritten as

$$\sum_{k=0}^{\infty}\frac{1+(-1)^k}{2}\frac{(2\rho)^k}{k!}\ \Gamma\left(\frac{m+k}{2}\right)\Gamma\left(\frac{k+1}{2}\right) = \frac{\pi\ \Gamma(m-1)}{2^{m-2}(1-\rho^2)^{m/2}\Gamma\left(\frac{m-1}{2}\right)}.$$

By the use of duplication formula of gamma function (1.2), we have

$$\sum_{k=0}^{\infty}\frac{1+(-1)^k}{2}\frac{(2\rho r)^k}{k!}\ \Gamma\left(\frac{m+k}{2}\right)\Gamma\left(\frac{k+1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma(m/2)}{(1-\rho^2)^{m/2}}$$

Note that there is a misprint in the definition of $\gamma_{k,m}$ in Joarder (2006). We define the mass function of the distribution as

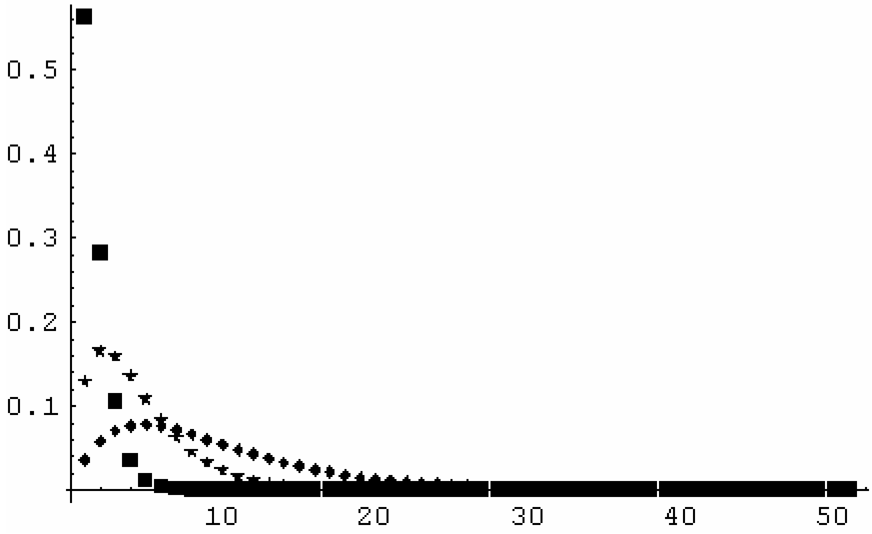
$$f(k;m,\rho) = \frac{(1-\rho^2)^{m/2}}{2\sqrt{\pi}\ \Gamma(m/2)}[1+(-1)^k]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+m}{2}\right)\Gamma\left(\frac{k+1}{2}\right), \quad k = 0,1,2,\cdots \tag{2.3}$$

$m > 2, \ -1 < \rho < 1$, which can be simply written as

$$f(k;m,\rho)=\frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}\,\Gamma(m/2)}\rho^k\gamma_{k,m}\ ,\ k=0,1,2,\cdots$$

$m>2,\ -1<\rho<1$, where $\gamma_{k,m}$ is defined in Theorem 2.1.

A graph of the probability mass function of k is provided in Figure 1 below for various values of ρ and when m = 4.



ρ = 0.90 (*), ρ = 0.80 (◆), and ρ = 0.50 (■)

Figure 1. Probability mass function of k when m =4 at different values of ρ.

3. Moments of the *K*-Distribution

Theorem 3.1 The first four factorial moments are given by

(i) $E(K)=m\rho^2(1-\rho^2)^{-1}$,

(ii) $E\left(K^{\{2\}}\right)=[m(m+1)\rho^4+m\rho^2](1-\rho^2)^{-2}$

(iii) $E\left(K^{\{3\}}\right)=[m(m+1)(m+2)\rho^6+3m(m+2)\rho^4](1-\rho^2)^{-3}$,

(iv) $E\left(K^{\{4\}}\right)=\left[m(m+1)(m+2)(m+3)\rho^8+6m(m+2)(m+3)\rho^6+3m(m+2)\rho^4\right](1-\rho^2)^{-4}$.

Proof. The a-th factorial moment of K is given by

$$E(K^{\{a\}})=\frac{(1-\rho^2)^{m/2}}{2\sqrt{\pi}\,\Gamma(m/2)}\sum_{k=0}^{\infty}k^{\{a\}}[1+(-1)^k]\frac{2^k\rho^k}{k!}\Gamma\left(\frac{k+m}{2}\right)\Gamma\left(\frac{k+1}{2}\right),$$

which can be written as

$$E(K^{\{a\}})=\frac{(1-\rho^2)^{m/2}}{2\sqrt{\pi}\,\Gamma(m/2)}\sum_{k=0}^{\infty}(2k)^{\{a\}}\frac{2^{2k}\rho^{2k}}{(2k)!}\Gamma\left(\frac{2k+m}{2}\right)\Gamma\left(\frac{2k+1}{2}\right).$$

Hence by (1.3), we have

$$E(K^{\{a\}}) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \sum_{k=0}^{\infty} (2k)^{\{a\}} \frac{\rho^{2k}}{k!} \Gamma\left(k + \frac{m}{2}\right). \tag{3.1}$$

From (3.1), the first factorial moment of K-distribution is given by

$$\begin{aligned} E(K^{\{1\}}) &= \frac{2(1-\rho^2)^{m/2}}{\Gamma(m/2)} \sum_{k=0}^{\infty} k \frac{\rho^{2k}}{k[(k-1)!]} \Gamma\left(k + \frac{m}{2}\right) \\ &= \frac{2(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[\rho^2 \sum_{j=0}^{\infty} \frac{\rho^{2j}}{j!} \Gamma\left(j + 1 + \frac{m}{2}\right)\right] \\ &= \frac{2(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[\rho^2 \Gamma\left(1 + \frac{m}{2}\right) {}_1F_0\left(1 + \frac{m}{2}; \rho^2\right)\right] \end{aligned}$$

where we have used the generalized hypergeometric function defined by (1.4). Hence by (1.5), we have $E(K^{\{1\}}) = m\rho^2(1-\rho^2)^{-1}$.

Since $(2k)^{\{2\}} = 4k^{\{2\}} + 2k$, the second factorial moment is given by

$$E(K^{\{2\}}) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \sum_{k=0}^{\infty} [4k^{\{2\}} + 2k] \frac{\rho^{2k}}{k!} \Gamma\left(k + \frac{m}{2}\right),$$

which, by virtue of (1.4), can be simplified as

$$\begin{aligned} E(K^{\{2\}}) &= \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[4\rho^4 \sum_{j=0}^{\infty} \frac{\rho^{2j}}{j!} \Gamma\left(j + 2 + \frac{m}{2}\right)\right] \\ &\quad + \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[2\rho^2 \sum_{j=0}^{\infty} \frac{\rho^{2j}}{j!} \Gamma\left(j + 1 + \frac{m}{2}\right)\right] \\ &= \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[4\rho^4 {}_1F_0\left(2 + \frac{m}{2}; \rho^2\right) \Gamma\left(2 + \frac{m}{2}\right)\right] \\ &\quad + \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[2\rho^2 {}_1F_0\left(1 + \frac{m}{2}; \rho^2\right) \Gamma\left(1 + \frac{m}{2}\right)\right]. \end{aligned}$$

Hence by virtue of (1.5), we have

$$E(K^{\{2\}}) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[4\rho^4 \left(1 + \frac{m}{2}\right) \frac{m}{2} (1-\rho^2)^{-2-(m/2)}\right] + \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \left[2\rho^2 \frac{m}{2} (1-\rho^2)^{-1-(m/2)}\right],$$

which simplifies to what we have in (ii) of the theorem.

Since $(2k)^{\{3\}}=8k^{\{3\}}+12k^{\{2\}}$, from (3.1), the third factorial moment of K-distribution is given by

$$\begin{aligned}E(K^{\{3\}})&=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{k=0}^{\infty}[8k^{\{3\}}+12k^{\{2\}}]\frac{\rho^{2k}}{k!}\Gamma\left(k+\frac{m}{2}\right)\\&=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{j=0}^{\infty}8\frac{\rho^{2(j+3)}}{j!}\Gamma\left(j+3+\frac{m}{2}\right)\\&+\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{j=0}^{\infty}12\frac{\rho^{2(j+2)}}{j!}\Gamma\left(j+2+\frac{m}{2}\right)\\&=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}8\rho^{2\times3}{}_1F_0\left(3+\frac{m}{2};\rho^2\right)\Gamma\left(3+\frac{m}{2}\right)\\&+\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}12\rho^{2\times2}{}_1F_0\left(2+\frac{m}{2};\rho^2\right)\Gamma\left(2+\frac{m}{2}\right),\end{aligned}$$

where we have used the generalized hypergeometric function defined by (1.4). Hence, by (1.5), we have

$$\begin{aligned}E(K^{\{3\}})&=\frac{8}{2^3}m(m+2)(m+4)(1-\rho^2)^{m/2}\rho^{2\times3}(1-\rho^2)^{-3-(m/2)}\\&+\frac{12}{2^2}m(m+2)(1-\rho^2)^{m/2}\rho^{2\times2}(1-\rho^2)^{-2-(m/2)}\\&=\frac{8}{2^3}m(m+2)(m+4)\rho^{2\times3}(1-\rho^2)^{-3}+\frac{12}{2^2}m(m+2)\rho^{2\times2}(1-\rho^2)^{-2},\end{aligned}$$

which simplifies to (iii) of the theorem.

Since $(2k)^{\{4\}}=16k^4-48k^3+44k^2-12k=16k^{\{4\}}+48k^{\{3\}}+12k^{\{2\}}$, from (3.1), we have the fourth factorial moment of the K-distribution

$$\begin{aligned}E(K^{\{4\}})&=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\sum_{k=0}^{\infty}[16k^{\{4\}}+48k^{\{3\}}+12k^{\{2\}}]\frac{\rho^{2k}}{k!}\Gamma\left(k+\frac{m}{2}\right)\\&=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(4)}\sum_{j=0}^{\infty}\frac{16\rho^{2j}}{j!}\Gamma\left(j+4+\frac{m}{2}\right)\\&+\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(3)}\sum_{j=0}^{\infty}\frac{48\rho^{2(j)}}{j!}\Gamma\left(j+3+\frac{m}{2}\right)\\&+\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(2)}\sum_{j=0}^{\infty}\frac{12\rho^{2(j)}}{j!}\Gamma\left(j+2+\frac{m}{2}\right),\end{aligned}$$

Then by virtue of (1.4), we have

$$E(K^{\{4\}}) = 16\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(4)}\,{}_1F_0\left(4+\frac{m}{2};\rho^2\right)\Gamma\left(4+\frac{m}{2}\right)$$
$$+48\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(3)}\,{}_1F_0\left(3+\frac{m}{2};\rho^2\right)\Gamma\left(3+\frac{m}{2}\right)$$
$$+12\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}\rho^{2(2)}\,{}_1F_0\left(2+\frac{m}{2};\rho^2\right)\Gamma\left(2+\frac{m}{2}\right).$$

Hence by virtue of (1.5), we have

$$E(K^{\{4\}}) = 16\frac{\Gamma\left(4+\frac{m}{2}\right)}{\Gamma(m/2)}(1-\rho^2)^{m/2}\rho^{2(4)}(1-\rho^2)^{-4-(m/2)}$$
$$+48\frac{\Gamma\left(3+\frac{m}{2}\right)}{\Gamma(m/2)}(1-\rho^2)^{m/2}\rho^{2(3)}(1-\rho^2)^{-3-(m/2)}$$
$$+12\frac{\Gamma\left(2+\frac{m}{2}\right)}{\Gamma(m/2)}(1-\rho^2)^{m/2}\rho^{2(2)}(1-\rho^2)^{-2-(m/2)},$$

which simplifies to what we have in (iv) of the theorem.

Theorem 3.2 The first four moments of the distribution are given by

$(i)\ E(K) = m\rho^2(1-\rho^2)^{-1}$,
$(ii)\ E(K^2) = (m^2\rho^4 + 2m\rho^2)(1-\rho^2)^{-2}$,
$(iii)\ E(K^3) = [(4m\rho^2 + (6m^2+4m)\rho^4 + m^3\rho^6](1-\rho^2)^{-3}$,
$(iv)\ E(K^4) = \left[m^4\rho^8 + (12m^3+16m^2+8m)\rho^6 + (28m^2+32m)\rho^4 + 8m\rho^2\right](1-\rho^2)^{-4}$.

Proof. Since

$K^2 = K^{\{2\}} + K$,
$K^3 = K^{\{3\}} + 3K^{\{2\}} + K$,
$K^4 = K^{\{4\}} + 6K^{\{3\}} + 7K^{\{2\}} + K$,

the moments follow from Theorem 3.1.

Theorem 3.3 The mean corrected moment of order 2, 3 and 4 of the K-Distribution are respectively given by

$$\mu_2 = \frac{2m\rho^2}{(1-\rho^2)^2},$$

$$\mu_3 = \frac{4m\rho^2(1+\rho^2)}{(1-\rho^2)^3} \text{ and}$$

$$\mu_4 = \frac{4m\rho^2[2+(3m+8)\rho^2+2\rho^4]}{(1-\rho^2)^4}.$$

Proof. The mean corrected moments are given by

$$\mu_a = E(K-\mu)^a, \quad a = 1, 2, \cdots,$$

That is, the second, the third and the fourth order mean corrected moments are

$$\mu_2 = E(K^2) - \mu^2,$$
$$\mu_3 = E(K^3) - 3E(K^2)\mu + 2\mu^3,$$
$$\mu_4 = E(K^4) - 4E(K^3)\mu + 6E(K^2)\mu^2 - 3\mu^4.$$

Routine algebraic simplifications lead to the theorem.

The coefficient of skewness and kurtosis of the K-Distribution are given by the following moment ratios:

$$\alpha_i(K) = \frac{\mu_i}{\mu_2^{i/2}}, \; i = 3, 4.$$

The coefficient of skewness and kurtosis of the distribution simplify to

$$\alpha_3(K) = \frac{\sqrt{2}(1+\rho^2)}{\sqrt{m}\,\rho} \quad \text{and}$$
$$\alpha_4(K) = \frac{2+(3m+8)\rho^2+2\rho^4}{m\rho^2}.$$

Note that as $m \to \infty, \; \alpha_3(K) \to 0$ and $m \to \infty, \; \alpha_4(K) \to 3$ coinciding with that of the univariate normal distribution.

4. Some Applications of the Distribution

The mass function is used to derive the product moments of the bivariate Wishart distribution, the bivariate chi-square distribution and the product moment correlation distribution. Product moments of bivariate Wishart distribution were derived by Joarder (2006). It has been noticed that there are some mistakes in the paper which can be corrected as follows:

There should be a term $[1+(-1)^{k+l}]$ inserted at the beginning of the summand in Theorem 2.1 (Joarder, 2006), in the line 3 from the bottom of page 236 and in the line 2 of page 237. The heading of Section 3 should end with $\mu'(l_1,l_2,l;\rho)$. Moreover, the quantity $b_{k,m}$ in Theorem 3.1 (Joarder, 2006) should be defined as $b_{k,m}=[1+(-1)^{k+l}]\frac{2^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+m}{2}\right)$ for any integer l. The rest of the Theorem 3.1 remains intact. For clarity, we correct the Thorem 2.1 (Joarder, 2006) and label it below as Theorem 4.1:

Theorem 4.1 For finite $a, b,$ and c, the product moments $E(S_1^{2a}S_2^{2b}R^c)$ denoted by $\mu'(a,b,c;\rho)$ are given by

$$\mu'(a,b,c;\rho)=\frac{2^{a+b-1}(1-\rho^2)^{a+b+(m/2)}}{m^{a+b}\Gamma(m/2)\sqrt{\pi}}\sigma_1^{2a}\sigma_2^{2b}$$

$$\times\sum_{k=0}^{\infty}[1+(-1)^{k+c}]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+m}{2}+a\right)\Gamma\left(\frac{k+m}{2}+b\right)\frac{\Gamma\left(\frac{k+1+c}{2}\right)}{\Gamma\left(\frac{k+m+c}{2}\right)}, \tag{4.1}$$

where $m>2,\ \sigma_1>0,\ \sigma_2>0,$ and $-1<\rho<1$.

Now we demonstrate how the moments of the K-Distribution can be exploited to derive closed form expressions of the above product moment.

(i) To derive $E(S_1^2)=\mu'(1,0,0;\rho)$, put $a=1,\ b=0$ and $c=0$. It can be checked from (4.1) that

$$\mu'(1,0,0;\rho)=\frac{1}{m}(1-\rho^2)\sigma_1^2\ [E(K)+m]$$

where K has the mass function given by (2.4). By the use of Theorem 3.1(i), we have $\mu'(1,0,0;\rho)=\sigma_1^2$.

(ii) To calculate $E(S_1^2S_2^2)=\mu'(1,1,0;\rho)$, put $a=1,\ b=1$ and $c=0$. The resulting expression can further be simplified to

$$\mu'(1,1,0;\rho)=\frac{(1-\rho^2)^2\sigma_1^2\sigma_2^2}{m^2}\frac{(1-\rho^2)^{(m/2)}}{2\sqrt{\pi}\Gamma(m/2)}$$

$$\times\sum_{k=0}^{\infty}[1+(-1)^k]\frac{(2\rho)^k}{k!}(k+m)^2\Gamma\left(\frac{k+m}{2}\right)\Gamma\left(\frac{k+1}{2}\right).$$

Since K has the mass function by (2.4), we have

$$\mu'(1,1,0;\rho)=\frac{(1-\rho^2)^2\sigma_1^2\sigma_2^2}{m^2}E(K+m)^2.$$

By the use of Theorem 3.1(i) and (ii) in $E(K+m)^2 = E(K^2)+2mE(K)+m^2$, we have

$$\mu'(1,1,0;\rho)=\frac{1}{m}(\sigma_1\sigma_2)^2(m+2\rho^2).$$

The product moment $E(UV)$ of the bivariate chi-square distribution can be derived from the above since $U=mS_1^2/\sigma_1^2$ and $V=mS_2^2/\sigma_2^2$.

(iii) To calculate $E(S_1^4S_2^4)=\mu'(2,2,0;\rho)$, put $a=2$, $b=2$ and $c=0$. The resulting expression can further be simplified to

$$\mu'(2,2,0;\rho)=\frac{m+2}{m^3}(1-\rho^2)^2\sigma_1^4\sigma_2^4\sum_{k=0}^{\infty}(k+m+2)(k+m)\frac{(1-\rho^2)^{(m+4)/2}}{\sqrt{\pi}\Gamma(\frac{m+4}{2})}\gamma_{k,m+4}.$$

That is

$$\mu'(2,2,0;\rho)=\frac{m+2}{m^3}(1-\rho^2)^2\sigma_1^4\sigma_2^4E[(K+m+2)(K+m)]$$

where K has the mass function given by (2.4). By the use of Theorem 3.1(i) and (ii) in $E[(K+m+2)(K+m)]$, we have

$$\mu'(2,2,0;\rho)=\frac{m+2}{m^3}(1-\rho^2)^2\sigma_1^4\sigma_2^4[E(K^2)+(2m+2)E(K)+m^2]$$

$$=\frac{m+2}{m^3}[8\rho^4+8(m+2)\rho^2+m(m+2)](\sigma_1\sigma_2)^4.$$

The product moment $E(U^2V^2)$ of bivariate chi-square distribution can be derived from the above since $U=mS_1^2/\sigma_1^2$ and $V=mS_2^2/\sigma_2^2$.

(iv) To calculate $\mu'(2,2,2;\rho)$, put $a=2$, $b=2$ and $c=2$. The resulting expression can further be simplified to

$$\mu'(2,2,2;\rho)=\frac{m+2}{m^3}(1-\rho^2)^2\sigma_1^4\sigma_2^4\sum_{k=0}^{\infty}(k+m+2)(k+1)\frac{(1-\rho^2)^{(m+4)/2}}{\sqrt{\pi}\Gamma((m+4)/2))}\gamma_{k,m+4}.$$

Since K has the mass function given by (2.4), we have

$$\mu'(2,2,2;\rho)=\frac{m+2}{m^3}(1-\rho^2)^2\sigma_1^4\sigma_2^4\ E[(K+m+2)(K+1)].$$

Then by the use of Theorem 3.1, we have

$$E[(K+m+2)(K+1)]$$
$$=E(K^2)+(m+3)E(K)+(m+2)$$
$$=[(m+4)^2\rho^4+2(m+4)\rho^2](1-\rho^2)^{-2}+(m+3)(m+4)\rho^2(1-\rho^2)^{-1}+(m+2),$$

so that

$$\mu'(2,2,2;\rho)=\frac{m+2}{m^3}\sigma_1^4\sigma_2^4[(2m+6)\rho^4+(m^2+7m+16)\rho^2+(m+2)]$$

(cf. Joarder , 2006).

(v) To derive $E(R)$, we proceed as follows:

$$\mu'(0,0,1;\rho)=\frac{(1-\rho^2)^{m/2}}{2\Gamma(m/2)\sqrt{\pi}}\sum_{k=0}^{\infty}[1+(-1)^{k+1}]\frac{(2\rho)^k}{k!}\Gamma^2\left(\frac{k+m}{2}\right)\frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+m+1}{2}\right)}.$$

By the use of (1.3), it can be checked that

$$\mu'(0,0,1;\rho)=\frac{(1-\rho^2)^{m/2}\rho}{\Gamma(m/2)}\sum_{k=0}^{\infty}\frac{\rho^{2k}}{k!}\Gamma^2\left(k+\frac{m+1}{2}\right)\frac{1}{\Gamma\left(k+\frac{m+2}{2}\right)}.$$

Then by using (1.4) and (1.6), we have

$$\mu'(0,0,1;\rho)=\frac{(1-\rho^2)^{m/2}\rho}{\Gamma(m/2)}\,{}_2F_1\left(\frac{m+1}{2},\frac{m+1}{2};\frac{m+2}{2};\rho^2\right)\frac{\Gamma^2\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$=2(1-\rho^2)^{-m/2}\rho\,\frac{\Gamma^2\left(\frac{m+1}{2}\right)}{m\Gamma^2\left(\frac{m}{2}\right)}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{m+2}{2};\rho^2\right),$$

which is the same as obtained by Ghosh (1966).

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