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On the Bivariate Dirichlet Distribution

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Abstract Raw and corrected product moments are calculated for bivariate Dirichlet distribution. Then, these are used to derive the second and third order standardized moments which are the kurtosis and skewness of the distribution. Some other characteristics of the distribution are also studied.

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1. Introduction

The product moment of order a and b for two random variables X_1 and X_2 are defined by $\mu'_{a,b} = E(X_1^a X_2^b)$ while the centered product moments (sometimes called central product moments or central mixed moments) are defined by

$$\mu_{a,b} = E\left[(X_1 - E(X_1))^a (X_2 - E(X_2))^b\right].$$

The former moment is often called product moments of order zero or raw product moments. Evidently $\mu'_{a,0} = E(X_1^a)$ is the a-th moment of X_1, and $\mu'_{0,b} = E(X_2^b)$ is the b-th moment of X_2. In case X_1 and X_2 are independent $\mu'_{a,b} = E(X_1^a)E(X_2^b) = \mu'_{a,0}\mu'_{0,b}$. Interested readers may go through Johnson, Kotz and Kemp (1993, 46) or Johnson, Kotz, and Balakrishnan (1997, 3).

The correlation coefficient ρ $(-1 < \rho < 1)$ between X_1 and X_2 is denoted by

$$\rho_{X_1,X_2} = \frac{\mu_{1,1}}{\sqrt{\mu_{2,0}\mu_{0,2}}}. \tag{1.1}$$

Note that $\mu_{2,0} = E(X_1 - E(X_1))^2 = \sigma_{20}$ which is popularly denoted by σ_1^2 while the central product moment $\mu_{1,1} = E[(X_1 - E(X_1))(X_2 - E(X_2))]$, denoted popularly by σ_{12}, is in fact the covariance between X_1 and X_2.

In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of measures for kurtosis and skewness based on Mahalanobis distance. As it is difficult to derive distribution of Mahalanobis distance for many distributions and calculate moments thereof, Joarder (2006) derived Mahalanobis moments in terms of central product moments. Central product moments that are deemed essential are derived with the help of product moments around zero. Mahalanobis moments, or standardized moments, for a bivariate Dirichlet distribution are calculated. It is worth mentioning that the second Mahalanobis moment accounts for kurtosis while the third for skewness. In addition, Shannon entropy and local dependence functions for the bivariate Dirichlet distribution are also discussed. It is worth reporting that the calculations for the central moments and the standardized moments are formidable.

2. The Bivariate Dirichlet Distribution

Kotz, Balakrishnan, and Johnson (2000, 487) discussed the multivariate Dirichlet distribution. The probability density function of the multivariate Dirichlet distribution is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{\Gamma(\sum_{i=0}^{k} \theta_i)}{\prod_{i=0}^{k} \Gamma(\theta_i)} (1 - \sum_{i=1}^{k} x_i)^{\theta_0 - 1} \prod_{i=1}^{k} x_i^{\theta_j - 1}, \tag{2.1}$$

where $\theta_0 > 0,\ \theta_i > 0,\ x_i \geq 0,\ i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} x_i \leq 1$.

So, the probability density function of the bivariate Dirichlet distribution is given by

$$f(x_1, x_2) = \frac{\Gamma(m + n + p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x_1^{m-1} x_2^{n-1} (1 - x_1 - x_2)^{p-1}, \tag{2.2}$$

where $m, n, p > 0,\ x_i \geq 0,\ i = 1, 2$ and $x_1 + x_2 \leq 1$.

In what follows we will rather use $X_1 = X$ and $X_2 = Y$ to avoid all confusion of a trivial nature.

For the above bivariate Dirichlet distribution, the marginal probability density functions are as follows:

(i) $X \sim Beta(m, n + p)$,

(ii) $Y \sim Beta(n, m + p)$.

Note that if $m = n$, then we have a special case where the marginal probability density functions are identical. Further, if $m = n = p$, then we have a more specialized version of the marginal density functions with each random variable is identically distributed as $Beta(m, 2m)$.

In what follows we will define

$$\mu_{a,b} = E[(X - \xi)^a (Y - \theta)^b] \tag{2.3}$$

where $\xi = E(X)$ and $\theta = E(Y)$.

3. Product Moments of Bivariate Dirichlet Distribution

For any non-negative integer a, we have Pochhammer factorials defined as

$c_{\{a\}} = c(c + 1)(c + 2)\cdots(c + a - 1)$ and

$c^{\{a\}} = c(c - 1)(c - 2)\cdots(c - a + 1)$.

Also, the $(a, b)^{th}$ raw product moment of X and Y of the bivariate Dirichlet distribution is given by

$$E(X^a Y^b) = \int_0^1 \int_0^1 x^a y^b f(x,y)\, dxdy. \tag{3.1}$$

Lemma 3.1 If X and Y has a joint probability density function defined by equation 2.2 then

(i) the marginal density function of $X \sim Beta(m, n+p)$, has an expected value of

$$E(X^a) = \frac{m_{\{a\}}}{t_{\{a\}}}$$

(ii) the marginal density function of $Y \sim Beta(n, m+p)$, has an expected value of

$$E(Y^b) = \frac{n_{\{b\}}}{t_{\{b\}}}$$

(iii) and the raw product moment of order (a,b) is

$$E(X^a Y^b) = \frac{m_{\{a\}} n_{\{b\}}}{t_{\{a+b\}}}, \qquad \text{where } t = m+n+p.$$

Lemma 3.1 gives rise to some useful raw moments that will be used further in the calculation of centered moments. In particular, some specific raw moments that are needed for calculation of centered moments are given below.

$$E(Y) = \frac{n}{t},$$

$$E(Y^2) = \frac{n(n+1)}{t(t+1)},$$

$$E(XY) = \frac{mn}{t(t+1)},$$

$$E(XY^2) = \frac{mn(n+1)}{t(t+1)(t+2)},$$

$$E(X^2 Y^3) = \frac{m(m+1)n(n+1)(n+2)}{t(t+1)(t+2)(t+3)(t+4)},$$

$$E(X^3 Y^2) = \frac{m(m+1)(m+2)n(n+1)}{t(t+1)(t+2)(t+3)(t+4)},$$

$$E(X^3 Y^3) = \frac{m(m+1)(m+2)(n+2)(n+1)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$

$$E(X^4 Y^2) = \frac{m(m+1)(m+2)(m+3)(n+1)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$

$$E(X^5 Y) = \frac{m(m+1)(m+2)(m+3)(m+4)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$

$$E(XY^5) = \frac{mn(n+1)(n+2)(n+3)(n+4)}{t(t+1)(t+2)(t+3)(t+4)},$$

$$E(X^2 Y^2) = \frac{m(m+1)n(n+1)}{t(t+1)(t+2)(t+3)},$$

$$E(X) = \frac{m}{t}.$$

4. Centered Moments of Bivariate Dirichlet Distribution

Details of the formidably painstaking calculations for the central product moments of the bivariate Dirichlet distribution can be found in Omar and Joarder (2006). Some product moments of order $a+b = 2, 3, 4, 5, 6$ that are needed for calculation of bivariate skewness and kurtosis moments are reproduced below:

$$
\begin{aligned}
\mu_{0,2} &= \frac{(t-n)n}{(t+1)t^2}, \\
\mu_{0,4} &= 3(t-n)n\frac{t^2(n+2)-n(n+6)t+6n^2}{(t+1)(t+2)(t+3)t^4}, \\
\mu_{0,6} &= \frac{5(t-n)n}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6} \\
&\quad \times (t^4(3n^2+26n+24)-2n(3n^2+56n+60)t^3 \\
&\qquad + n^2(3n^2+172n+240)t^2-2n^3(43n+120)t+120n^4),
\end{aligned}
$$

$$
\begin{aligned}
\mu_{1,1} &= -\frac{nm}{(t+1)t^2}, \\
\mu_{1,3} &= n\frac{(n+2)(n+1)(m-n)t^3+3n(n+2)(m-n-1)t^2+3nm(n+6)t-18mn^2}{(t+1)(t+2)(t+3)t^4}, \\
\mu_{1,5} &= \frac{mn}{(t+1)t^2} \\
&\quad \times (5\frac{(t(n+2)-n)n^3}{t^4}+5\frac{(n^2-4(n+1)t)(n+1)n^2}{(t+2)t^3} \\
&\qquad -5\frac{(t(n+3)-3n-3)(n+1)(n+2)n}{(t+2)(t+3)t}+(t-1)\frac{(n+1)(n+2)(n+3)(n+4)}{(t+2)(t+3)(t+4)}),
\end{aligned}
$$

$$
\begin{aligned}
\mu_{2,2} &= mn\frac{t^3-(m+n)t^2+3(2m+2n+mn)t-18mn}{(t+1)(t+2)(t+3)t^4}, \\
\mu_{2,4} &= \frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)} \\
&\quad \times (-2m(n+3)(n+2)(n+1)+\frac{(n+2)(3-16m(n+3)(n+1))}{t} \\
&\qquad + \frac{(-3m(n+2)(10n^2+40n+31)-2n(3n+20))}{t^2} \\
&\qquad + \frac{3n^2(n+40)+2m(9n^2+82n+60)}{t^3}-\frac{n(86n^2+3m(5n^2+172n+160))}{t^4} \\
&\qquad + \frac{10n^2(12n+m(43n+72))}{t^5}-\frac{600mn^3}{t^6}),
\end{aligned}
$$

$$
\begin{aligned}
\mu_{3,3} &= \frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)t} \\
&\quad \times (4-3\frac{10m+10n+3mn}{t}+\frac{(26n^2+m^2(9n+26)+9mn(n+20))}{t^2} \\
&\qquad -3\frac{2n^2(43m+20)+m^2(86n+5n^2+40)}{t^3} \\
&\qquad +10\frac{mn(36m+36n+43mn)}{t^4}-600\frac{m^2n^2}{t^5}).
\end{aligned}
$$

Similar expressions for moments $\mu(b,a) = E[(X-\xi)^b(Y-\theta)^a]$ are provided below.

$$
\begin{aligned}
\mu_{2,0} &= \frac{(t-m)m}{(t+1)t^2}, \\
\mu_{4,0} &= 3(t-m)m\frac{t^2(m+2)+-m(m+6)t+6m^2}{(t+1)(t+2)(t+3)t^4},
\end{aligned}
$$

$$\mu_{6,0} = \frac{5(t-m)m}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$\times ((3m^2+26m+24)t^4 - 2m(3m^2+56m+60)t^3$$
$$+ m^2(3m^2+172m+240)t^2 - 2m^3(43m+120)t + 120m^4),$$

$$\mu_{3,1} = 3mn\frac{-(m+2)t^2 + m(m+6)t - 6m^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{5,1} = \frac{5}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$\times (m^5(n-m)t^5 + m(n(15m^4 - 3m^2 - 26m - 24) - 15m^5)t^4$$
$$+ m^2(n(112m + 6m^2 + 85m^3 + 120) - 85m^4)t^3$$
$$+ m^3(2n(111m^2 - 86m - 120) - 225m^3)t^2$$
$$+ 2m^4(60n(3m+2) - 137m^2)t - 120m^6),$$

$$\mu_{4,2} = \frac{mn}{(t+1)(t+2)(t+3)t^3}$$
$$\times (3n(m(2n+2n^2+1) - 4m^3 - 6m^2 + 2) + 2mn\frac{15n - 36m - 19m^2 + 15n^2 - 12}{t}$$
$$+ mn\frac{36n - 7m^2 + 36n^2}{t^2} - 30\frac{m^3 n}{t^3} - 86m^3\frac{n+1}{(t+4)(t+5)t}$$
$$+ 120m^3\frac{(n+1)}{(t+4)(t+5)t^2} + m\frac{(320n + m120(5n+1) + m^2(163n+3))}{(t+4)(t+5)}$$
$$+ 2\frac{n(31m^3 - 30) - (3n+20)m + 3(25n-1)m^2}{(t+4)(t+5)}t + 3\frac{(m+2)(2m^2 n - n + 1)}{(t+4)(t+5)}t^2).$$

5. Standardized Moments for any Bivariate Distribution

Nadarajah and Mitov (2003) presented an elegant technique for product moments of multivariate random vectors in terms of cumulative distribution function or survival function. It appears that if the cumulative distribution function or the survival function has a closed form, their technique works well. For Marshall-Olkin bivariate exponential distribution with survival function

$$P(X \geq x, Y \geq y) = \begin{cases} e^{-x-(1+\lambda)y}, & 0 \leq x \leq y \\ e^{-y-(1+\lambda)x}, & 0 \leq y \leq x \end{cases}, \quad (5.1)$$

Kotz, Balakrishnan and Johnson (2000, 82) mentioned that the coefficient of kurtosis is given by

$$\beta_2 = 2(1+\rho)^{-3}(3\rho^4 + 9\rho^3 + 15\rho^2 + 12\rho + 4)$$

where the correlation coefficient ρ is given by $(\lambda+2)\rho = \lambda.$ They also mentioned that in case $\rho = 0$, the components X and Y become independent, in which case $\beta_2 = 8$ (which is the same as that of the bivariate normal distribution). Interested readers may also go through Nadarajah and Mitov (2003) for a useful formula for product moments for univariate distributions. They also calculated raw product moments of general order from which it is possible to calculate skewness and kurtosis of the distribution in (5.1).

For a bivariate random vector $W = \begin{bmatrix} X \\ Y \end{bmatrix}$, with mean vector $\mu = \begin{bmatrix} \xi \\ \theta \end{bmatrix}$ and covariance matrix Σ, the standardized distance is defined by

$$\begin{aligned} Q &= (W-\mu)'\Sigma^{-1}(W-\mu) \\ &= ((X-\xi)(Y-\theta))\Sigma^{-1}((X-\xi)(Y-\theta))' \end{aligned} \tag{5.2}$$

with

$$\Sigma = \begin{bmatrix} \mu_{2,0} & \mu_{1,1} \\ \mu_{1,1} & \mu_{0,2} \end{bmatrix}.$$

The quantity Q is also known to be the generalized distance or Mahalanobis distance (Rencher, 1998, 40). In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of measures for kurtosis and skewness based on Mahalanobis distance.

The coefficient of kurtosis and skewness are $\beta_2 = E(Q^2)$ and $\beta_3 = E(Q^3)$ respectively (Kotz, Balakrishnan, and Johnson, 2000, 77). For a bivariate random vector W with $E(W) = \mu$ and $Cov(W) = E(W-\mu)(W-\mu)' = \Sigma$, we define standardized moments by $\beta_i = E(Q^i)$, $i = 1, 2, \cdots$ where $Q = (W-\mu)'\Sigma^{-1}(W-\mu) = \|\Sigma^{-1/2}(W-\mu)\|^2$.

It is known that for a p-variate normal distribution, $W \sim N_p(\mu, \Sigma)$, the standardized distance $Q = (W-\mu)'\Sigma^{-1}(W-\mu) \sim \chi_p^2$ so that $\beta_1 = E(Q) = p$, $\beta_2 = E(Q^2) = p(p+2)$ and $\beta_3 = E(Q^3) = p(p+2)(p+4)$. That is for a univariate normal distribution, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 15$ and for bivariate normal distribution,

$$\beta_1 = 2, \beta_2 = 8, \beta_3 = 48. \tag{5.3}$$

As it is difficult to derive distribution of Mahalanobis distance for many bivariate distributions and calculate moments thereof, Joarder (2006) derived standardized moments in terms of central product moments just to demonstrate the potential of an alternative way. As discussed in Joarder (2006), $E(Q) = 2$ for any bivariate distribution, including that for the bivariate Dirichlet distribution represented in equation 2.2.

Theorem 5.1 Let $\mu_{a,b}$ be the centered product moment and $(\mu_{2,0}\mu_{0,2})^{-1/2}\mu_{1,1} = \rho$ be the correlation coefficient between X and Y. Then the second and third order Mahalanobis moments are respectively given by

$$\begin{aligned} (i)\quad ((\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2)^2 E(Q^2) &= \mu_{4,0}\mu_{0,2}^2 + \mu_{0,4}\mu_{2,0}^2 + 2(\mu_{2,2})(2\mu_{1,1}^2 + \mu_{0,2}\mu_{2,0}) \\ &\quad - 4(\mu_{1,1})(\mu_{2,0}\mu_{1,3} - \mu_{0,2}\mu_{3,1}), \end{aligned}$$

$$\begin{aligned} (ii)((\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2)^3 E(Q^3) &= \mu_{6,0}\mu_{0,2}^3 + \mu_{0,6}\mu_{2,0}^3 - 6\mu_{1,1}(\mu_{0,2}^2\mu_{5,1} + \mu_{2,0}^2\mu_{1,5}) \\ &\quad + 4\mu_{1,1}^2(3\mu_{0,2}\mu_{4,2} + 3\mu_{2,0}\mu_{2,4} - 2\mu_{1,1}\mu_{3,3}) \\ &\quad + 3\mu_{2,0}\mu_{0,2}(\mu_{0,2}\mu_{4,2} + \mu_{2,0}\mu_{2,4} - 4\mu_{1,1}\mu_{3,3}). \end{aligned}$$

The proof for this theorem can be found in Joarder (2006).

6. Covariance Matrix and Product Moment Correlation Coefficient of Bivariate Dirichlet Distribution

By applying the central moments of order 2, we have the following lemma.

Lemma 6.1 For a bivariate Dirichlet distribution defined in (2.2),

(i) the covariance matrix is given by

$$\Sigma = \frac{1}{(t+1)t^2}\begin{bmatrix} (t-m)m & -nm \\ -nm & (t-n)n \end{bmatrix},$$

(ii) and the product moment correlation coefficient is given by

$$\rho = -\sqrt{\frac{mn}{(n+p)(m+p)}}\,.$$

Note that Lemma 6.1 is a special case of results in Kotz, Balakrishnan & Johnson (2000, 488).

Proof (i) Substituting the variance of each random variable and its covariance, we have the following variance covariance matrix Σ

$$\Sigma = \begin{bmatrix} \mu_{2,0} & \mu_{1,1} \\ \mu_{1,1} & \mu_{0,2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(t-m)m}{(t+1)t^2} & -\frac{nm}{(t+1)t^2} \\ -\frac{nm}{(t+1)t^2} & \frac{(t-n)n}{(t+1)t^2} \end{bmatrix}$$

which can further be simplified as Lemma 6.1(i).

(ii) Substituting the variance of each random variable and their covariance into the product moment correlation coefficient, we have the following result.

$$\begin{aligned}
\rho &= \frac{\mu_{1,1}}{\sqrt{\mu_{2,0}\mu_{0,2}}} \\
&= \frac{\frac{-nm}{(t+1)t^2}}{\sqrt{\frac{1}{(t+1)^2t^4}(t-n)n(t-m)m}} \\
&= \frac{-nm}{\sqrt{\frac{((t+1)t^2)^2}{(t+1)^2t^4}(t-n)n(t-m)m}} \\
&= -\frac{mn}{\sqrt{mn(t-m)(t-n)}} \\
&= -\sqrt{\frac{mn}{(t-m)(t-n)}} \\
&= -\sqrt{\frac{mn}{(m+n+p-m)(m+n+p-n)}}
\end{aligned}$$

which can be simplified to Lemma 6.1(ii).

Some special cases are discussed below.

a) For a special case of a bivariate Dirichlet distribution defined in (2.2) where $m = n$, that is $(t = 2m + p)$ then the covariance matrix is

$$\Sigma = \frac{1}{(t+1)t^2}\begin{bmatrix} (m+p)m & -m^2 \\ -m^2 & (m+p)m \end{bmatrix}$$

and the product moment correlation coefficient is $\rho = -\frac{m}{m+p}$.
These results can be obtained simply by substituting $m = n$ and $t = 2m + p$ into Lemma 6.1. Note that when $m = n$, we have the case discussed in Section 2 where the two bivariate marginal probability density functions are identical.

b) For a special case of a bivariate Dirichlet distribution defined in (2.2) where $m = n = p$, that is $t = 3m$ then the covariance matrix is

$$\Sigma = \frac{m^2}{(t+1)t^2} \times \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and the product moment correlation coefficient $\rho = -\frac{1}{2}$.
These results can be obtained by simply substituting $m = n$ and $t = 2m + p$ into Lemma 6.1. Note that here we have a special case of Lemma 6.1 when the two bivariate marginal pdfs are identical with $m = n = p$.

7. Skewness and Kurtosis of Bivariate Dirichlet Distribution

As mentioned in Section 1, a series of papers by Mardia (1970,1974,1975) considered the properties of measures for kurtosis and skewness based on Mahalanobis distance. Specifically, the coefficient of kurtosis and skewness are $\beta_2 = E(Q^2)$ and $\beta_3 = E(Q^3)$ respectively (Kotz, Balakrishnan, and Johnson, 2000, 77).

We derive standardized moments in terms of centered product moments along the lines of Theorem 5.1 just to demonstrate the potential of an alternative way.

Also as pointed out in Section 5, the Mahalanobis moment $E(Q) = 2$ for any bivariate distribution. In particular, this is true as well for the bivariate Dirichlet distribution and can be easily checked. However, for the higher order Mahalanobis moments, formidable calculations are done. The following theorem provides results for the second and third order Mahalanobis moments.

Theorem 7.1 Let $\mu_{a,b}$ be the centered product moment of the bivariate dirichlet distribution. Then

(i) The second Mahalanobis moment which is the kurtosis of the distribution is

$$
\begin{aligned}
E(Q^2) = {} & 2\frac{k}{nm}t^2 \times ((3m+3n+4mn)t^2 \\
& \qquad - 2((-n^3-3n^2+2n+3)m^2 + n(n^3+3n^2+6n+12)m + 3n^2)t \\
& \qquad + (m^3(-2n^3-6n^2+3) + 2n(21n+6n^2+n^3+24)m^2 \\
& \qquad - 2n^2(7n+3n^2-24)m + 3n^3)) \\
& + 2\frac{k}{t} \times (-3((10n+2n^2+9)m^2 - 2n(n^2+n-15)m + 13n^2)t^2 \\
& \qquad + (12)nm(6m+6n+mn)t - (72)m^2n^2)
\end{aligned}
$$

where $k = \dfrac{(t+1)^2}{t_{\{4\}}p^2}$ and $t_{\{4\}} = t(t+1)(t+2)(t+3)$.

(ii) The third Mahalanobis moment which is the skewness of the distribution is

$$
\begin{aligned}
E(Q^3) = {} & -6\frac{m(t+1)^3t^7}{n^2p^3}\frac{n_{\{4\}}}{t_{\{6\}}} \\
& - 2\frac{(t+1)^3t^6}{m^2n^2p^3t_{\{6\}}} \\
& \times (-3n(2n^3+9n^2+22n+12)m^4 + 3n(3n^4+22n^3+60n^2+62n+24)m^3 \\
& \quad - (9n^5+9n^4+24n^2+74n+60)m^2 - 74mn^2 - 60n^2) \\
& - 6m\frac{(t+1)^3t^5}{n^2p^3t_{\{6\}}} \\
& \times (5n^2m^3 + (11n-2n^2+n^3+6)nm^2 \\
& \quad - (100n+31n^2+26n^3+2n^4+48)nm \\
& \quad + (68n+181n^2+344n^3+164n^4+36n^5+60)) \\
& - 12\frac{(t+1)^3t^5}{m^2np^3t_{\{6\}}} \\
& \times ((134n+12n^2-21n^3-18n^4+3n^5+60)m^2 + 30n^2 + (37n+60)nm) \\
& + 2\frac{(t+1)^3t^4}{m^2p^3t_{\{6\}}} \\
& \times (6(37n+150)nm + 180n^2 \\
& \quad + (1820n+711n^2+387n^3-243n^4+9n^5+1800)m^2) \\
& - 2\frac{(t+1)^3t^4}{n^2p^3t_{\{6\}}} \\
& \times (-15(2n-15)n^2m^4 - 3(104n^2-38n-26n^3+n^4-24)nm^3 \\
& \quad - 3(62n+146n^2+81n^3+208n^4+31n^5+60)m^2 \\
& \quad - (740n-2475n^2-954n^3-456n^4+69n^5+900)mn) \\
& - 2\frac{(t+1)^3t^3}{m^2n^2p^3t_{\{6\}}} \\
& \times (15(n^2-30n+85)n^2m^6 \\
& \quad + (56n-81n^2-2064n^3+1131n^4-57n^5+60)m^5 \\
& \quad + 3(148n-387n^2-534n^3-163n^4+10n^5+240)nm^4 \\
& \quad + (4286n-567n^2+894n^3-891n^4+45n^5+3000)n^2m^3 \\
& \quad - 6(14n^2-23n-192n^3+21n^4-500)n^3m^2 \\
& \quad + 2(37n+360)mn^4 + 60n^5)
\end{aligned}
$$

$$
\begin{aligned}
&- 2m^3 \frac{(t+1)^3 t^2}{np^3 t_{\{6\}}} \\
&\times (75(3n^2 - 34n + 45)mn \\
&\quad - (434n + 4761n^2 - 6384n^3 + 660n^4 + 15n^5 + 180)) \\
&- 6m\frac{(t+1)^3 t^2}{p^3 t_{\{6\}}} \\
&\times ((354n^2 - 536n - 587n^3 + 120n^4 - 660)m \\
&\quad + (210n^4 - 68n^2 - 1303n^3 - 896n - 840)n) \\
&- 2n^2 \frac{(t+1)^3 t^2}{mp^3 t_{\{6\}}} \times ((1258n + 2133n^2 - 639n^3 - 3060)m - 180n) \\
&- 6\frac{(t+1)^3 t}{p^3 t_{\{6\}}} \\
&\times (m^3(5(85n^2 - 450n + 274)m + (5022n^2 - 1730n - 1195n^3 - 60n^4 + 160)) \\
&\quad + n(2052n - 1673n^2 + 470n^3 + 120)m^2 \\
&\quad + 3n^2(355n^3 - 669n^2 - 550n + 40)m - 2n^3(231n^2 - 129n - 440)) \\
&- 30\frac{(t+1)^3}{p^3 t_{\{6\}}} \\
&\times (m^4(225n^2 - 548n + 120) - n(580n^2 - 980n + 47n^3 + 120)m^3 \\
&\quad + 6n^2(20n^2 - 85n + 72)m^2 + 6n^3(5n + 77n^2 - 60)m - 72n^4(n+1)) \\
&- 60mn\frac{(t+1)^3}{p^3 t_{\{6\}} t} \\
&\times (m^3(137n - 120) + n(120 - 30n^2 - 287n)m^2 - 180mn^2 + 180n^3(n+1)) \\
&+ 3600\frac{(t+1)^3}{p^3 t_{\{6\}} t^2} m^3 n^2 (n - m)
\end{aligned}
$$

where $t_{\{6\}} = t(t+1)(t+2)(t+3)(t+4)(t+5)$.

Proof. In Theorem 5.1, the left handside of both the skewness and kurtosis equations involves the expression, $[(\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2]$. Because we will be needing the value of this expression in the following proof, we will start the proof by evaluating this expression.

$$
\begin{aligned}
[(\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2] &= \left(\frac{(t-n)n}{(t+1)t^2}\right)\left(\frac{(t-m)m}{(t+1)t^2}\right) - \left(-\frac{nm}{(t+1)t^2}\right)^2 \\
&= \frac{1}{t^4 + 2t^5 + t^6} nm(m-t)(n-t) - \frac{m^2 n^2}{t^4 + 2t^5 + t^6} \\
&= \frac{nm(m-t)(n-t) - m^2 n^2}{t^4 + 2t^5 + t^6} \\
&= \frac{tnm(t-n-m)}{t^4(t+1)^2} \\
&= \frac{nmp}{t^3(t+1)^2} \qquad (7.1)
\end{aligned}
$$

because $p = t - m - n$.

We will use $[(\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2] = \dfrac{nmp}{t^3(t+1)^2}$ in our calculations of (i) and (ii) of Theorem 7.1. From Theorem 5.1,

(i) $[(\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2]^2 E(Q^2) = \mu_{4,0}\mu_{0,2}^2 + \mu_{0,4}\mu_{2,0}^2 + 2(\mu_{2,2})(2\mu_{1,1}^2 + \mu_{0,2}\mu_{2,0}) - 4(\mu_{1,1})(\mu_{2,0}\mu_{1,3} - \mu_{0,2}\mu_{3,1})$

That is,

$$\left[\frac{nmp}{t^3(t+1)^2}\right]^2 E(Q^2) = \mu_{4,0}\mu_{0,2}^2 + \mu_{0,4}\mu_{2,0}^2 + 2(\mu_{2,2})(2\mu_{1,1}^2 + \mu_{0,2}\mu_{2,0}) - 4(\mu_{1,1})(\mu_{2,0}\mu_{1,3} - \mu_{0,2}\mu_{3,1}) \tag{7.2}$$

We first focus on the right handside of the above equality. By substituting relevant central moments from Section 4, the right handside of (7.2) simplifies to the following.

$$\begin{aligned}
&\left(3(t-m)m\frac{t^2(m+2)+t(-6m-m^2)+6m^2}{(t+1)(t+2)(t+3)t^4}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)^2 \\
&+\left(3(t-n)n\frac{(6n^2-6nt+2t^2+nt^2-n^2t)}{(t+1)(t+2)(t+3)t^4}\right)\left(\frac{(t-m)m}{(t+1)t^2}\right)^2 \\
&+2\left(mn\frac{t^3+t^2(-m-n)+t(6m+6n+3mn)-18mn}{(t+1)(t+2)(t+3)t^4}\right) \\
&\quad\times\left(2\left(-\frac{nm}{(t+1)t^2}\right)^2+\left(\frac{(t-n)n}{(t+1)t^2}\right)\left(\frac{(t-m)m}{(t+1)t^2}\right)\right) \\
&-4\left(-\frac{nm}{(t+1)t^2}\right)\left(\frac{(t-m)m}{(t+1)t^2}\right)\times\frac{n}{(t+1)(t+2)(t+3)t^4} \\
&\quad\times\left(t^3(2m-2n+3mn-3n^2-n^3+mn^2)+t^2(6mn-6n-9n^2-3n^3+3mn^2)\right. \\
&\qquad\left.+t(18mn+3mn^2)-18mn^2\right) \\
&+4\left(-\frac{nm}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)\left(3mn\frac{t^2(-m-2)+t(6m+m^2)-6m^2}{(t+1)(t+2)(t+3)t^4}\right) \\
&=6\frac{(t-n)(t-m)mn}{(t+1)^3(t+2)(t+3)t^8} \\
&\quad\times\left(t^3(m+n+mn)+t^2(-6mn-m^2-n^2-mn^2-m^2n)+t(6mn^2+6m^2n+m^2n^2)\right. \\
&\qquad\left.-6m^2n^2\right) \\
&\quad+2\frac{(3mn-mt-nt+t^2)(6mt-18mn+6nt+3mnt+t^3-mt^2-nt^2)n^2m^2}{(t+3)(t+2)(t+1)^3t^8} \\
&\quad+4\frac{(n-m)m^2n^2}{(t+1)^3(t+2)(t+3)t^6} \\
&\quad\times\left(t^2(-3n-n^2-2)+t(2m-9n+3mn-3n^2+mn^2)+9mn-6n+3mn^2\right) \\
&=2\frac{mn}{(t+1)^3(t+2)(t+3)t^8} \\
&\quad\times\left(t^5(3m+3n+4mn)+t^4(-24mn-6m^2-6n^2-8mn^2-8m^2n)\right. \\
&\qquad+t^3(3m^3+3n^3+48mn^2+48m^2n+4mn^3+4m^3n+20m^2n^2) \\
&\qquad+t^2(-27mn^3-27m^3n-102m^2n^2-12m^2n^3-12m^3n^2) \\
&\qquad\left.+t(72m^2n^3+72m^3n^2+12m^3n^3)-72m^3n^3\right) \\
&\quad+4\frac{(n-m)m^2n^2}{(t+1)^3(t+2)(t+3)t^6} \\
&\quad\times\left(t^2(-3n-n^2-2)+t(2m-9n+3mn-3n^2+mn^2)+9mn-6n+3mn^2\right) \\
&=2mn\frac{t^5(3m+3n+4mn)+t(72m^2n^3+72m^3n^2+12m^3n^3)-72m^3n^3}{(t+1)^3(t+2)(t+3)t^8}
\end{aligned}$$

$$+\,2mn\frac{m^2(6n^2-4n+2n^3-6)-2mn(6n+3n^2+n^3+12)-6n^2}{(t+1)^3(t+2)(t+3)t^4}$$
$$+\,2\frac{mn}{(t+1)^3(t+2)(t+3)t^5}$$
$$\times\,(m^3(3-2n^3-6n^2)+2m^2n(n^3+6n^2+21n+24)-2mn^2(3n^2+7n-24)+3n^3)$$
$$+\,2mn\frac{-3m^3n(10n+2n^2+9)+6m^2n^2(n+n^2-15)-39mn^3}{(t+1)^3(t+2)(t+3)t^6}$$
$$=\frac{m^2n^2p^2}{t^6(t+1)^4}$$
$$\times\,(2(t+1)\frac{(3m+3n+4mn)t^5+12n^2m^2(6m+6n+mn)t-72m^3n^3}{p^2nm(t+3)(t+2)t^2}$$
$$-\,4(t+1)t^2\frac{3n^2+(6n+3n^2+n^3+12)nm+(2n-3n^2-n^3+3)m^2}{p^2nm(t+3)(t+2)}$$
$$+\,2\frac{(t+1)t}{p^2nm(t+3)(t+2)}$$
$$\times\,((3-2n^3-6n^2)m^3+2(21n+6n^2+n^3+24)nm^2-2(7n+3n^2-24)n^2m+3n^3)$$
$$+\,6(t+1)\frac{(-2n^2-10n-9)m^2+2(n+n^2-15)nm-13n^2}{(t+3)(t+2)p^2}).$$

Dividing both sides of (7.2) by $\left(\frac{nmp}{t^3(t+1)^2}\right)^2$, we have the following kurtosis index

$$E(Q^2)=2(t+1)\frac{(3m+3n+4mn)t^5+12n^2m^2(6m+6n+mn)t-72m^3n^3}{p^2nm(t+3)(t+2)t^2}$$
$$-\,4(t+1)t^2\frac{3n^2+(6n+3n^2+n^3+12)nm+(2n-3n^2-n^3+3)m^2}{p^2nm(t+3)(t+2)}$$
$$+\,2\frac{(t+1)t}{p^2nm(t+3)(t+2)}$$
$$\times\,((3-2n^3-6n^2)m^3+2(21n+6n^2+n^3+24)nm^2-2(7n+3n^2-24)n^2m+3n^3)$$
$$+\,6(t+1)\frac{(-2n^2-10n-9)m^2+2(n+n^2-15)nm-13n^2}{(t+3)(t+2)p^2}$$
$$=2(t+1)^2\frac{(3m+3n+4mn)t^5+12n^2m^2(6m+6n+mn)t-72m^3n^3}{p^2nm(t+3)(t+2)(t+1)t^2}$$
$$-\,4(t+1)^2t^3\frac{3n^2+(6n+3n^2+n^3+12)nm+(2n-3n^2-n^3+3)m^2}{p^2nm(t+3)(t+2)(t+1)t}$$
$$+\,2\frac{(t+1)^2t^2}{p^2nm(t+3)(t+2)(t+1)t}$$
$$\times\,((3-2n^3-6n^2)m^3+2(21n+6n^2+n^3+24)nm^2-2(7n+3n^2-24)n^2m+3n^3)$$
$$+\,6(t+1)^2t\frac{(-2n^2-10n-9)m^2+2(n+n^2-15)nm-13n^2}{(t+3)(t+2)(t+1)tp^2}.$$

Writing the above with Pochhammer factorials gives us the following

$$E(Q^2)=2\frac{(t+1)^2}{t_{\{4\}}p^2nmt}((3m+3n+4mn)t^5+12n^2m^2(6m+6n+mn)t-72m^3n^3)$$
$$-\,4\frac{(t+1)^2}{t_{\{4\}}p^2nm}t^3(3n^2+(n^3+3n^2+6n+12)nm+(-n^3-3n^2+2n+3)m^2)$$

$$+ 2\frac{(t+1)^2}{t_{\{4\}}p^2nm}t^2$$
$$\times ((-2n^3 - 6n^2 + 3)m^3 + 2(n^3 + 6n^2 + 21n + 24)nm^2 - 2(3n^2 + 7n - 24)n^2m + 3n^3)$$
$$+ 6\frac{(t+1)^2}{t_{\{4\}}p^2}t((-2n^2 - 10n - 9)m^2 + 2(n^2 + n - 15)nm - 13n^2).$$

Letting $k = \dfrac{(t+1)^2}{t_{\{4\}}p^2}$,

$$E(Q^2) = 2\frac{k}{nmt}((3m + 3n + 4mn)t^5 + 12n^2m^2(6m + 6n + mn)t - 72m^3n^3)$$
$$- 4\frac{k}{nm}t^3(3n^2 + (n^3 + 3n^2 + 6n + 12)nm + (-n^3 - 3n^2 + 2n + 3)m^2)$$
$$+ 2\frac{k}{nm}t^2$$
$$\times ((-2n^3 - 6n^2 + 3)m^3 + 2(n^3 + 6n^2 + 21n + 24)nm^2 - 2(3n^2 + 7n - 24)n^2m + 3n^3)$$
$$+ 6kt((-2n^2 - 10n - 9)m^2 + 2(n^2 + n - 15)nm - 13n^2)$$
$$= 2k$$
$$\times (\frac{1}{nmt}((3m + 3n + 4mn)t^5 + 12n^2m^2(6m + 6n + mn)t - 72m^3n^3)$$
$$- 2\frac{t^3}{nm}(3n^2 + (n^3 + 3n^2 + 6n + 12)nm + (-n^3 - 3n^2 + 2n + 3)m^2)$$
$$+ \frac{t^2}{nm}$$
$$\times ((-2n^3 - 6n^2 + 3)m^3 + 2(n^3 + 6n^2 + 21n + 24)nm^2 - 2(3n^2 + 7n - 24)n^2m$$
$$+ 3n^3)$$
$$+ 3t((-2n^2 - 10n - 9)m^2 + 2(n^2 + n - 15)nm - 13n^2))$$

which simplifies to part (i) of Theorem 7.1.

(ii) $$[(\mu_{2,0}\mu_{0,2}) - \mu_{1,1}^2]^3 E(Q^3) = \mu_{6,0}\mu_{0,2}^3 + \mu_{0,6}\mu_{2,0}^3$$
$$- 6\mu_{1,1}(\mu_{0,2}^2\mu_{5,1} + \mu_{2,0}^2\mu_{1,5})$$
$$+ 4\mu_{1,1}^2(3\mu_{0,2}\mu_{4,2} + 3\mu_{2,0}\mu_{2,4} - 2\mu_{1,1}\mu_{3,3})$$
$$+ 3\mu_{2,0}\mu_{0,2}(\mu_{0,2}\mu_{4,2} + \mu_{2,0}\mu_{2,4} - 4\mu_{1,1}\mu_{3,3}).$$

That is,
$$\left[\frac{nmp}{t^3(t+1)^2}\right]^3 E(Q^3) = \mu_{6,0}\mu_{0,2}^3 + \mu_{0,6}\mu_{2,0}^3$$
$$- 6\mu_{1,1}(\mu_{0,2}^2\mu_{5,1} + \mu_{2,0}^2\mu_{1,5})$$
$$+ 4\mu_{1,1}^2(3\mu_{0,2}\mu_{4,2} + 3\mu_{2,0}\mu_{2,4} - 2\mu_{1,1}\mu_{3,3})$$
$$+ 3\mu_{2,0}\mu_{0,2}(\mu_{0,2}\mu_{4,2} + \mu_{2,0}\mu_{2,4} - 4\mu_{1,1}\mu_{3,3}) \tag{7.3}$$

We first focus on the right handside of the above equality. Substituting the relevant central moments from Section 4, we have the following expression for the right handside of (7.3).

$$5(t - m)m$$
$$\times (\frac{(26m + 3m^2 + 24)t^4 + (-120m - 112m^2 - 6m^3)t^3}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$+ \frac{(240m^2 + 172m^3 + 3m^4)t^2 + (-240m^3 - 86m^4)t + 120m^4}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6})\left(\frac{(t-n)n}{(t+1)t^2}\right)^3$$
$$+ 5(t - n)n \times (\frac{t^4(26n + 3n^2 + 24) + t^3(-120n - 112n^2 - 6n^3)}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$+ \frac{t^2(240n^2 + 172n^3 + 3n^4) + 120n^4 + t(-240n^3 - 86n^4)}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6})\left(\frac{(t-m)m}{(t+1)t^2}\right)^3$$

$$
\begin{aligned}
&-6\left(-\frac{nm}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)^2\\
&\times(5\frac{m(m^4n-m^5)t^5}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}+5\frac{m(15m^4n-26mn-15m^5-3m^2n-24n)t^4}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}\\
&\quad+5\frac{m(120mn-85m^5+112m^2n+6m^3n+85m^4n)t^3}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6})\\
&-6\left(-\frac{nm}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)^2\\
&\times(5\frac{m(222m^4n-240m^2n-172m^3n-225m^5)t^2}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}\\
&\quad+5\frac{m(240m^3n-274m^5+360m^4n)t-120m^6}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6})\\
&-6\left(-\frac{nm}{(t+1)t^2}\right)\left(\frac{(t-m)m}{(t+1)t^2}\right)^2\\
&\times(5\frac{(n^2-4nt-4t)(n+1)mn^3}{(t+1)(t+2)t^5}+5\frac{(2t-n+nt)mn^4}{(t+1)t^6}-5\frac{(3t-3n+nt-3)(n+1)(n+2)mn^2}{(t+1)(t+2)(t+3)t^3}\\
&\quad+(t-1)\frac{(n+1)(n+2)(n+3)(n+4)mn}{(t+1)(t+2)(t+3)(t+4)t^2})\\
&+4\left(-\frac{nm}{(t+1)t^2}\right)^2\left(3\left(\frac{(t-n)n}{(t+1)t^2}\right)\right)\\
&\times mn^2(\frac{(3m+6mn-18m^2-12m^3+6mn^2+6)t^3}{(t+1)(t+2)(t+3)t^6}+\frac{t^2(30mn-24m-72m^2-38m^3+30mn^2)}{(t+1)(t+2)(t+3)t^6}\\
&\qquad+\frac{t(36mn-7m^3+36mn^2)-30m^3}{(t+1)(t+2)(t+3)t^6})\\
&+4\left(-\frac{nm}{(t+1)t^2}\right)^2\left(3\left(\frac{(t-n)n}{(t+1)t^2}\right)\right)\\
&\times mn(\frac{(3m-6n-3mn+12m^2n+6m^3n+6)t^4}{(t+1)(t+2)(t+3)(t+4)(t+5)t^5}+\\
&\qquad+\frac{(150m^2n-60n-6mn-6m^2-40m+62m^3n)t^3}{(t+1)(t+2)(t+3)(t+4)(t+5)t^5}\\
&\qquad+\frac{(320mn+120m^2+3m^3+600m^2n+163m^3n)t^2}{(t+1)(t+2)(t+3)(t+4)(t+5)t^5}\\
&\qquad+\frac{(-86m^3-86m^3n)t+(120m^3+120m^3n)}{(t+1)(t+2)(t+3)(t+4)(t+5)t^5})\\
&+4\left(-\frac{nm}{(t+1)t^2}\right)^2 3\left(\frac{(t-m)m}{(t+1)t^2}\right)\times\frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)}\\
&\times(-2m(n+3)(n+2)(n+1)+\frac{(3n-96m-176mn-96mn^2-16mn^3+6)}{t}\\
&\quad+\frac{(-186m-40n-333mn-6n^2-180mn^2-30mn^3}{t^2}\\
&\quad+\frac{(120m+164mn+120n^2+3n^3+18mn^2)}{t^3}+\frac{(-480mn-86n^3-516mn^2-15mn^3)}{t^4}\\
&\quad+\frac{(120n^3+720mn^2+430mn^3)}{t^5}-\frac{600mn^3}{t^6})
\end{aligned}
$$

$$+4\left(-\frac{nm}{(t+1)t^2}\right)^2\left(2\frac{nm}{(t+1)t^2}\right)\times\frac{1}{(t+1)(t+2)(t+3)(t+4)(t+5)}$$
$$\times\left(\frac{4mn}{t}+\frac{mn(-30m-30n-9mn)}{t^2}+\frac{mn(180mn+26m^2+26n^2+9mn^2+9m^2n)}{t^3}\right.$$
$$+\frac{mn(-120m^2-120n^2-258mn^2-258m^2n-15m^2n^2)}{t^4}$$
$$\left.+\frac{mn(360mn^2+360m^2n+430m^2n^2)}{t^5}-600\frac{m^3n^3}{t^6}\right)$$
$$+3\left(\frac{(t-m)m}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)\left(\left(\frac{(t-n)n}{(t+1)t^2}\right)\times\frac{mn^2}{(t+1)(t+2)(t+3)t^6}\right.$$
$$\times((3m+6mn-18m^2-12m^3+6mn^2+6)t^3+t^2(30mn-24m-72m^2-38m^3+30mn^2)$$
$$+t(36mn-7m^3+36mn^2)-30m^3)$$
$$+\frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)t^5}$$
$$\times((3m-6n-3mn+12m^2n+6m^3n+6)t^4$$
$$+(150m^2n-60n-6mn-6m^2-40m+62m^3n)t^3$$
$$+(320mn+120m^2+3m^3+600m^2n+163m^3n)t^2+(-86m^3-86m^3n)t$$
$$\left.+(120m^3+120m^3n))\right)$$
$$+3\left(\frac{(t-m)m}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)\times\left(\left(\frac{(t-m)m}{(t+1)t^2}\right)\times\left(\frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)}\right.\right.$$
$$\times((-12m-22mn-12mn^2-2mn^3)+\frac{(3n-96m-176mn-96mn^2-16mn^3+6)}{t}$$
$$+\frac{(-186m-40n-333mn-6n^2-180mn^2-30mn^3)}{t^2}$$
$$+\frac{(120m+164mn+120n^2+3n^3+18mn^2)}{t^3}+\frac{(-480mn-86n^3-516mn^2-15mn^3)}{t^4}$$
$$\left.\left.+\frac{(120n^3+720mn^2+430mn^3)}{t^5}-\frac{600mn^3}{t^6}\right)\right)$$
$$+3\left(\frac{(t-m)m}{(t+1)t^2}\right)\left(\frac{(t-n)n}{(t+1)t^2}\right)\times\left(-4\left(-\frac{nm}{(t+1)t^2}\right)\times\frac{mn}{t(t+1)(t+2)(t+3)(t+4)(t+5)}\right.$$
$$\times\left(4+\frac{(-30m-30n-9mn)}{t}+\frac{(180mn+26m^2+26n^2+9mn^2+9m^2n)}{t^2}\right.$$
$$+\frac{(-120m^2-120n^2-258mn^2-258m^2n-15m^2n^2)}{t^3}$$
$$\left.\left.+\frac{(360mn^2+360m^2n+430m^2n^2)}{t^4}-600\frac{m^2n^2}{t^5}\right)\right)$$
$$=5\frac{(t-m)(t-n)^3mn^3}{(t+5)(t+4)(t+3)(t+2)(t+1)^4t^{12}}$$
$$\times((26m+3m^2+24)t^4+(-120m-112m^2-6m^3)t^3+(240m^2+172m^3+3m^4)t^2$$
$$+(-240m^3-86m^4)t+120m^4)$$
$$+5\frac{(t-n)n(t-m)^3m^3}{(t+5)(t+4)(t+3)(t+2)(t+1)^4t^{12}}$$
$$\times(t^4(26n+3n^2+24)+t^3(-120n-112n^2-6n^3)+t^2(240n^2+172n^3+3n^4)+120n^4$$
$$+t(-240n^3-86n^4))$$
$$+6\frac{m^2n^2}{(t+1)^4(t+2)(t+3)(t+4)(t+5)t^{12}}$$
$$\times(t^8(24m^2+20m^2n-20m^2n^2-20m^2n^3-4m^2n^4)$$

$$
\begin{aligned}
&+ t^7(96m^2 - 48m^3 - 40m^2n - 40m^3n - 5m^5n - 300m^2n^2 - 200m^2n^3 + 40m^3n^2 - 41m^2n^4\\
&\quad + 40m^3n^3 + 5m^4n^2 + 8m^3n^4)\\
&+ t^6(24m^4 - 192m^3 - 120n^2 - 120m^2 - 130mn^2 - 580m^2n + 80m^3n + 20m^4n - 75m^5n\\
&\quad - 855m^2n^2 - 450m^2n^3 + 600m^3n^2 - 140m^2n^4 + 400m^3n^3 + 55m^4n^2 + 5m^2n^5\\
&\quad + 82m^3n^4 - 30m^4n^3 + 10m^5n^2 - 4m^4n^4)\\
&+ t^5(240m^3 + 96m^4 + 240n^3 + 600mn^2 + 600m^2n + 260mn^3 + 1160m^3n - 40m^4n - 425m^5n\\
&\quad + 1120m^2n^2 + 60m^2n^3 + 1710m^3n^2 - 295m^2n^4 + 900m^3n^3 + 125m^4n^2 + 60m^2n^5\\
&\quad + 280m^3n^4 - 350m^4n^3 + 150m^5n^2 - 10m^3n^5 - 36m^4n^4 - 5m^5n^3)\\
&+ t^4(270m^4n^2 - 120n^4 - 1200mn^3 - 1200m^3n - 130mn^4 - 580m^4n - 1125m^5n - 2400m^2n^2\\
&\quad - 1980m^2n^3 - 1980m^3n^2 - 565m^2n^4 - 120m^3n^3 - 120m^3n^3 - 120m^4 + 235m^2n^5\\
&\quad + 590m^3n^4 - 1300m^4n^3 + 850m^5n^2 - 120m^3n^5 - 65m^4n^4 - 75m^5n^3 + 5m^4n^5)\\
&+ t^3(-2190m^4n^3 + 2250m^5n^2 - 470m^3n^5 + 130m^4n^4 - 425m^5n^3 + 60m^4n^5 + 600mn^4\\
&\quad + 600m^4n - 1370m^5n + 3600m^2n^3 + 3600m^3n^2 + 690m^2n^4 + 3440m^3n^3 + 2360m^4n^2\\
&\quad + 300m^2n^5 + 1130m^3n^4)\\
&+ t^2(2740m^5n^2 - 1800m^2n^4 - 4800m^3n^3 - 1200m^4n^2 - 1120m^3n^4 - 4460m^4n^3 - 600m^5n\\
&\quad - 600m^3n^5 + 560m^4n^4 - 1125m^5n^3 + 235m^4n^5)\\
&+ t(2400m^3n^4 + 1200m^4n^3 + 1200m^5n^2 + 1930m^4n^4 - 1370m^5n^3 + 300m^4n^5)\\
&- 600m^4n^3(n+m))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{8m^3n^3}{(t+5)(t+4)(t+3)(t+2)(t+1)^4t^{12}}\\
&\times (t^7(18m^2 + 33m^2n + 18m^2n^2 + 3m^2n^3)\\
&\quad + t^6(144m^2 - 9n - 9mn - 9m - 18m^3 + 264m^2n - 9mn^3 - 33m^3n - 9mn^4 + 153m^2n^2\\
&\qquad + 24m^2n^3 - 9m^3n^2 - 3m^3n^3)\\
&\quad + t^5(116mn + 288m^2 - 144m^3 + 18n^2 + 18mn^2 + 513m^2n - 126mn^3 - 264m^3n - 117mn^4\\
&\qquad + 9mn^5 + 396m^2n^2 + 36m^2n^3 - 18m^3n^2 - 33m^3n^3)\\
&\quad + t^4(126mn^5 - 279m^3 - 180n^2 - 9n^3 - 456mn^2 - 456m^2n - 648mn^3 - 504m^3n - 513mn^4\\
&\qquad - 180m^2 + 576m^2n^2 - 126m^2n^3 + 369m^3n^2 - 171m^3n^3)\\
&\quad + t^3(180m^3 + 180n^3 + 720mn^2 + 720m^2n - 1037mn^3 + 349m^3n - 747mn^4 + 639mn^5\\
&\qquad + 3114m^2n^2 - 594m^2n^3 + 1431m^3n^2 - 639m^3n^3)\\
&\quad + t^2(306mn^4 - 780m^3n - 1860mn^3 + 1386mn^5 - 1080m^2n^2 - 2676m^2n^3 - 210m^3n^2\\
&\qquad - 1416m^3n^3)\\
&\quad + t(1080mn^4 + 1080mn^5 + 720m^2n^3 + 1800m^3n^2 - 220m^3n^3)\\
&\quad - 150m^3n^3)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-2mn}{(t+5)(t+4)(t+3)(t+2)(t+1)^4t^{12}}\\
&\quad \times (t^9(18m^3n + 33m^3n^2 + 18m^3n^3 + 3m^3n^4)\\
&\qquad + t^8(72m^3n - 60n^2 - 74mn^2 - 74m^2n - 60m^2 - 36m^4n - 24m^2n^2 + 186m^3n^2 - 9m^2n^4\\
&\qquad\quad + 180m^3n^3 - 66m^4n^2 - 9m^2n^5 + 66m^3n^4 - 27m^4n^3 + 9m^3n^5 - 6m^4n^4)\\
&\qquad + t^7(180m^3 + 180n^3 + 360mn^2 + 360m^2n + 222mn^3 + 204m^3n - 144m^4n + 18m^5n\\
&\qquad\quad + 804m^2n^2 + 72m^2n^3 + 543m^3n^2 - 126m^2n^4 + 1032m^3n^3 - 300m^4n^2\\
&\qquad\quad - 108m^2n^5 + 492m^3n^4 - 93m^4n^3 + 33m^5n^2 + 18m^2n^6 + 108m^3n^5\\
&\qquad\quad - 78m^4n^4 - 6m^5n^3 + 15m^6n^2 - 6m^4n^5 + 3m^5n^4)\\
&\qquad + t^6(72m^5n - 180n^4 - 900mn^3 - 900m^3n - 222mn^4 - 186m^4n - 180m^4 - 1800m^2n^2\\
&\qquad\quad - 1820m^2n^3 - 740m^3n^2 - 711m^2n^4 + 2475m^3n^3 - 438m^4n^2 - 387m^2n^5 + 954m^3n^4\\
&\qquad\quad - 243m^4n^3 + 114m^5n^2 + 243m^2n^6 + 456m^3n^5 - 624m^4n^4 - 312m^5n^3 + 225m^6n^2\\
&\qquad\quad - 9m^2n^7 - 69m^3n^6 - 93m^4n^5 + 78m^5n^4 - 30m^6n^3 - 3m^5n^5)\\
&\qquad + t^5(m^6(1275n^2 - 450n^3 + 15n^4) + m^5(56n - 81n^2 - 2064n^3 + 1131n^4 - 57n^5 + 60)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + m^4(720n + 444n^2 - 1161n^3 - 1602n^4 - 489n^5 + 30n^6) \\
&\quad + m^3(3000n^2 + 4286n^3 - 567n^4 + 894n^5 - 891n^6 + 45n^7) \\
&\quad + m^2(3000n^3 + 138n^4 - 84n^5 + 1152n^6 - 126n^7) \\
&\quad + m(720n^4 + 74n^5) + 60n^5) + mn(1800m^6 n^4 - 1800m^5 n^5)) \\
&+ t^4(1258m^2 n^5 - 180m^5 n - 3060m^2 n^4 - 2520m^3 n^3 - 1980m^4 n^2 - 180mn^5 - 2688m^3 n^4 \\
&\quad - 1608m^4 n^3 - 434m^5 n^2 + 2133m^2 n^6 - 204m^3 n^5 + 1062m^4 n^4 \\
&\quad - 4761m^5 n^3 + 3375m^6 n^2 - 639m^2 n^7 - 3909m^3 n^6 - 1761m^4 n^5 + 6384m^5 n^4 \\
&\quad - 2550m^6 n^3 + 630m^3 n^7 + 360m^4 n^6 - 660m^5 n^5 + 225m^6 n^4 - 15m^5 n^6) \\
&+ t^3(2640m^2 n^5 + 360m^3 n^4 + 360m^4 n^3 + 480m^5 n^2 + 774m^2 n^6 - 4950m^3 n^5 + 6156m^4 n^4 \\
&\quad - 5190m^5 n^3 + 4110m^6 n^2 - 1386m^2 n^7 - 6021m^3 n^6 - 5019m^4 n^5 + 15\,066m^5 n^4 \\
&\quad - 6750m^6 n^3 + 3195m^3 n^7 + 1410m^4 n^6 - 3585m^5 n^5 + 1275m^6 n^4 - 180m^5 n^6) \\
&+ t^2(6480m^4 n^4 - 5400m^3 n^5 - 1080m^2 n^6 - 1800m^5 n^3 + 1800m^6 n^2 - 1080m^2 n^7 \\
&\quad + 450m^3 n^6 - 7650m^4 n^5 + 14\,700m^5 n^4 - 8220m^6 n^3 + 6930m^3 n^7 + 1800m^4 n^6 \\
&\quad - 8700m^5 n^5 + 3375m^6 n^4 - 705m^5 n^6) \\
&+ t(5400m^3 n^6 - 5400m^4 n^5 + 3600m^5 n^4 - 3600m^6 n^3 + 5400m^3 n^7 - 8610m^5 n^5 \\
&\quad + 4110m^6 n^4 - 900m^5 n^6) \\
&+ (1800m^5 n^3 - 1800m^4 n^4)
\end{aligned}
$$

Dividing both sides of (7.3) by $\left[\dfrac{nmp}{t^3(t+1)^2}\right]^3$, or equivalently $\left(m^3 n^3 \dfrac{p^3}{t^9(t+1)^6}\right)$, we have the following skewness index

$$
\begin{aligned}
E(Q^3) = &-6\frac{m(n+1)(n+2)(n+3)}{np^3}\frac{(t+1)^2 t^6}{(t+2)(t+3)(t+4)(t+5)} \\
&- \frac{2t^5(t+1)^2}{m^2 n^2 p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times ((72m^3 n - 60n^2 - 74mn^2 - 74m^2 n - 60m^2 - 36m^4 n - 24m^2 n^2 + 186m^3 n^2 - 9m^2 n^4 \\
&\quad + 180m^3 n^3 - 66m^4 n^2 - 9m^2 n^5 + 66m^3 n^4 - 27m^4 n^3 + 9m^3 n^5 - 6m^4 n^4)) \\
&- \frac{6t^4(t+1)^2}{m^2 n^2 p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times ((60m^3 + 60n^3 + 120mn^2 + 120m^2 n + 74mn^3 + 68m^3 n - 48m^4 n + 6m^5 n + 268m^2 n^2 \\
&\quad + 24m^2 n^3 + 181m^3 n^2)) \\
&- \frac{6t^4(t+1)^2}{p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (344mn - 100m^2 + 11m^3 - 42n^2 + 5m^4 - 36n^3 + 6n^4 + 164mn^2 - 31m^2 n + 36mn^3 \\
&\quad - 2m^3 n - 26m^2 n^2 - 2m^2 n^3 + m^3 n^2) \\
&+ \frac{2t^3(t+1)^2}{m^2 n^2 p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (180m^4 + 180n^4 + 900mn^3 + 900m^3 n + 222mn^4 + 186m^4 n - 72m^5 n + 1800m^2 n^2 \\
&\quad + 1820m^2 n^3 + 740m^3 n^2 + 711m^2 n^4 - 2475m^3 n^3 + 438m^4 n^2) \\
&+ \frac{6t^3(t+1)^2}{p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (129n^3 - 75m^4 - 38m^3 - 81n^4 + 3n^5 - 318mn^2 + 81m^2 n - 152mn^3 + 104m^3 n \\
&\quad + 23mn^4 + 10m^4 n + 208m^2 n^2 + 31m^2 n^3 - 26m^3 n^2 + m^3 n^3) \\
&- 2\frac{m^2 t^2(t+1)^2}{n^2 p^3(t+2)(t+3)(t+4)(t+5)} \times
\end{aligned}
$$

$$
\begin{aligned}
&\times (60m+720n+56mn+444n^2-1161n^3-1602n^4-489n^5+30n^6-81mn^2 \\
&\quad -2064mn^3+1131mn^4-57mn^5+1275m^2n^2-450m^2n^3+15m^2n^4) \\
&-\frac{2t^2(t+1)^2}{m^2p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (1800m^7n^3-1800m^6n^4+m^3(4286n-567n^2+894n^3-891n^4+45n^5+3000) \\
&\quad +m^2(3000n+138n^2-84n^3+1152n^4-126n^5)+m(720n^2+74n^3)+60n^3) \\
&-\frac{2t(t+1)^2}{mnp^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (1258mn^4-180n^4-3060mn^3-1980m^3n-180m^4-434m^4n+2133mn^5 \\
&\quad -2520m^2n^2-2688m^2n^3-1608m^3n^2-204m^2n^4) \\
&-\frac{6t(t+1)^2}{p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (1125m^4-213n^5-1587m^3n-1303mn^4-850m^4n \\
&\quad +210mn^5+354m^2n^2-587m^2n^3+2128m^3n^2 \\
&\quad +120m^2n^4-220m^3n^3+75m^4n^2-5m^3n^4) \\
&+\frac{12(t+1)^2}{p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (231n^5-685m^4-440n^3-129n^4-80m^3-60mn^2-60m^2n+825mn^3 \\
&\quad +865m^3n-1026m^2n^2) \\
&-6\frac{mn(t+1)^2}{p^3(t+2)(t+3)(t+4)(t+5)} \\
&\times (1065n^4-2007n^3-2250m^3-1673mn^2+5022m^2n+470mn^3+425m^3n \\
&\quad -1195m^2n^2-60m^2n^3) \\
&+\frac{720(t+1)^2(3n^4-5m^4+3n^5+15mn^3+5m^3n-18m^2n^2)}{p^3t(t+2)(t+3)(t+4)(t+5)} \\
&-30\frac{mn(t+1)^2}{p^3t(t+2)(t+3)(t+4)(t+5)} \\
&\times (30n^3-548m^3+462n^4-510mn^2+980m^2n+120mn^3+225m^3n-580m^2n^2 \\
&\quad -47m^2n^3) \\
&-60\frac{mn(t+1)^2}{p^3t^2(t+2)(t+3)(t+4)(t+5)} \\
&\times (180n^3-120m^3+180n^4-180mn^2+120m^2n+137m^3n-287m^2n^2-30m^2n^3) \\
&+3600\frac{m^2n(n-m)}{p^3}\frac{(t+1)^2}{t^3(t+2)(t+3)(t+4)(t+5)} \\
=\ &-6m(t+1)^2t^6\frac{(n+3)(n+2)(n+1)}{np^3(t+5)(t+4)(t+3)(t+2)} \\
&-2\frac{(t+1)^2t^5}{m^2n^2p^3(t+5)(t+4)(t+3)(t+2)} \\
&\times (-3n(22n+9n^2+2n^3+12)m^4+3n(62n+60n^2+22n^3+3n^4+24)m^3 \\
&\quad -(60+74n+24n^2+9n^4+9n^5)m^2-74mn^2-60n^2) \\
&-6\frac{m(t+1)^2t^4}{n^2p^3(t+5)(t+4)(t+3)(t+2)} \\
&\times (5m^3n^2+(11n-2n^2+n^3+6)nm^2-(100n+31n^2+26n^3+2n^4+48)nm \\
&\quad +(68n+181n^2+344n^3+164n^4+36n^5+60))
\end{aligned}
$$

$$
\begin{aligned}
&- 12(t+1)^2 t^4 \frac{(134n + 12n^2 - 21n^3 - 18n^4 + 3n^5 + 60)m^2 + 30n^2 + (37n + 60)nm}{m^2 np^3(t+5)(t+4)(t+3)(t+2)} \\
&+ 2(t+1)^2 t^3 \frac{6(37n + 150)nm + 180n^2 + (1820n + 711n^2 + 387n^3 - 243n^4 + 9n^5 + 1800)m^2}{m^2 p^3(t+5)(t+4)(t+3)(t+2)} \\
&- 2\frac{(t+1)^2 t^3}{n^2 p^3(t+5)(t+4)(t+3)(t+2)} \\
&\quad \times (-15(2n - 15)n^2 m^4 - 3(104n^2 - 38n - 26n^3 + n^4 - 24)nm^3 \\
&\qquad - 3(62n + 146n^2 + 81n^3 + 208n^4 + 31n^5 + 60)m^2) \\
&- 2(t+1)^2 t^3 \frac{-(740n - 2475n^2 - 954n^3 - 456n^4 + 69n^5 + 900)mn}{n^2 p^3(t+5)(t+4)(t+3)(t+2)} \\
&- 2\frac{(t+1)^2 t^2}{m^2 n^2 p^3(t+5)(t+4)(t+3)(t+2)} \\
&\quad \times (15(n^2 - 30n + 85)n^2 m^6 + (56n - 81n^2 - 2064n^3 + 1131n^4 - 57n^5 + 60)m^5 \\
&\qquad + 3(148n - 387n^2 - 534n^3 - 163n^4 + 10n^5 + 240)nm^4) \\
&- 2\frac{(t+1)^2 t^2}{m^2 n^2 p^3(t+5)(t+4)(t+3)(t+2)} \\
&\quad \times ((4286n - 567n^2 + 894n^3 - 891n^4 + 45n^5 + 3000)n^2 m^3 \\
&\qquad - 6(14n^2 - 23n - 192n^3 + 21n^4 - 500)n^3 m^2 + 2(37n + 360)mn^4 + 60n^5) \\
&- 2m^3(t+1)^2 t \frac{75(3n^2 - 34n + 45)mn - (434n + 4761n^2 - 6384n^3 + 660n^4 + 15n^5 + 180)}{np^3(t+5)(t+4)(t+3)(t+2)} \\
&- 6m(t+1)^2 t \\
&\quad \times \frac{(354n^2 - 536n - 587n^3 + 120n^4 - 660)m + (210n^4 - 68n^2 - 1303n^3 - 896n - 840)n}{p^3(t+5)(t+4)(t+3)(t+2)} \\
&- 2n^2(t+1)^2 t \frac{(1258n + 2133n^2 - 639n^3 - 3060)m - 180n}{mp^3(t+5)(t+4)(t+3)(t+2)} \\
&- 6m^3(t+1)^2 \frac{(5(85n^2 - 450n + 274)m + (5022n^2 - 1730n - 1195n^3 - 60n^4 + 160))}{p^3(t+5)(t+4)(t+3)(t+2)} \\
&- 6\frac{(t+1)^2}{p^3(t+5)(t+4)(t+3)(t+2)} \\
&\quad \times (n(2052n - 1673n^2 + 470n^3 + 120)m^2 + 3n^2(355n^3 - 669n^2 - 550n + 40)m \\
&\qquad - 2n^3(231n^2 - 129n - 440)) \\
&- 30\frac{(t+1)^2}{p^3(t+5)(t+4)(t+3)(t+2)t} \\
&\quad \times (m^4(225n^2 - 548n + 120) - n(580n^2 - 980n + 47n^3 + 120)m^3 \\
&\qquad + 6n^2(20n^2 - 85n + 72)m^2 + 6n^3(5n + 77n^2 - 60)m - 72n^4(n+1)) \\
&- 60mn(t+1)^2 \frac{m^3(137n - 120) + n(120 - 30n^2 - 287n)m^2 - 180mn^2 + 180n^3(n+1)}{p^3(t+5)(t+4)(t+3)(t+2)t^2} \\
&+ 3600m^3 n^2(t+1)^2 \frac{(n-m)}{p^3(t+5)(t+4)(t+3)(t+2)t^3}.
\end{aligned}
$$

Writing the above with Pochhammer factorials gives us Theorem 7.1(ii).

Corollary 7.1. In particular if $n = m$, that is ($t = 2m + p$) and

$$k = \frac{(t+1)^2}{t_{\{4\}} p^2} = \frac{(2m + p + 1)}{(2m + p)(2m + p + 2)(2m + p + 3)p^2} \text{ then}$$

(i) $$E(Q^2) = 4\frac{2m+p+1}{mp^2(2m+p)^2(2m+p+2)(2m+p+3)}(12m^6 + 6(5p+2)m^5 + 4p(9+10p$$
$$+ 6p^2(6+5p)m^3 + 3p^3(9+4p)m^2 + 2p^4(6+p)m + 3p^5),$$

(ii) $$E(Q^3) = -2\frac{(2m+p+1)^2}{m^2p^3(2m+p)^2(2m+p+2)(2m+p+3)(2m+p+4)(2m+p+5)}$$
$$\times (864m^{13} + 72(50p+119)m^{12}$$
$$+ 12(536p^2 + 2821p + 2730)m^{11}$$
$$+ 6(1076p^3 + 9493p^2 + 20\,270p + 10\,035)m^{10}$$
$$+ 3(1330p^4 + 17\,835p^3 + 63\,282p^2 + 69\,405p + 17\,682)m^9$$
$$+ 3(519p^5 + 10\,261p^4 + 54\,417p^3 + 99\,007p^2 + 56\,475p + 5964)m^8$$
$$+ 3p(125p^5 + 3715p^4 + 28\,285p^3 + 76\,127p^2 + 72\,408p + 17\,496)m^7$$
$$+ p(51p^6 + 2493p^5 + 27\,480p^4 + 103\,635p^3 + 144\,621p^2 + 58\,324p - 960)m$$
$$+ p^2(3p^6 + 318p^5 + 5490p^4 + 28\,590p^3 + 53\,787p^2 + 31\,712p - 480)m^5$$
$$+ p^3(18p^5 + 633p^4 + 4740p^3 + 11\,082p^2 + 7678p - 240)m^4$$
$$+ p^4(33p^4 + 438p^3 + 1143p^2 - 640p - 840)m^3$$
$$+2p^5(9p^3 + 24p^2 - 421p - 540)m^2$$
$$- 4(37p+150)p^6m - 120p^7).$$

Proof. The case for equal marginal probability density functions occurs when $n = m$, that is, when $(t = 2m+p)$. In this case,

$$k = \frac{(t+1)^2}{t_{\{4\}}p^2} = \frac{(2m+p+1)}{(2m+p)(2m+p+2)(2m+p+3)p^2}.$$

(i) Substituting $m = n$, that is $t = 2m+p$ and k into Theorem 7.1(i), we obtain the following expression for kurtosis.

$$E(Q^2) = 2\frac{k}{mm}t^2$$
$$\times ((3m+3m+4mm)t^2$$
$$- 2((-m^3 - 3m^2 + 2m + 3)m^2 + m(m^3 + 3m^2 + 6m + 12)m + 3m^2)t$$
$$+ (m^3(-2m^3 - 6m^2 + 3) + 2m(21m + 6m^2 + m^3 + 24)m^2$$
$$- 2m^2(7m + 3m^2 - 24)m + 3m^3))$$
$$+ 2\frac{k}{t}$$
$$\times (-3((10m + 2m^2 + 9)m^2 - 2m(m^2 + m - 15)m + 13m^2)t^2$$
$$+ (12)mm(6m + 6m + mm)t - (72)m^2m^2)$$
$$= 2\frac{k}{m^2}t^2 \times (2m(2m+3)t^2 - 4m^2(4m+9)t + 2m^3(14m+51))$$
$$+ 2\frac{k}{t} \times (-12m^2(2m+13)t^2 + 12(m+12)m^3t - 72m^4)$$
$$= 2\frac{k}{m^2}(2m+p)^2 \times (2m(2m+3)(2m+p)^2 - 4m^2(4m+9)(2m+p) + 2m^3(14m+51))$$
$$+ 2\frac{k}{(2m+p)} \times (-12m^2(2m+13)(2m+p)^2 + 12(m+12)m^3(2m+p) - 72m^4)$$
$$= 4\frac{k}{m(2m+p)}$$
$$\times (12m^6 + m^5(30p+12) + m^4(36p + 40p^2) + m^3(36p^2 + 30p^3) + m^2(27p^3 + 12p^4)$$
$$+ m(12p^4 + 2p^5) + 3p^5)$$

$$= 4\frac{2m+p+1}{mp^2(2m+p)^2(2m+p+2)(2m+p+3)}$$
$$\times (12m^6 + m^5(30p+12) + m^4(36p+40p^2) + m^3(36p^2+30p^3) + m^2(27p^3+12p^4)$$
$$+ m(12p^4+2p^5) + 3p^5)$$
which simplifies to Corollary 7.1(i).

(ii) Substituting $m = n$, $t = 2m+p$ and $k = \dfrac{(2m+p+1)}{(2m+p)(2m+p+2)(2m+p+3)p^2}$ into

Theorem 7.1(ii), we obtain the following expression for skewness.

$$E(Q^3) = -6m\frac{((2m+p)+1)^2(2m+p)^6(m+3)(m+2)(m+1)}{mp^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$-2\frac{((2m+p)+1)^2(2m+p)^5}{m^2m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(-3m(22m+9m^2+2m^3+12)m^4+3m(62m+60m^2+22m^3+3m^4+24)m^3$$
$$-(60+74m+24m^2+9m^4+9m^5)m^2-74mm^2-60m^2)$$
$$-6\frac{m((2m+p)+1)^2(2m+p)^4}{m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(5m^3m^2+(11m-2m^2+m^3+6)mm^2-(100m+31m^2+26m^3+2m^4+48)mm$$
$$+(68m+181m^2+344m^3+164m^4+36m^5+60))$$
$$-12\frac{((2m+p)+1)^2(2m+p)^4}{m^2mp^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times((134m+12m^2-21m^3-18m^4+3m^5+60)m^2+30m^2+(37m+60)mm)$$
$$+2\frac{((2m+p)+1)^2(2m+p)^3}{m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(6(37m+150)mm+180m^2+(1820m+711m^2+387m^3-243m^4+9m^5+1800)m^2$$
$$-2\frac{((2m+p)+1)^2(2m+p)^3}{m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(-15(2m-15)m^2m^4-3(104m^2-38m-26m^3+m^4-24)mm^3$$
$$-3(62m+146m^2+81m^3+208m^4+31m^5+60)m^2)$$
$$-2((2m+p)+1)^2\frac{-(2m+p)^3(740m-2475m^2-954m^3-456m^4+69m^5+900)mm}{m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$-2\frac{((2m+p)+1)^2(2m+p)^2}{m^2m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(15(m^2-30m+85)m^2m^6+(56m-81m^2-2064m^3+1131m^4-57m^5+60)m^5$$
$$+3(148m-387m^2-534m^3-163m^4+10m^5+240)mm^4)$$
$$-2\frac{((2m+p)+1)^2(2m+p)^2}{m^2m^2p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times((4286m-567m^2+894m^3-891m^4+45m^5+3000)m^2m^3$$
$$-6(14m^2-23m-192m^3+21m^4-500)m^3m^2+2(37m+360)mm^4+60m^5)$$
$$-2m^3\frac{((2m+p)+1)^2(2m+p)}{mp^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$
$$\times(75(3m^2-34m+45)mm-(434m+4761m^2-6384m^3+660m^4+15m^5+180))$$
$$-6m\frac{((2m+p)+1)^2(2m+p)}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)}$$

$$
\begin{aligned}
&\times ((354m^2 - 536m - 587m^3 + 120m^4 - 660)m \\
&\quad + (210m^4 - 68m^2 - 1303m^3 - 896m - 840)m) \\
&- 2m^2((2m+p)+1)^2 \frac{(2m+p)((1258m + 2133m^2 - 639m^3 - 3060)m - 180m)}{mp^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)} \\
&- 6m^3 \frac{((2m+p)+1)^2}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)} \\
&\times (5(85m^2 - 450m + 274)m + (5022m^2 - 1730m - 1195m^3 - 60m^4 + 160)) \\
&- 6 \frac{((2m+p)+1)^2}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)} \\
&\times (m(2052m - 1673m^2 + 470m^3 + 120)m^2 + 3m^2(355m^3 - 669m^2 - 550m + 40)m \\
&\quad - 2m^3(231m^2 - 129m - 440)) \\
&- 30 \frac{((2m+p)+1)^2}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)(2m+p)} \\
&\times (m^4(225m^2 - 548m + 120) - m(580m^2 - 980m + 47m^3 + 120)m^3 \\
&\qquad + 6m^2(20m^2 - 85m + 72)m^2 + 6m^3(5m + 77m^2 - 60)m - 72m^4(m+1)) \\
&- 60mm \frac{((2m+p)+1)^2}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)(2m+p)^2} \\
&\times (m^3(137m - 120) + m(120 - 30m^2 - 287m)m^2 - 180mm^2 + 180m^3(m+1)) \\
&+ 3600m^3m^2 \frac{((2m+p)+1)^2(m-m)}{p^3((2m+p)+5)((2m+p)+4)((2m+p)+3)((2m+p)+2)(2m+p)^3} \\
&= -\frac{2}{m^2p^3(2m+p)^2} \frac{(2m+p+1)^2}{(2m+p+2)(2m+p+3)(2m+p+4)(2m+p+5)} \\
&\times (17\,892m^8 + 53\,046m^9 + 60\,210m^{10} - 120p^7 + 32\,760m^{11} + 8568m^{12} \\
&+ 864m^{13} - 600mp^6 - 960m^6p - 148mp^7 + 52\,488m^7p + 169\,425m^8p \\
&+ 208\,215m^9p + 121\,620m^{10}p + 33\,852m^{11}p + 3600m^{12}p - 1080m^2p^5 \\
&- 840m^3p^4 - 240m^4p^3 - 480m^5p^2 - 842m^2p^6 - 640m^3p^5 + 7678m^4p^4 \\
&+ 31\,712m^5p^3 + 58\,324m^6p^2 + 48m^2p^7 + 1143m^3p^6 + 11\,082m^4p^5 + 53\,787m^5p^4 \\
&+ 144\,621m^6p^3 + 217\,224m^7p^2 + 18m^2p^8 + 438m^3p^7 + 4740m^4p^6 + 28\,590m^5p^5 \\
&+ 103\,635m^6p^4 + 228\,381m^7p^3 + 297\,021m^8p^2 + 33m^3p^8 + 633m^4p^7 + 5490m^5p^6 \\
&+ 27\,480m^6p^5 + 84\,855m^7p^4 + 163\,251m^8p^3 + 189\,846m^9p^2 + 18m^4p^8 + 318m^5p^7 \\
&+ 2493m^6p^6 + 11\,145m^7p^5 + 30\,783m^8p^4 + 53\,505m^9p^3 + 56\,958m^{10}p^2 + 3m^5p^8 \\
&+ 51m^6p^7 + 375m^7p^6 + 1557m^8p^5 + 3990m^9p^4 + 6456m^{10}p^3 + 6432m^{11}p^2) \\
&= -\frac{2}{m^2p^3(2m+p)^2} \frac{(2m+p+1)^2}{(2m+p+2)(2m+p+3)(2m+p+4)(2m+p+5)} \\
&\times (864m^{13} + m^{12}(3600p + 8568) \\
&+ m^{11}(33\,852p + 6432p^2 + 32\,760) \\
&+ m^{10}(121\,620p + 56\,958p^2 + 6456p^3 + 60\,210) \\
&+ m^9(208\,215p + 189\,846p^2 + 53\,505p^3 + 3990p^4 + 53\,046) \\
&+ m^8(169\,425p + 297\,021p^2 + 163\,251p^3 + 30\,783p^4 + 1557p^5 + 17\,892) \\
&+ m^7(52\,488p + 217\,224p^2 + 228\,381p^3 + 84\,855p^4 + 11\,145p^5 + 375p^6) \\
&+ m^6(58\,324p^2 - 960p + 144\,621p^3 + 103\,635p^4 + 27\,480p^5 + 2493p^6 + 51p^7) \\
&+ m^5(31\,712p^3 - 480p^2 + 53\,787p^4 + 28\,590p^5 + 5490p^6 + 318p^7 + 3p^8) \\
&+ m^4(7678p^4 - 240p^3 + 11\,082p^5 + 4740p^6 + 633p^7 + 18p^8) \\
&+ m^3(1143p^6 - 640p^5 - 840p^4 + 438p^7 + 33p^8) \\
&+ m^2(48p^7 - 842p^6 - 1080p^5 + 18p^8)
\end{aligned}
$$

$$+ m(-600p^6 - 148p^7)$$
$$- 120p^7)$$
Factoring and collecting terms yield Corollary 7.1(ii).

Corollary 7.2. If $m = n = p$, that is $t = 3m$ and $k = \dfrac{(t+1)^2}{t_{\{4\}}p^2} = \dfrac{(3m+1)}{(3m^3)(3m+2)(3m+3)}$

then (i) $E(Q^2) = \dfrac{56}{3}\dfrac{(3m+1)}{(3m+2)}$

(ii) $E(Q^3) = -\dfrac{16}{3}(3m+1)^2\dfrac{(324m^5 + 2421m^4 + 6273m^3 + 6661m^2 + 2372m - 60)}{(3m+5)(3m+4)(3m+2)}$.

Proof. (i) Substituting $m = n = p$, that is $t = 3m$ and $k = \dfrac{(t+1)^2}{t_{\{4\}}p^2} = \dfrac{(3m+1)}{(3m^3)(3m+2)(3m+3)}$

into Theorem 7.1(i), we obtain the following expression for kurtosis.

$$\begin{aligned}
E(Q^2) &= 2((3m)+1)\frac{(3m+3m+4mm)(3m)^5 + 12m^2m^2(6m+6m+mm)(3m) - 72m^3m^3}{m^2mm((3m)+3)((3m)+2)(3m)^2} \\
&\quad - 4((3m)+1)(3m)^2\frac{3m^2 + (6m+3m^2+m^3+12)mm + (2m-3m^2-m^3+3)m^2}{m^2mm((3m)+3)((3m)+2)} \\
&\quad + 2\frac{((3m)+1)(3m)}{m^2mm((3m)+3)((3m)+2)} \times ((3-2m^3-6m^2)m^3 \\
&\qquad + 2(21m+6m^2+m^3+24)mm^2 - 2(7m+3m^2-24)m^2m + 3m^3) \\
&\quad + 6((3m)+1)\frac{(-10m-2m^2-9)m^2 + 2(m+m^2-15)mm - 13m^2}{((3m)+3)((3m)+2)m^2} \\
&= \frac{1}{5m+3m^2+2}(-328m - 48m^2 - 104) \\
&\quad + \frac{1}{5m+3m^2+2}(668m + 168m^2 + 204) \\
&\quad + \frac{1}{5m+3m^2+2}(-744m - 288m^2 - 216) \\
&\quad + \frac{1}{15m+9m^2+6}(1436m + 672m^2 + 404).
\end{aligned}$$

With some algebraic manipulations, the above simplifies to Corollary 7.2(i).

(ii) Substituting $p = m$, that is ($t = 3m,\ m = n = p$ and $k = \dfrac{(3m+1)}{(3m^3)(3m+2)(3m+3)}$)

into Corollary 7.1(ii), we obtain the following expression for skewness.

$$\begin{aligned}
E(Q^3) &= -\frac{2}{m^2m^3(2m+m)^2}\frac{(2m+m+1)^2}{(2m+m+2)(2m+m+3)(2m+m+4)(2m+m+5)} \\
&\quad \times (864m^{13} + m^{12}(3600m + 8568) + m^{11}(33\,852m + 6432m^2 + 32\,760) \\
&\qquad + m^{10}(121\,620m + 56\,958m^2 + 6456m^3 + 60\,210) \\
&\qquad + m^9(208\,215m + 189\,846m^2 + 53\,505m^3 + 3990m^4 + 53\,046) \\
&\qquad + m^8(169\,425m + 297\,021m^2 + 163\,251m^3 + 30\,783m^4 + 1557m^5 + 17\,892) \\
&\qquad + m^7(52\,488m + 217\,224m^2 + 228\,381m^3 + 84\,855m^4 + 11\,145m^5 + 375m^6) \\
&\qquad + m^6(58\,324m^2 - 960m + 144\,621m^3 + 103\,635m^4 + 27\,480m^5 + 2493m^6 + 51m
\end{aligned}$$

$$+ m^5(31\,712m^3 - 480m^2 + 53\,787m^4 + 28\,590m^5 + 5490m^6 + 318m^7 + 3m^8)$$
$$+ m^4(7678m^4 - 240m^3 + 11\,082m^5 + 4740m^6 + 633m^7 + 18m^8)$$
$$+ m^3(1143m^6 - 640m^5 - 840m^4 + 438m^7 + 33m^8)$$
$$+ m^2(48m^7 - 842m^6 - 1080m^5 + 18m^8) + m(-600m^6 - 148m^7) - 120m^7).$$

Some algebraic manipulations of the above yield corollary 7.2(ii).

8. The Shannon's Entropy Function

The Shannon Entropy for any bivariate density function is defined by $H(f) = -E[\ln f(x,y)]$.

Theorem 8.1. Let $f(x,y)$ be the pdf of bivariate Dirichlet distribution given by (2.2). Then the Shannon Entropy for $f(x,y)$ is defined by Nadarajah and Zogfaros (2005) as

$$\begin{aligned} H(f) &= \ln(\Gamma(m)\Gamma(n)\Gamma(p)) + (m+n+p-3)\Psi(m+n+p) \\ &\quad - \ln\Gamma(m+n+p) - (m-1)\Psi(m) - (n-1)\Psi(n) - (p-1)\Psi(p). \end{aligned}$$

where $\Psi(w) = \frac{d}{dw}\ln\Gamma(w)$.

9. Bivariate Dependence Function

Sankaran and Gupta (2004) defined the local dependence function for any bivariate pdf as

$$\frac{\partial^2 \ln(f(x,y))}{\partial x \partial y}.$$

Theorem 9.1. Let $f(x,y)$ be the pdf of bivariate Dirichlet distribution given by (2.2). Then, the local dependence function for the bivariate Dirichlet distribution is

$$\frac{\partial^2 \ln(f(x,y))}{\partial x \partial y} = \frac{1-p}{(1-x-y)^2} \qquad \text{where } p > 0, x \ge 0, y \ge 0, \text{ and } x+y \le 1.$$

Proof. The local dependence function for the bivariate Dirichlet distribution is

$$\begin{aligned} \frac{\partial^2 \ln(f(x,y))}{\partial x \partial y} &= \frac{\partial^2 \ln\left(\frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1} (1-x-y)^{p-1}\right)}{\partial x \partial y} \\ &= \frac{\partial^2\left(\ln\left(\frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)}\right) + (m-1)\ln(x) + (n-1)\ln(y) + (p-1)\ln(1-y-x)\right)}{\partial x \partial y} \\ &= \frac{1-p}{2xy - 2y - 2x + x^2 + y^2 + 1} \end{aligned}$$

which is equivalent to the theorem.

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