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# On the Distributions of Norms of Spherical Distributions

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The paper deals with the norms of distributions of several members of spherical distributions. Some functions of norms have standard distributions. Moments of the norms of different spherical distributions are derived. Covariance matrices of related elliptical distributions are also derived with the moments of norms in an accessible and elegant manner.

**Key words:** Spherical distributions, multivariate normal distribution, uniform distributions on or inside spheres, multivariate  $t$ -distribution, multivariate Cauchy distribution, multivariate Pearson type II distribution.

**AMS Mathematics Subject Classification:** 60-01, 60E10

## 1 Introduction

The distribution of the norm of spherical distribution is known in its general form. We specialize it to several members of spherical distributions namely multivariate normal distribution, uniform distribution on or inside hyper spheres, multivariate  $t$ -distribution, multivariate Cauchy distribution and multivariate Pearson type II distribution. Some functions of norms are found to have standard distributions. Moments of norms of different spherical distributions are derived. They are then used to derive covariance matrices of related elliptical distributions.

A  $p$ -dimensional random variable  $\mathbf{Z}$  is said to have a spherical distribution if its

probability density function (p.d.f.) is given by

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}). \quad (1.1)$$

Muirhead (1982) is the first to discuss spherical and elliptical distributions in a text book of multivariate analysis. Much of the theoretical development are available in Fang and Anderson (1990) and Fang, Kotz and Ng (1990). For applications of such distributions we refer to Lange, Little and Taylor (1989), Billah and Saleh (2000), Kibria and Saleh (2000), Kibria (2003), Kotz and Nadarajah (2004) and Kibria (2006) and the references therein.

The distribution of  $R = (\mathbf{Z}'\mathbf{Z})^{1/2} = \|\mathbf{Z}\|$ , called the norm of the distribution of  $\mathbf{Z}$ , is known (see e.g. Muirhead, 1982, pp.36-37). The following theorem of the distribution of  $R$  is originally due to Goldman (1974) and Goldman (1976).

**Theorem 1.1.** Let  $\mathbf{z}' = (z_1, z_2, \dots, z_p)$ ,  $p \geq 2$ . Consider the transformation to spherical coordinates for  $\mathbf{z}$

$$\begin{aligned} z_i &= r \left( \prod_{K=1}^p \sin \theta_k \right) \cos \theta_i, \quad i = 1, 2, \dots, p-1 \\ z_p &= r \left( \prod_{K=1}^p \sin \theta_k \right) \sin \theta_{p-1} \end{aligned} \quad (1.2)$$

where  $0 \leq r < \infty$ ;  $0 \leq \theta_k < \pi$ ,  $k = 1, 2, \dots, p-2$ ;  $0 \leq \theta_{p-1} < 2\pi$ . Then  $R, \Theta_1, \Theta_2, \dots, \Theta_{p-1}$  are independent and have respectively the following probability functions:

$$\begin{aligned} h(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} g(r^2), \quad 0 \leq r < \infty \\ u(\theta_K) &= \frac{1}{B\left(\frac{1}{2}, \frac{p-k}{2}\right)} \sin^{p-k-1} \theta_k, \quad 0 \leq \theta_k < \pi, \\ &\quad k = 1, 2, \dots, p-2, \\ v(\theta_{p-1}) &= \frac{1}{2\pi} \quad 0 \leq \theta_{p-1} < 2\pi. \end{aligned} \quad (1.3)$$

Conversely, if  $R, \Theta_1, \Theta_2, \dots, \Theta_{p-1}$  are independent and have probability density functions given by (1.3) and  $\mathbf{z}$  is defined by (1.2), then  $\mathbf{Z}$  has a spherical distribution.

The distribution of  $Y = R^2$  is obviously

$$w(y) = \frac{\pi^{p/2}}{r(p/2)} y^{p/2-1} g(y). \quad (1.4)$$

The  $k$ -th moment of  $R$  is given by

$$E(R^k) = \int_0^\infty h(r) dr = \int_0^\infty \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p+k-1} g(r^2) dr. \quad (1.5)$$

It should be warned that the integral

$$\int_0^\infty r^{p-1} g(r^2) dr = \frac{\Gamma(p/2)}{2\pi^{p/2}}$$

which follows from (1.3) is exclusive for  $p - 1$  as the exponent of  $r$ , and as such cannot be used for the evaluation of the integral in (1.5). Other alternative is to label the quantity  $g(r^2)$  as  $g_p(r^2)$  (see Joarder and Ali, 1992).

In section 2 we provide the distribution of norm or some functions of norm and derive moments for norms of spherical distribution. In section 3 we find product moments of some spherical distributions. In section 4 we derive covariance matrices of related elliptical distributions of  $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \mathbf{Z}$  with the help of the moment of norm or of characteristic functions of the elliptical distributions.

## 2 Distributions of the Norms of Spherical Distributions

In this section we sketch distributions of some functions of norm of different spherical distributions. Most of these results are originally due to Fang, Kotz and Ng (1990). Throughout the book, the authors emphasized the so called characteristic generator rather than the usual probability density function. The purpose of this section is to sketch accessible proofs of the well known results on norms.

**Theorem 2.1** Let  $\mathbf{Z}$  have the multivariate normal distribution given by

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}) = (2\pi)^{-p/2} \exp\left(-\frac{1}{2} \mathbf{z}'\mathbf{z}\right). \quad (2.1)$$

Then  $R^2 \sim \chi_p^2$  and that

$$E(R^k) = 2^{k/2} \frac{\Gamma((p+k)/2)}{\Gamma(p/2)} \quad (2.2)$$

(cf. Fang, Kotz and Ng, 1990, 32).

**Proof.** It follows from (1.3) that the p.d.f. of  $R$  is given by

$$\begin{aligned} h(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} \left( (2\pi)^{-p/2} e^{-\frac{1}{2} r^2} \right) \\ &= \frac{2^{-p/2+1}}{\Gamma(p/2)} r^{p-1} e^{-\frac{1}{2} r^2}. \end{aligned}$$

The rest can be proved by making a transformation. In particular  $E(R^2) = p$ .

**Theorem 2.2** Let  $\mathbf{Z}$  have the uniform distribution on unit hypersphere with p.d.f.

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}) = \frac{I_T(\mathbf{z})}{S(p, 1)} \quad (2.3)$$

where  $S(p, 1) = \frac{2\pi^{p/2}}{\Gamma(p/2)}$  is the surface area of a unit hypersphere and  $I_T(\mathbf{z})$  is the indicators function of the set  $T = \{\mathbf{z} : \mathbf{z}'\mathbf{z} = 1\}$ . Then  $P(R = 1) = 1$ .

**Proof.** It follows from (1.3) that the p.d.f. of  $R$  is given by

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} 1^{p-1} \left( \frac{\Gamma(p/2)}{2\pi^{p/2}} \right)$$

i.e.

$$h(r) = 1 \quad \text{for } r = 1. \quad (2.4)$$

Obviously the  $k$ -th moment of  $R$  is  $E(R^k) = 1$ .

**Theorem 2.3** Let  $\mathbf{Z}$  have the uniform distribution inside unit hyper-sphere with p.d.f.

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}) = \frac{I_T(\mathbf{z})}{V(p, 1)} \quad (2.5)$$

where  $V(p, 1) = \frac{\pi^{p/2}}{\Gamma(p/2+1)}$ , the volume of a  $p$ -dimensional unit hyper-sphere, and  $I_T(\mathbf{z})$  is the indicators function of the set  $T = \{\mathbf{z} : \mathbf{z}'\mathbf{z} \leq 1\}$ .

Then  $R$  has a Beta distribution with parameter  $p$  and 1 i.e.  $R \sim B(p, 1)$ , and that

$$E(R^k) = \frac{B(k+p, 1)}{B(p, 1)} = \frac{p}{p+k}. \quad (2.6)$$

**Proof.** It follows from (1.3) that the p.d.f. of  $R$  is given by

$$\begin{aligned} h(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} \left( \frac{r(p/2+2)}{\pi^{p/2}} \right) \\ &= p r^{p-1}, \quad p \geq 2. \end{aligned}$$

**Theorem 2.4** Let  $\mathbf{Z}$  have the multivariate  $t$ -distribution with p.d.f.

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}) = \frac{1}{K(\nu, p)\pi^{p/2}} \left( 1 + \frac{\mathbf{z}'\mathbf{z}}{\nu} \right)^{-(\nu+p)/2} \quad (2.7)$$

where  $K(\nu, p)$  is given by  $c(\nu, p)\Gamma((\nu+p)/2) = \nu^{p/2}\Gamma(\nu/2)$ . Then

$$\frac{R^2}{p} \sim F(p, \nu)$$

and that

$$E(R^k) = \nu^{k/2} \frac{\Gamma((p+k)/2)\Gamma((\nu-k)/2)}{\Gamma(p/2)\Gamma(\nu/2)}, \quad \nu > k \quad (2.8)$$

(cf. Fang, Kotz and Ng, 1990, 22).

**Proof.** It follows from (1.3) that the p.d.f. of  $R$  is given by

$$\begin{aligned} h(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} \left( \frac{1}{K(\nu, p)\pi^{p/2}} \right) \left( 1 + \frac{r^2}{\nu} \right)^{-(\nu+p)/2} \\ &= \frac{2r^{p-1}}{\nu^{p/2}B(p/2, \nu/2)} \left( 1 + \frac{r^2}{\nu} \right)^{-(\nu+p)/2}. \end{aligned}$$

The rest is immediate by virtue of a simple transformation, and routine algebra. In particular,

$$E(R^2) = \frac{\nu p}{\nu-2}, \quad \nu > 2 \quad (2.9)$$

and  $V(R^2) = \frac{2p(p+\nu-2)\nu^2}{(\nu-2)(\nu-4)^2}$ ,  $\nu > 4$ .

As  $\nu \rightarrow \infty$ ,  $E(R^2) = p$  and  $v(R^2) = 2p$  which are the mean and variance respectively of  $R^2 \sim \chi_p^2$ .

**Theorem 2.5** Let  $\mathbf{Z}$  have the Multivariate Pearson Type II distribution with p.d.f.

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z}) = K(\alpha, p)(1 - \mathbf{z}'\mathbf{z})^{(\alpha-p)/2} \quad (2.10)$$

where

$$K(\alpha, p) = \frac{\Gamma(\alpha/2 + 1)}{\pi^{\alpha/2} \Gamma((\alpha - p)/2 + 1)}.$$

Then

$$R^2 \sim \text{Beta}\left(\frac{p}{2}, \frac{\alpha - p}{2} + 1\right)$$

and that

$$E(R^k) = \frac{B\left(\frac{k+p}{2}, \frac{\alpha-p}{2} + 1\right)}{B\left(\frac{p}{2}, \frac{\alpha-p}{2} + 1\right)} \quad (2.11)$$

(cf. Fang, Kotz and Ng, 1990, 89).

**Proof.** It follows from (1.3) that the p.d.f. of  $R$  is given by

$$h(r) = \frac{2}{B\left(\frac{p}{2}, \frac{\alpha - p + 2}{2}\right)} r^{p-1} (1 - r^2)^{(\alpha-p)/2}, 0 \leq r \leq 1.$$

The rest of the proof is immediate by virtue of a simple transformation. In particular  $E(R^2) = p/\alpha + 2$ .

### 3 Deriving Product Moments with the Help of Norms

In this section we discuss the mixed moments of spherical distributions most of which are discussed in Fang, Kotz and Ng (1990).

**Theorem 3.1.** Let  $Z$  have a spherical distribution given by (1.1). Then for any integers  $k_1, k_2, \dots, k_p$  with  $k = \sum_{i=1}^p k_i$ , the product moment is given by

$$E\left(\prod_{i=1}^p \mathbf{z}_i^{k_i}\right) = \begin{cases} 0 & \text{if at least one } k_i (i = 1, 2, \dots, p) \text{ is odd} \\ E(R^k) \frac{\Gamma(p/2)}{2^K \Gamma((k+p)/2)} \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!} & \text{if all } k_i (i = 1, 2, \dots, p) \text{ are even.} \end{cases}$$

1. Product moment of spherical normal distribution (see equation 2.1).

$E\left(\prod_{i=1}^p \mathbf{Z}_i^{k_i}\right) = 0$  if any  $k_i (i = 1, 2, \dots, p)$  is odd. However if all  $k_i$ 's are even, then it follows from Theorem 3.1 that

$$E\left(\prod_{i=1}^p \mathbf{Z}_i^{k_i}\right) = \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!}.$$

2. Product moments of uniform distribution on unit hyper-sphere (see equation 2.4).

If all  $k_i$ 's are even then it follows from theorem 3.1 and equation (2.5) that

$$E\left(\prod_{i=1}^p \mathbf{Z}_i^{k_i}\right) = \frac{\Gamma(p/2)}{2^K \Gamma((k+p)/2)} \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!},$$

(cf. Fang, Kotz and Ng, 1990, 72).

3. Product moments of uniform distribution inside unit hyper-sphere (see equation 2.6).

If all  $k_i$ 's are even then

$$E\left(\prod_{i=1}^p \mathbf{Z}_i^{k_i}\right) = \frac{p}{p+k} \frac{\Gamma(p/2)}{2^k \Gamma((K+p)/2)} \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!},$$

(cf. Fang, Kotz and Ng, 1990, 75).

4. Product moments of multivariate  $t$ -distribution (see equation 2.7).

If all  $k_i$ 's are even then, it follows from Theorem 3.1 and equation (2.9) that

$$E\left(\prod_{i=1}^p \mathbf{Z}_i^{k_i}\right) = \frac{\nu^{k/2} \Gamma((\nu-k)/2)}{2^K \Gamma(\nu/2)} \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!},$$

(Fang, Kotz and Ng, 1990, 88) which was also derived by Joarder (1992) laboriously by differentiating the characteristic function of the multivariate  $t$ -distribution.

5. Product moment of Multivariate Pearson Type II distribution (see equation 2.10).

If all  $k_i$ 's are even, it follows from Theorem 3.1 and (2.11) that

$$E\left(\prod_{i=1}^p \mathbf{Z}_i^{m_i}\right) = \frac{\Gamma(\alpha/2 + 1)}{2^k \Gamma((k+\alpha)/2 + 1)} \prod_{i=1}^p \frac{k_i!}{\left(\frac{k_i}{2}\right)!}.$$



## 4 Covariance Matrices of Related Elliptical Distributions

Consider the elliptical random variable  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (1.1). It is well known (Cambanis, Hunag and Simons, 1981) that the covariance matrix of  $\mathbf{X}$  is given by  $Cov(\mathbf{X}) = -2\psi'_{\mathbf{X}}(\mathbf{0})\boldsymbol{\Sigma}$  where  $\psi_{\mathbf{X}}(t) = \exp(it'\boldsymbol{\theta})\psi(\|\boldsymbol{\Sigma}^{1/2}t\|)$  is the characteristic function of  $\mathbf{X}$ .

Since most elliptical distributions do not have closed form for characteristic functions, an easy way out is to exploit stochastic decomposition  $\boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\theta}) = \mathbf{Z} = R\mathbf{U}$  where  $R = (\mathbf{Z}'\mathbf{Z})^{1/2}$  is independent of  $\mathbf{U}$  and the random variable  $\mathbf{U}$  is uniformly distributed on the surface of unit sphere is  $\mathcal{R}^p$ .

For any elliptical random variable  $\mathbf{X}$  where  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  with  $\mathbf{Z}$  having the p.d.f.(1.1), it is well known that  $Cov(\mathbf{X}) = \frac{1}{p}E(R^2)\boldsymbol{\Sigma}$  (Cambanis, Huang and Hsu, 1989, or Joarder, 1992). In this section we outline how the covariance matrix of elliptical distributions can be derived by the above result.

(i) Multivariate normal distribution

Let  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (2.1). Then it follows from (2.2) that  $Cov(\mathbf{X}) = \frac{1}{p}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}$ .

(ii) Uniform distribution on unit hyper-ellipse.

Let  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (2.3) then  $Cov(\mathbf{X}) = \frac{1}{p}(1)\boldsymbol{\Sigma} = \frac{1}{p}\boldsymbol{\Sigma}$ .

(iii) Uniform distribution inside unit hyper-ellipse.

Let  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (2.5). Then it follows from (2.6) that  $Cov(\mathbf{X}) = \frac{1}{p}\left(\frac{p}{p+2}\right)\boldsymbol{\Sigma} = \frac{1}{p+2}\boldsymbol{\Sigma}$ .

(iv) Multivariate  $t$ -distribution

Let  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (2.7). Then it follows from (2.8) that  $Cov(\mathbf{X}) = \frac{1}{p}\left(\frac{\nu p}{\nu-2}\right)\boldsymbol{\Sigma} = \frac{\nu}{\nu-2}\boldsymbol{\Sigma}$ .

(v) Multivariate Pearson type II distribution

Let  $\mathbf{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2} \mathbf{Z}$  where  $\mathbf{Z}$  has the p.d.f. given by (2.10). Then it follows from (2.11) that  $Cov(\mathbf{X}) = \frac{1}{p} \left( \frac{p}{p+2} \right) \boldsymbol{\Sigma} = \frac{1}{p+2} \boldsymbol{\Sigma}$ .

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