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Suliman Al-Homidan and Mohammad M. Alshahrani

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: c-math@kfupm.edu.sa

Positive Definite Hankel Matrices using Cholesky Factorization

Suliman Al-Homidan* Mohammad. M. Alshahrani†

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Abstract

Real positive definite Hankel matrices arise in many important applications. They have spectral condition numbers which exponentially increase with their orders. This paper gives a structural algorithm for finding positive definite Hankel matrices using the Cholesky factorization. We also compute it for orders less than or equal to 30 and compare our result with earlier results.

1 Introduction

An $n \times n$ Hankel matrix H is one with constant anti-diagonal elements. It has the following structure

$$H = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_n & h_{n+1} \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ h_n & h_{n+1} & \cdots & \cdots & \cdots & h_{2n-1} \end{bmatrix}.$$

The general complex-valued elements can be well-conditioned while, real positive definite Hankel matrices are known to be very ill-conditioned since their spectral condition numbers increase exponentially with n. The condition number of a positive definite Hankel matrix is bounded below by $3 \cdot 2^{(n-6)}$ which is very large even for relatively small orders [10]. Beckermann [6] gave a better bound $\gamma^{n-1}/(16n)$ with $\gamma \approx 3.210$. Here, we try to force the condition number of the constructed matrix to be as small as possible to reach the latter, or at least the former, experimentally.

*Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Saudia Arabia. Email homidan@kfupm.edu.sa

†Department of Mathematics, Dammam Teachers' College, P.O. Box 14262, Dammam 31424, Saudia Arabia Email mmogib@@awalnet.net.sa

Several other authors have developed algorithms for the factorization or inversion of Hankel matrices [7, 9, 11]. In some application areas such as digital signal processing and control theory it is required to compute the closest, in some sense, positive semidefinite Hankel matrix with no restriction on its rank, to a given data covariance matrix computed from a data sequence. The problem of preserving the structure while approximating low rank also arises in many applications (see [1, 2, 3]). When using the interior point method in solving this problem, it is important for some algorithms to start from within the cone of positive semidefinite Hankel matrices, i.e., the initial point must be a positive definite Hankel matrix [4, 5].

In the following, we construct a positive definite Hankel matrix. The Cholesky factorization theorem [8] states that for any $n \times n$ real symmetric positive definite matrix A, there exists a unique lower triangular matrix L with positive diagonal enteries such that $A = LL^T$. This provides a means to construct a symmetric positive definite matrix by building up a lower triangular L with positive diagonal enteries. The martix LL^T is certainly symmetric positive definite but not Hankel. To impose the Hankel structure, we force the anti-diagonal elements of LL^T to be equal. This produces a system of nonlinear equations with infinitely many solutions depending on the diagonal elements of L and the elements of the first diagonal (the diagonal just below the main diagonal). A closed formula for finding the elements of L that makes LL^T positive definite Hankel matrix is introduced. An algorithm is proposed and implemented using MatLab. Finally, some numerical comparisons and experiments are presented to show how different choices of starting elements affect the smallest eigenvalue and subsequently the condition number of the resulting Hankel matrix.

2 Generating Formula

Let us consider the Cholesky factorization $H = LL^T$ and write down the product. The resulting matrix $L_{n\times n}L^T_{n\times n}$ is as follows:

$$\begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & \cdots & l_{11}l_{n1} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & \cdots & l_{21}l_{n1} + l_{22}l_{n2} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 & \cdots & l_{31}l_{n1} + l_{32}l_{n2} + l_{33}l_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{11}l_{n1} & l_{21}l_{n1} + l_{22}l_{n2} & l_{31}l_{n1} + l_{32}l_{n2} + l_{33}l_{n3} & \cdots & l_{n1}^2 + l_{n2}^2 + \cdots + l_{nn}^2 \end{bmatrix}.$$

This is a dense matrix. Now, we show how to find the elements of L which make LL^T positive definite Hankel.

- First, we assign values to all the elements l_{ii} and $l_{i\ i-1}$.
- Each computed element l_{ij} has the form $(S - D)/l_{jj}$.
- S is composed of summing products of pairs of elements $l_{hs}l_{ks}$, where $h = [n/2]$ ($[x]$ the integer smaller than or equal to x) and $k = n - h$ and the index s runs from 1 to h. So S is related to $i + j$.
- D is also a sum of products of pairs $l_{jr}l_{ir}$, where r runs from 1 to $j - 1$. So D is related to j only.
- The last two elements $l_{n\ n-1}$ and l_{nn} are free in the sense that no other elements depend on them.

In light of the above insights, the following two steps construct L such that LL^T is a positive definite Hankel matrix

- Assign values to l_{ii} and $l_{i\ i-1}$.
- Copmute the other elements usig the formula

$$l_{ij} = \frac{\sum_{s=1}^{q} l_{q,s}l_{rs} - \sum_{t=1}^{j-1} l_{it}l_{jt}}{l_{jj}}, \tag{2.1}$$

where $j \le i-2$ and

$$\begin{aligned} q &= \begin{cases} \frac{i+j}{2}, & \text{if } i+j \text{ even} \\ \frac{i+j-1}{2}, & \text{if } i+j \text{ odd} \end{cases} \\ r &= \begin{cases} q, & \text{if } i+j \text{ even} \\ q+1, & \text{if } i+j \text{ odd} . \end{cases} \end{aligned} \tag{2.2}$$

3 Algorithm and Implementation

The following algorithm presents formula (2.1) in a more programmable way. It produces all the elements of L such that LL^T is a positive definite Hankel matrix.

Algorithm 3.1 *(Constructing Positive Difinite Hankel Matrix)*

begin

    for $i := 1$ **to** n
        populate l_{ii} *and* $l_{i\ i-1}$
    for $i := 3$ **to** n
        for $j := 1$ **to** $i-2$
            set $q := [(i+j)/2]$
            if n *is even*
                set $r := q$
            else
                set $r := q+1$
            set $S := 0$
            set $D := 0$
            for $s := 1$ **to** q
                $S = S + l_{qs} l_{rs}$
            for $t := 1$ **to** $j-1$
                $D = D + l_{it} l_{jt}$
            set $l_{ij} := (S - D)/l_{jj}$.

4 Numerical Experiments

The first experiment is to find the highest order of H we can produce. Varah [11] managed to get up to $n = 16$, whereas our program managed to construct a positive definite Hankel matrix H up to $n = 30$ with smallest eigenvalue $e = 5.18 \times 10^{-5}$. With our program, the condition number is very large, almost infinity. Hence, we managed with a higher accuracy to push n to 30.

The remaining experiments will be improving the condition number as much as one can by choosing different values for l_{ii} and $l_{i\ i-1}$ and the required Hankel matrix will be of the order 20. The lower bound mentioned in [3] for $n = 20$ is 49152 while the bound in [1] is 3624.29. We change l_{11} only and hold the other elements constant (with 1 as initial value). Table 1 shows that the condition number improves when l_{11} is changed from 1 to 10, but then gets worse when we go higher.

l_{11}	e	cond(H,2)
1	6.27×10^{-8}	7.55×10^{17}
10	1.65×10^{-7}	4.58×10^{16}
100	1.61×10^{-7}	7.17×10^{16}
1000	1.60×10^{-7}	7.56×10^{16}

Table 1: Changing l_{11}

Now, we change l_{11} and $L(n, n)$ togother and hold the other elements constant. Clearly, from Table 2, the best choice is, $l_{11} = L(n, n) = 10$.

l_{11}	$L(n,n)$	e	cond(H,2)
10	10	4.20×10^{-7}	1.80×10^{16}
100	100	4.09×10^{-7}	2.82×10^{16}
1000	1000	4.07×10^{-7}	2.97×10^{16}

Table 2: Changing l_{11} and l_{nn}

After some testing, the following combination in Table 3 seems to give better results:

a	e	cond(H,2)
[100000 5000 -90 10 100000]	14.14	9.15×10^{14}
[1000000 5000 -90 10 1000000]	14.18	9.11×10^{14}
[10000000 50000 -90 10000 10000000]	1417.8	9.09×10^{14}

Table 3: Changing All

After testing different values of l_{ii} and $l_{i\ i-1}$ with different n, we find that if we normalize all values of the matrix L so that all the elements of L and H are less than or equal to one, then the best condition number can be reached by selecting the elements of $l_{i\ i-1}$ equal to either one or zero. The elements of l_{ii} are selected according to the following (these are approximated values and not exact):

$$l_{11} \approx 0.9,\ l_{22} \approx 0.45,\ l_{33} \approx 0.23,\ l_{44} \approx 0.12, \ldots.$$

In other words, l_{ii} is made smaller by a factor of $f = 0.5$ as i increases. However, when i reaches $n/2$, the factor should be increased from $f = 0.5$ to $f = 1$ when $i = 2n/3$, and then we start increasing values of l_{ii} such that the value of $l_{nn} = 150 * l_{ii}$ when $i = 2n/3$.

We implement this strategy for the above case ($n = 20$) and find a remarkable improvement to the condition number with a small decrease in the smallest eigenvalue as follows:

$$l_{11} \ldots l_{nn} =$$

0.9000 0.4500 0.2300 0.1200 0.0600 0.0300 0.0150 0.0080 0.0040 0.0025 0.0020 0.0018 0.0018 0.0020 0.0030 0.0050 0.0100 0.0200 0.0500 0.2500

and the condition number we find is 7.2776×10^8 which is much better while the smallest eigenvalue is 2.2945×10^{-9}. It is worth mentioning that the condition number we find for $n = 3, \ldots, 16$ is very close to the one found by [11].

5 Conclusion

In this paper, we managed to construct a positive definite Hankel matrix with a smaller eigenvalue greater than zero, and we improved the condition number by a factor of 100 times. However, the produced matrices are extremely ill-conditioned and more work needs to be done to deduce a pattern that improves the results. Also a higher dimension of H is needed. One way to do it is to decrease the tolerance by giving the program more accuracy but this will increase the dimension by one or two since it is clear that the condition number increases exponentially while the smallest eigenvalue also decreases exponentially.

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