King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 325

March 2005

Hopf Pairings and (Co)induction Functors over Commutative Rings

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Abstract

(Co)induction functors appear in several areas of Algebra in different forms. Interesting examples are the so called induction functors in the Theory of Affine Algebraic Groups. In this paper we investigate Hopf pairings (bialgebra pairings) and use them to study (co)induction functors for affine group schemes over arbitrary commutative ground rings. We present also a special type of Hopf pairings (bialgebra pairings) satisfying the so called α-condition. For those pairings the *coinduction functor* is studied and nice descriptions of it are obtained. Along the paper several interesting results are generalized from the case of base fields to the case of arbitrary commutative (Noetherian) ground rings.

Introduction

Hopf pairings (respectively *bialgebra pairings*) were presented by M. Takeuchi [35, Page 15] (respectively S. Majid [26, 1.4]). With the help of these, several authors studied affine group schemes and quantum groups over arbitrary commutative ground rings (e.g. [16], [33], [31]). In this paper we study the category of Hopf pairings $\mathcal{P}_{Hopf}$ and the category of bialgebra pairings $\mathcal{P}_{Big}$ over an arbitrary commutative base ring. In the case of Noetherian base rings we present the *full* subcategories $\mathcal{P}^{\alpha}_{Hopf} \subset \mathcal{P}_{Hopf}$ (respectively $\mathcal{P}^{\alpha}_{Big} \subset \mathcal{P}_{Big}$) of Hopf pairings (respectively bialgebra pairings) satisfying the so called *α-condition,* see 1.4. For those a *coinduction functor* is presented and an interesting description of it is obtained.

The paper is divided into seven sections. The first section includes some preliminaries about the so called *measuring α-pairings*, *rational modules* and *dual coalgebras*. In the

*MSC (2000): 16W30, 14L17, 16W35, 20G42
Keywords: Hopf Pairings, Bialgerba Pairings, Dual Bialgebras, Dual Hopf Algebras, Affine Group Schemes, Quantum Groups, Coinduction Functors, Cotensor Product.

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second section we consider the cotensor functor and prove some properties of it that will be used in later sections. In the third section we introduce the coinduction functor in the category of measuring α-pairings and prove mainly that it can be obtained as a special case of a more general coinduction functor between categories of type $\sigma[M]$. Hopf pairings (bialgebra pairings) are presented in the fourth section, where an algebraically topological approach is used and several duality theorems are proved. In the fifth section we consider the category of Hopf α-pairings (bialgebra α-pairings) and generalize known results on *admissible* Hopf pairings over Dedekind domains to the case of *quasi admissible* bialgebra α-pairings and Hopf α-pairings over arbitrary commutative Noetherian ground rings. There the coinduction functor is also studied and different forms of it that appear in the literature are shown to be equivalent. The classical duality between groups and commutative Hopf algebras (e.g. [28], [30]) is the subject of the sixth section. In the seventh and last section we apply results obtained in the previous sections to affine group schemes over arbitrary commutative rings.

Throughout this paper R denotes a commutative ring with $1_R \neq 0_R$. We consider R as a left (and a right) linear topological ring with the discrete topology. All R-modules are assumed to be unital and category of R-(bi)modules will be denoted by $\mathcal{M}_R$. With unadorned $\mathrm{Hom}(-,)$ and $- \otimes -$ we mean $\mathrm{Hom}_R(-,)$ and $- \otimes_R -$ respectively. For an R-module M we call an R-submodule $K \subset M$ *pure* (in the sense of Cohn), if the canonical mapping $\iota_K \otimes_R id_N : K \otimes_R N \to M \otimes_R N$ is injective for every R-module N. For an R-module M and subsets $X \subset M$ (respectively $Y \subset M^*$) we set

$$\mathrm{An}(X) := \{f \in M^* : f(X) = 0\} \text{ (respectively } \mathrm{Ke}(Y) := \{m \in M : f(m) = 0 \ \forall \ f \in Y\}).$$

Let A be an R-algebra (not necessarily with unity). A left A-module is said to be *faithful*, if $\mathrm{An}({}_AM) := \{a \in A : aM = 0\} = (0_A)$. We define a left A-module M to be *A-faithful* (respectively *unital*), if the canonical map $\rho : M \to \mathrm{Hom}_R(A, M)$ is injective (respectively, if $AM = M$). With $\widetilde{{}_A\mathcal{M}}$ (respectively ${}_A\mathcal{M}$) we denote the category of A-faithful (respectively *unital*) left A-modules and left A-linear maps. The categories of A-faithful (respectively unital) right A-modules $\widetilde{\mathcal{M}_A}$ (respectively $\mathcal{M}_A$) are analogously defined. If A has unity, then obviously every unital left (or right) A-module is A-faithful. For an R-algebra A and an A-module M, an A-submodule $N \subset M$ will be called *R-cofinite*, if M/N is finitely generated in $\mathcal{M}_R$. Unless otherwise explicitly mentioned, we assume that all R-algebras have unities respected by R-algebra morphisms and that all modules of R-algebras are unital.

We assume the reader is familiar with the theory and notation of Hopf Algebras. For any needed definitions the reader may refer to any of the classical books on the subject (e.g. [1], [28], [30]) or to the recent monograph [12] for the theory of coalgebras over arbitrary base rings. For an R-coalgebra C, we call a right (respectively a left) C-comodule (M, ϱ_M) *counital* if its structure map ϱ_M is injective, compare [13, Lemma 1.1.]. For an R-coalgebra C we denote with $\mathcal{M}^C$ (respectively ${}^C\mathcal{M}$) the category of *counital* right (respectively left) C-comodules. For an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an R-algebra (A, μ_A, η_A) we consider $\mathrm{Hom}_R(C, A)$ as an R-algebra with multiplication the so called *convolution product* $(f \star g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$.

1 Preliminaries

In this section we present some definitions and lemmas to be referred to later in the paper.

1.1. Subgenerators. Let A be an R-algebra (not necessarily with unity) and K be a left A-module. We say a left A-module N is *K-subgenerated*, if N is isomorphic to a submodule of a K-generated left A-module (equivalently, if N is kernel of a morphism between K-generated left A-modules). The *full* subcategory of ${}_A\mathcal{M}$, whose objects are the K-subgenerated left A-modules is denoted by $\sigma[{}_AK]$. In fact $\sigma[{}_AK] \subseteq {}_A\mathcal{M}$ is the *smallest* Grothendieck full subcategory that contains K. If M is a left A-module, then

$$\mathrm{Sp}(\sigma[{}_AK], M) := \sum\{f(N) \mid \; f \in \mathrm{Hom}_{A-}(N, M),\; N \in \sigma[{}_AK]\}$$

is the largest A-submodule of M that belongs to $\sigma[{}_AK]$. The subcategory $\sigma[{}_AK] \subseteq {}_A\mathcal{M}$ can also be seen as the category of discrete left A-modules, where A is considered as a left linear topological R-algebra with the *K-adic topology* (e.g. [10]). The reader is referred to [38] and [37] for the well developed theory of categories of this type.

An important result to which we will often refer is

Lemma 1.2. ([38, 15.8], [12, 42.2]) *Let A be a ring, K be a faithful left A-module and $B \subset A$ be a subring. Then $\sigma[{}_BK] = \sigma[{}_AK]$ if and only if $B \subset A$ is K-dense.*

Remark 1.3. Let C be an R-algebra. Then C^* becomes two (left) linear topologies, the C-adic topology induced by ${}_{C^*}C$ and the finite topology induced by the embedding $C^* \hookrightarrow R^C$. By [4, Lemma 2.2.4] the two topologies coincide.

1.4. The α-condition. With an *R-pairing* $P = (V, W)$ we mean R-modules V, W with an R-linear map $\kappa_P : V \to W^*$ (equivalently $\chi_P : W \to V^*$). For R-pairings (V, W) and (V', W') a morphism $(\xi, \theta) : (V', W') \to (V, W)$ consists of R-linear mappings $\xi : V \to V'$ and $\theta : W' \to W$, such that the induced R-bilinear map

$$V \times W \to R,\; (v, w) \mapsto \;\; < v, w >:= \kappa_P(v)(w) = \chi_P(w)(v)$$

has the property

$$< \xi(v), w' >=< v, \theta(w') > \;\text{ for all } v \in V \text{ and } w' \in W'.$$

We say an R-pairing $P = (V, W)$ satisfies the *α-condition* (or P is an *α-pairing*), if for every R-module M the following mapping is injective:

$$\alpha_M^P : M \otimes_R W \to \mathrm{Hom}_R(V, M),\; \sum m_i \otimes w_i \mapsto [v \mapsto \sum m_i < v, w_i >]. \qquad (1)$$

We say an R-module W satisfies the *α-condition*, if (W^*, W) satisfies the α-condition (equivalently, if ${}_RW$ is *locally projective* in the sense of Zimmermann-Huisgen [39, Theorem 2.1], [17, Theorem 3.2]). With $\mathcal{P}$ we denote the category of R-pairing with morphisms of pairings described above and with $\mathcal{P}^\alpha \subseteq \mathcal{P}$ the full subcategory of R-pairings satisfying the α-condition.

Remark 1.5. ([2, Remark 2.2]) Let $P = (V, W)$ be an α-pairing. Then $_RW$ is flat and R-cogenerated. If moreover $_RW$ is finitely presented or R is perfect, then $_RW$ turns to be projective.

Lemma 1.6. ([2, Lemma 2.3]) *Let $P = (V, W) \in \mathcal{P}^\alpha$. For every R-module M and every R-submodule $N \subset M$ we have for $\sum m_i \otimes w_i \in M \otimes_R W$:*

$$\sum m_i \otimes w_i \in N \otimes_R W \Leftrightarrow \sum m_i < v, w_i > \quad \in N \text{ for all } v \in V. \tag{2}$$

1.7. Measuring R-pairings. For an R-coalgebra C and an R-algebra A (not necessarily with unity) we call an R-pairing $P = (A, C)$ a *measuring R-pairing*, if the induced mapping $\kappa_P : A \to C^*$ is an R-algebra morphism. In this case C is an A-bimodule through the left and the right A-actions

$$a \rightharpoonup c := \sum c_1 < a, c_2 > \text{ and } c \leftharpoonup a := \sum < a, c_1 > c_2 \text{ for all } a \in A,\ c \in C. \tag{3}$$

Let (A, C) and (B, D) be measuring R-pairings (A and B not necessarily with unities). Then we say an R-pairings morphism $(\xi, \theta) : (B, D) \to (A, C)$ is a *morphism of measuring R-pairings,* if $\xi : A \to B$ is an R-algebra morphism and $\theta : D \to C$ is an R-coalgebra morphism. The category of measuring R-pairings and morphisms described above will be denoted by $\mathcal{P}_m$. If $P = (A, C)$ is a measuring R-pairing, $D \subset C$ is a (pure) R-subcoalgebra and $I \lhd A$ is an ideal with $< I, D >= 0$, then $Q := (A/I, D)$ is a measuring R-pairing, $(\pi_I, \iota_D) : (A/I, D) \to (A, C)$ is a morphism in $\mathcal{P}_m$ and we call $Q \subset P$ a (pure) *measuring R-subpairing.* With $\mathcal{P}_m^\alpha \subset \mathcal{P}_m$ we denote the *full* subcategory of measuring R-pairings satisfying the α-condition. Obviously $\mathcal{P}_m^\alpha \subset \mathcal{P}_m$ is closed under pure measuring R-subpairings.

Rational modules

1.8. Let $P = (A, C)$ be a measuring α-pairing (A not necessarily with unity). Let M be an A-faithful left A-module and consider the injective canonical A-linear mapping $\rho_M : M \to \mathrm{Hom}_R(A, M)$. We put $\mathrm{Rat}^C(_AM) := \rho_M^{-1}(M \otimes_R C)$. If $\mathrm{Rat}^C(_AM) = M$, then M is said to be *C-rational* and we define

$$\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \to M \otimes_R C.$$

For $m \in \mathrm{Rat}^C(_AM)$ with $\varrho_M(m) = \sum_{i=1}^{k} m_i \otimes c_i$ we call $\{(m_i, c_i)\}_{i=1}^k \subset M \times C$ a *rational system* for m. With $\mathrm{Rat}^C(_A\widetilde{\mathcal{M}}) \subseteq {}_A\widetilde{\mathcal{M}}$ we denote the *full* subcategory of C-rational left A-modules. The *full* subcategory of C-rational right A-modules ${}^C\mathrm{Rat}(\widetilde{\mathcal{M}}_A) \subseteq \widetilde{\mathcal{M}}_A$ is analogously defined (we will show in Theorem 1.14 that every C-rational left, respectively right, A-module is unital).

As a preparation for the proof of the main results in this section (Theorems 1.14 and 1.15) and to make the paper more self-contained we begin with some technical lemmas.

Lemma 1.9. *Let $P = (A, C)$ be a measuring α-pairing (A not necessarily with unity). For every A-faithful left A-module M we have:*

1. $\mathrm{Rat}^C({}_AM) \subseteq M$ *is an A-submodule.*

2. *For every A-submodule $N \subset M$ we have* $\mathrm{Rat}^C({}_AN) = N \cap \mathrm{Rat}^C({}_AM)$.

3. $\mathrm{Rat}^C(\mathrm{Rat}^C({}_AM)) = \mathrm{Rat}^C({}_AM)$.

4. *For every $L \in {}_A\widetilde{\mathcal{M}}$ and $f \in \mathrm{Hom}_{A-}(M, L)$ we have* $f(\mathrm{Rat}^C({}_AM)) \subseteq \mathrm{Rat}^C({}_AL)$.

Proof. 1. Let $b \in A$ and $m \in \mathrm{Rat}^C({}_AM)$ with rational system $\{(m_i, c_i)\}_{i=1}^k \subset M \times C$. Then we have for arbitrary $a \in A$:

$$a(bm) = (ab)m = \sum_{i=1}^{k} m_i < ab, c_i >= \sum_{i=1}^{k} m_i < a, bc_i >$$

and so $bm \in \mathrm{Rat}^C({}_AM)$ with rational system $\{(m_i, bc_i)\}_{i=1}^k \subset M \times C$.

2. Clearly $\mathrm{Rat}^C({}_AN) \subseteq N \cap \mathrm{Rat}^C({}_AM)$. On the other hand take $n \in N \cap \mathrm{Rat}^C({}_AM)$ with rational system $\{(m_i, c_i)\}_{i=1}^k \subset M \times C$. Then for arbitrary $a \in A$ we have $\sum_{i=1}^{k} m_i < a, c_i >= an \in N$ and so $n \in \mathrm{Rat}^C({}_AN)$ by Lemma 1.6.

3. Follows from 1. and 2.

4. Let $f : M \to L$ be a morphism of A-faithful left A-modules and take $m \in \mathrm{Rat}^C({}_AM)$ with rational system $\{(m_i, c_i)\}_{i=1}^k \subset M \times C$. Then for arbitrary $a \in A$ we have

$$af(m) = f(am) = f(\sum_{i=1}^{k} m_i < a, c_i >) = \sum_{i=1}^{k} f(m_i) < a, c_i >,$$

i.e. $f(m) \in \mathrm{Rat}^C({}_AL)$ with rational system $\{(f(m_i), c_i)\}_{i=1}^k \subset L \times C$.■

Lemma 1.10. *Let $P = (A, C) \in \mathcal{P}_m$ (A not necessarily with unity).*

1. *If (M, ϱ_M) is a right C-comodule, then M is a left A-module through*

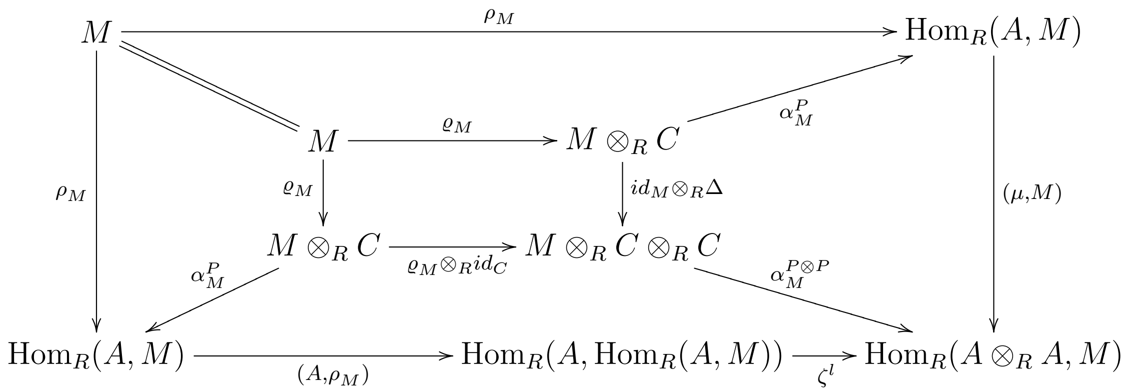
$$\rho_M : M \xrightarrow{\alpha_M^P \circ \varrho_M} \mathrm{Hom}_R(A, M) \qquad (4)$$

If M is counital and A has unity, then ${}_AM$ is unital (and A-faithful).

2. *Let $(M, \varrho_M), (N, \varrho_N)$ be right C-comodules and consider the induced left A-module structures $(M, \rho_M), (N, \rho_N)$ as in (4). If $f : M \to N$ is C-colinear, then f is A-linear.*

3. *Let N be a right C-comodule, $K \subset N$ be a right C-subcomodule and consider the induced left A-module structures (N, ρ_N), (K, ρ_K) as in (4). Then $K \subset N$ is an A-submodule.*

Proof. 1. Set $P \otimes P := (A \otimes_R A, C \otimes_R C)$ and consider the following diagram with commutative trapezoids (where ζ^l the isomorphism given by $\zeta^l(\delta)(a \otimes b) := \delta(b)(a)$):



(5)

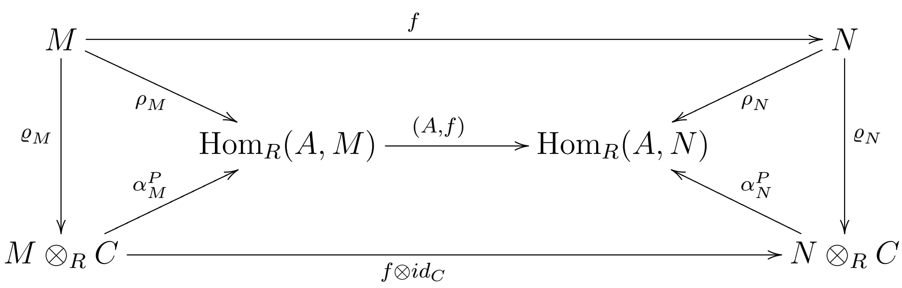
By assumption the internal rectangle is commutative and consequently the outer rectangle is commutative, i.e. (M, ρ_M) is a left A-module.

If M is counital and A has unity, then for every $m \in M$:

$$1_A m = \varepsilon_C m = \sum m_{<0>} \varepsilon_C(m_{<1>}) = m,$$

i.e. ${}_A M$ is unital *(*and A-faithful*)*.

2. Consider the diagram



(6)

The lower trapezoid is obviously commutative. Moreover both triangles are commutative by the definition of ρ_M and ρ_N (4). If f is C-colinear, then the outer rectangle is commutative and consequently the upper trapezoid is commutative, i.e. f is A-linear.

3. Trivial.■

Lemma 1.11. *Let $P = (A, C) \in \mathcal{P}_m^\alpha$ (A not necessarily with unity).*

1. *If* $(M, \rho_M) \in {}_A\widetilde{\mathcal{M}}$ *is* C*-rational, then* M *is a* counital *right* C*-comodule through*

$$\varrho_M : M \xrightarrow{(\alpha_M^P)^{-1} \circ \rho_M} M \otimes_R C \tag{7}$$

2. *Let* $(M, \rho_M), (N, \rho_N) \in {}_A\widetilde{\mathcal{M}}$ *be* C*-rational an consider the induced right* C*-comodule structures* (M, ϱ_M), (N, ϱ_N) *as in* (7). *Then* $\mathrm{Hom}^C(M, N) = \mathrm{Hom}_{A-}(M, N)$.

3. *Let* $(N, \rho_N) \in {}_A\widetilde{\mathcal{M}}$ *be* C*-rational and consider the induced right* C*-comodule structure* (N, ϱ_N) *as in* (7). *If* $K \subset N$ *is an* A*-submodule, then* K *is a counital right* C*-subcomodule and moreover* $\varrho_K = (\varrho_N)_{|K}$.

Proof. 1. If (M, ρ_M) is C-rational, then $\rho_M(M) \subset \alpha_M^P(M \otimes_R C)$ (by definition). Moreover α_M^P is injective, hence $\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \to M \otimes_R C$ is well defined and we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(A, M) & \xleftarrow{\ \alpha_M^P\ } & \\ {\scriptstyle \rho_M}\uparrow & & \\ M & \overset{\varrho_M}{\dashrightarrow} & M \otimes C \end{array}$$

The right trapezoid in diagram (5) is obviously commutative and by definition of ϱ_M (7) all other trapezoids are commutative. By assumption M is a left A-module and so the outer rectangle is also commutative. By [2, Lemma 2.8] $\alpha_M^{P \otimes P}$ is injective and consequently the internal rectangle is commutative, i.e. (M, ϱ_M) is a right C-comodule. Moreover, ρ_M and α_M^P are by assumption injective and so $\varrho_M := \alpha_M^P \circ \rho_M$ is injective, i.e. M is counital.

2. Let $M, N \in \mathrm{Rat}^C({}_A\mathcal{M})$ and $f : M \to N$ be A-linear. The lower trapezoid in diagram (6) is obviously commutative and by definition of ϱ_M, ϱ_N all triangles are commutative. If f is A-linear then, by the injectivity of α_N^P, the upper trapezoid is commutative and consequently the outer triangle is commutative, i.e. f is C-colinear. So $\mathrm{Hom}_{A-}(M, N) \subseteq \mathrm{Hom}^C(M, N)$ and the equality follows from Lemma 1.10 (2).

3. Let (N, ρ_N) be a C-rational left A-module. If $K \subset N$ is an A-submodule, then by Lemma 1.9 (2) $\mathrm{Rat}^C(K) = K \cap \mathrm{Rat}^C(M) = K$, i.e. K is a C-rational left A-module. By (1) it follows that K is a counital right C-comodule through some R-linear map $\varrho_K : K \to K \otimes_R C$. Moreover $K \overset{\iota_K}{\hookrightarrow} N$ is by assumption A-linear and so C-colinear (by 2.), i.e. $K \subset N$ is a C-subcomodule. By remark 1.5 ${}_RC$ is flat and so $\varrho_K = (\varrho_N)_{|K}$.■

Remark 1.12. Let C be an R-coalgebra and (N, ϱ_N) be an arbitrary right C-comodule. Let $R^{(\Lambda)} \xrightarrow{\pi} N \longrightarrow 0$ be a free representation of N in $\mathcal{M}_R$. Then

$$C^{(\Lambda)} \simeq R^{(\Lambda)} \otimes_R C \xrightarrow{\pi \otimes id} N \otimes_R C \longrightarrow 0$$

is an epimorphism in $\mathcal{M}^C$. Moreover the injective comodule structure map $\varrho_N : N \to N \otimes_R C$ is C-colinear, i.e. N is a C-subcomodule of the C-generated C-comodule $N \otimes_R C$ and so C-subgenerated. Since $N \in \mathcal{M}^C$ is arbitrary, we conclude that C is a subgenerator in $\mathcal{M}^C$.

1.13. For every R-coalgebra C we have an R-algebra isomorphism:

$$\Psi : (C^*, \star) \to (\mathrm{End}^C(C, C)^{op}, \circ),\ f \mapsto [c \mapsto \sum f(c_1)c_2]$$

with inverse $\Phi : g \mapsto \varepsilon \circ g$. Analogously $({}^C\mathrm{End}(C, C), \circ) \simeq (C^*, \star)$ as R-algebras. If $P = (A, C) \in \mathcal{P}_m^\alpha$, then we have isomorphisms of R-algebras:

$$C^* \simeq {}^C\mathrm{End}(C) = \mathrm{End}(C_{C^*}) = \mathrm{End}(C_{\mathrm{End}^C(C)^{op}}) = \mathrm{End}(C_{\mathrm{End}({}_AC)^{op}}) := \mathrm{Biend}({}_AC)$$

and

$$C^* \simeq \mathrm{End}^C(C)^{op} = \mathrm{End}({}_{C^*}C)^{op} = \mathrm{End}({}_{{}^C\mathrm{End}(C)}C)^{op} = \mathrm{End}({}_{\mathrm{End}(C_A)}C)^{op} := \mathrm{Biend}(C_A),$$

where $\mathrm{Biend}({}_AC)$ and $\mathrm{Biend}(C_A)$ are the *biendomorphism rings* of ${}_AC$ and C_A, respectively (compare [38, 6.4]).

We are now ready to prove the main result in this section:

Theorem 1.14. *Let $P = (A, C) \in \mathcal{P}_m$, A not necessarily with unity. If ${}_RC$ is locally projective and $\kappa_P(A) \subset C^*$ is dense with respect to the finite topology on $C^* \hookrightarrow C^C$, then every right (respectively left) C-comodule is a* unital *left (respectively right) A-module and we have category isomorphisms*

$$\begin{array}{ccccccc} \mathcal{M}^C & \simeq & \mathrm{Rat}^C({}_A\widetilde{\mathcal{M}}) & = & \mathrm{Rat}^C({}_A\mathcal{M}) & = & \sigma[{}_AC] \\ & \simeq & \mathrm{Rat}^C({}_{C^*}\widetilde{\mathcal{M}}) & = & \mathrm{Rat}^C({}_{C^*}\mathcal{M}) & = & \sigma[{}_{C^*}C] \end{array} \tag{8}$$

and

$$\begin{array}{ccccccc} {}^C\mathcal{M} & \simeq & {}^C\mathrm{Rat}(\widetilde{\mathcal{M}}_A) & = & {}^C\mathrm{Rat}(\mathcal{M}_A) & = & \sigma[C_A] \\ & \simeq & {}^C\mathrm{Rat}(\widetilde{\mathcal{M}}_{C^*}) & = & {}^C\mathrm{Rat}(\mathcal{M}_{C^*}) & = & \sigma[C_{C^*}]. \end{array} \tag{9}$$

Proof. We prove the category isomorphisms (8). The isomorphisms of categories (9) follow by symmetry.

Step 1. $\mathcal{M}^C \simeq \mathrm{Rat}^C({}_A\widetilde{\mathcal{M}})$.

Since ${}_RC$ satisfies the α-condition and $\kappa_P(A) \subseteq C^*$ is dense, it follows by [2, Proposition 2.4 (2)] that $P \in \mathcal{P}_m^\alpha$. For every counital $(M, \varrho_M) \in \mathcal{M}^C$ we conclude that $\rho_M := \alpha_M^P \circ \varrho_M$ is injective, i.e. the induced left A-module is A-faithful. By Lemmas 1.10 and 1.11 we have covariant functors

$$\begin{array}{llllllll} {}_A(-): & \mathcal{M}^C & \to & \mathrm{Rat}^C({}_A\widetilde{\mathcal{M}}), & (-)^C: & \mathrm{Rat}^C({}_A\widetilde{\mathcal{M}}) & \to & \mathcal{M}^C, \\ & (M, \varrho_M) & \mapsto & (M, \alpha_M^P \circ \varrho_M), & & (M, \rho_M) & \mapsto & (M, (\alpha_M^P)^{-1} \circ \rho_M), \end{array} \tag{10}$$

acting as the identity on morphisms. Obviously

$$(-)^C \circ \ _A(-) \simeq id_{\mathcal{M}^C} \text{ and } _A(-) \circ (-)^C \simeq id_{\mathrm{Rat}^C(_A\widetilde{\mathcal{M}})},$$

i.e. $\mathcal{M}^C \simeq \mathrm{Rat}^C(_A\widetilde{\mathcal{M}})$.

Step 2. $\mathrm{Rat}^C(_A\widetilde{\mathcal{M}}) = \mathrm{Rat}^C(_A\mathcal{M})$.

Let $(N, \rho_N) \in \mathrm{Rat}^C(_A\widetilde{\mathcal{M}})$ and $n \in N$ with $\varrho_N(n) = \sum_{i=1}^{k} n_i \otimes c_i$. By assumption $\kappa_P(A) \subset C^*$ is dense and so there exists some $a \in A$, so that $\kappa_P(a)(c_i) = \varepsilon(c_i)$ for $i = 1, ..., k$. Hence

$$n = \sum_{i=1}^{k} n_i \varepsilon(c_i) = \sum_{i=1}^{k} n_i < a, c_i >= an \in N \text{ (i.e. } _AN \text{ is } \textit{unital}).$$

Step 3. $\mathcal{M}^C = \sigma[_AC]$.

By Remark 1.5 $_RC$ is flat and so $\mathcal{M}^C$ is a Grothendieck category (e.g. [12, 3.13]). Moreover by the previous lemmas and Remark 1.12 $\mathcal{M}^C \subseteq \sigma[_AC]$ is a full subcategory of $_A\mathcal{M}$. The equality follows now by the fact that $\sigma[_AC] \subseteq {}_A\mathcal{M}$ is the smallest Grothendieck full subcategory of $_A\mathcal{M}$ that contains C.

Step 4. C^* has unity ε_C and by assumption $(C^*, C) \in \mathcal{P}_m^\alpha$, so the proof above can be repeated to get

$$\mathcal{M}^C \simeq \mathrm{Rat}^C(_{C^*}\widetilde{\mathcal{M}}) = \mathrm{Rat}^C(_{C^*}\mathcal{M}) = \sigma[_{C^*}C].\blacksquare$$

Theorem 1.15. *For a measuring R-pairing P = (A,C) (A not necessarily with unity), the following are equivalent:*

1. *P satisfies the α-condition;*
2. $\mathcal{M}^C \simeq \sigma[_AC] = \sigma[_{C^*}C]$;
3. *$_RC$ is locally projective and $\kappa_P(A) \subseteq C^*$ is dense;*
4. $^C\mathcal{M} \simeq \sigma[C_A] = \sigma[C_{C^*}]$.

Proof. 1. ⇒ 2. Follows from Theorem 1.14.

2. ⇒ 3. By assumption we have

$$\sigma[_AC] = \sigma[_{C^*}C] = \sigma[_{\mathrm{Biend}(_AC)}C]$$

and the density of $\kappa_P(A) \subseteq C^*$ follows by Lemma 1.2. The proof of "$\mathcal{M}^C = \sigma[_{C^*}C] \Rightarrow {}_RC$ locally projective" follows from [36, 3.5] (which appeared also as [12, 4.3]).

3. ⇒ 1. Follows from general theory of dual pairings over rings (e.g. [2, Proposition 2.4 (2)]).

1. ⇔ 4. Follows by symmetry.■

Example 1.16. An interesting example for a measuring pairing for which Theorem 1.15 applies is $P := (C^\square, C)$, where C is a locally projective R-coalgebra and $C^\square := \mathrm{Rat}^C(_{C^*}C)$.

Dual coalgebras

Definition 1.17. Let A be an R-algebra and consider the class of R-cofinite A-ideals $\mathcal{K}_A$. For every class $\mathcal{F}$ of R-cofinite A-ideals we define the *set*

$$A^{\circ}_{\mathcal{F}} := \{f \in A^* \mid f(I) = 0 \text{ for some } I \in \mathcal{F}\}. \tag{11}$$

1. A filter $\mathfrak{F} = \{I_\lambda\}_\Lambda$ consisting of R-cofinite A-ideals will be called

   an *α-filter*, if the R-pairing $(A, A^{\circ}_{\mathfrak{F}})$ satisfies the α-condition;

   cofinitary, if for every $I_\lambda \in \mathfrak{F}$ there exists $I_\varkappa \subset I_\lambda$ for some $\varkappa \in \Lambda$, such that $A/I_\varkappa$ is finitely generated and projective in $\mathcal{M}_R$;

   cofinitely R-cogenerated, if A/I is R-cogenerated for every $I \in \mathfrak{F}$.

2. We call A :

   an *α-algebra*, if $\mathcal{K}_A$ is an α-filter;

   cofinitary, if $\mathcal{K}_A$ is a cofinitary filter;

   cofinitely R-cogenerated, if A/I is R-cogenerated for every $I \in \mathcal{K}_A$.

Notation. With $\mathbf{Cog}_R$ (respectively $\mathbf{Big}_R$, $\mathbf{Hopf}_R$) denote the category of R-coalgebras (respectively R-bialgebras, Hopf R-algebras) and with $\mathbf{CAlg}_R$ (respectively $\mathbf{CCog}_R$) the category of *commutative* R-algebras (respectively *cocommutative* R-coalgebras). With $\mathbf{CBig}_R$ (respectively $\mathbf{CCBig}_R$) we denote the category of commutative (respectively cocommutative) R-bialgebras and with $\mathbf{CHopf}_R$ (respectively $\mathbf{CCHopf}_R$) the category of commutative (respectively cocommutative) Hopf R-algebras.

For two R-coalgebras C, D we denote with $\mathrm{Cog}_R(C, D)$ the set of all R-coalgebra morphisms from C to D. For two R-algebras (respectively R-bialgebras, Hopf R-algebras) H, K we denote with $\mathrm{Alg}_R(H, K)$ (respectively $\mathrm{Big}_R(H, K)$, $\mathrm{Hopf}_R(H, K)$) the set of all R-algebra morphisms (respectively R-bialgebra morphisms, Hopf R-morphisms) from H to K.

Remark 1.18. We make the convention that an R-bialgebra (respectively a Hopf R-algebra) is an *α-bialgebra* (respectively a *Hopf α-algebra*), is *cofinitary* or is *cofinitely R-cogenerated*, if it is so as an R-algebra. With $\mathbf{Big}^{\alpha}_R \subset \mathbf{Big}_R$ (respectively $\mathbf{Hopf}^{\alpha}_R \subset \mathbf{Hopf}_R$) we denote the full subcategory of α-bialgebras (respectively Hopf α-algebras).

Lemma 1.19. ([6, Proposition 2.6]) *Let R be Noetherian and A be an R-algebra. Then*

$$\begin{aligned} A^{\circ} &:= \{f \in A^* \mid f(I) = 0 \textit{ for some } R\textit{-cofinite ideal } I \lhd A\}; \\ &= \{f \in A^* \mid f(I) = 0 \textit{ for some } R\textit{-cofinite left (right) } A\textit{-ideal}\}; \\ &= \{f \in A^* \mid Af \ (fA) \textit{ is f.g. in } \mathcal{M}_R\} \\ &= \{f \in A^* \mid AfA \textit{ is f.g. in } \mathcal{M}_R\}. \end{aligned}$$

Theorem 1.20. ([3, Theorem 3.3.]) *Let R be Noetherian, A be an R-algebra and consider $A^\circ \subseteq A^*$ as an A-bimodule under the left and the right* regular A-actions

$$(af)(\widetilde{a}) = f(\widetilde{a}a) \text{ and } (fa)(\widetilde{a}) = f(a\widetilde{a}) \text{ for all } a, \widetilde{a} \in A \text{ and } f \in A^*. \qquad (12)$$

*For an A-*subbimodule *$C \subseteq A^\circ$ and $P := (A, C)$ the following are equivalent:*

1. *${}_RC$ is locally projective and $\kappa_P(A) \subset C^*$ is dense;*
2. *${}_RC$ satisfies the α-condition and $\kappa_P(A) \subset C^*$ is dense;*
3. *(A, C) is an α-pairing;*
4. *$C \subset R^A$ is pure;*
5. *C is an R-coalgebra and $(A, C) \in \mathcal{P}^m_\alpha$.*

   If R is a QF Ring, then these are moreover equivalent to

6. *${}_RC$ is projective.*

Corollary 1.21. ([3, Corollary 3.16]) *Let A be an R-algebra and $\mathfrak{F}$ be a filter consisting of R-cofinite A-ideals. If R is Noetherian and $\mathfrak{F}$ is an α-filter, or if $\mathfrak{F}$ is cofinitary then we have isomorphisms of categories*

$$\begin{array}{llllllll} \mathcal{M}^{A^\circ_{\mathfrak{F}}} & \simeq & \mathrm{Rat}^{A^\circ_{\mathfrak{F}}}({}_A\mathcal{M}) & = & \sigma[{}_AA^\circ_{\mathfrak{F}}] & \multirow{2}{*}{\&} & {}^{A^\circ_{\mathfrak{F}}}\mathcal{M} & \simeq \ {}^{A^\circ_{\mathfrak{F}}}\mathrm{Rat}(\mathcal{M}_A) \ = \ \sigma[A^\circ_{\mathfrak{F}A}] \\ & \simeq & \mathrm{Rat}^{A^\circ_{\mathfrak{F}}}({}_{A^{\circ *}_{\mathfrak{F}}}\mathcal{M}) & = & \sigma[{}_{A^{\circ *}_{\mathfrak{F}}}A^\circ_{\mathfrak{F}}] & & & \simeq \ {}^{A^\circ_{\mathfrak{F}}}\mathrm{Rat}(\mathcal{M}_{A^{\circ *}_{\mathfrak{F}}}) \ = \ \sigma[A^\circ_{\mathfrak{F}A^{\circ *}_{\mathfrak{F}}}]. \end{array}$$

2 The cotensor functor

Dual to the *tensor product* of modules, J. Milnor and J. Moore introduced in [27] the *cotensor product* of comodules. For a closer look on the properties of the cotensor product over arbitrary (commutative) base rings the interested reader may refer to [20] (and [7]).

2.1. Let C be an R-coalgebra, $(M, \varrho_M) \in \mathcal{M}^C$, $(N, \varrho_N) \in {}^C\mathcal{M}$ and consider the R-linear mapping

$$\overline{\varrho}_{M,N} := \varrho_M \otimes id_N - id_M \otimes \varrho_N : M \otimes_R N \to M \otimes_R C \otimes_R N.$$

The *cotensor product* of M and N (denoted with $M \Box_C N$) is defined through the exactness of the following sequence in $\mathcal{M}_R$:

$$0 \to M\Box_C N \to M \otimes_R N \overset{\overline{\varrho}_{M,N}}{\to} M \otimes_R C \otimes_R N.$$

For $M, M' \in \mathcal{M}^C$ and $N, N' \in {}^C\mathcal{M}$, the *cotensor product* of $f \in \mathrm{Hom}^C(M, M')$ and $g \in {}^C\mathrm{Hom}(N, N')$ is defined as the R-linear mapping

$$f\Box_C g : M\Box_C N \to M'\Box_C N',$$

that completes the following diagram commutatively

$$\begin{array}{ccccccc} 0 & \longrightarrow & M\Box_C N & \longrightarrow & M\otimes_R N & \xrightarrow{\overline{\varrho}_{M,N}} & M\otimes_R C\otimes_R N \\ & & \Big\downarrow{\scriptstyle f\Box_C g} & & \Big\downarrow{\scriptstyle f\otimes g} & & \Big\downarrow{\scriptstyle f\otimes id_C\otimes g} \\ 0 & \longrightarrow & M'\Box_C N' & \longrightarrow & M'\otimes_R N' & \xrightarrow{\overline{\varrho}_{M',N'}} & M'\otimes_R C\otimes_R N' \end{array} \qquad (13)$$

In this way we get the *cotensor functor*

$$M\Box_C - :\ {}^C\mathcal{M} \to \mathcal{M}_R \ \ (\text{respectively } \ -\Box_C N : \mathcal{M}^C \to \mathcal{M}_R),$$

which is left exact if ${}_RC$ and M_R (respectively ${}_RC$ and ${}_RN$) are flat.

Definition 2.2. Let C be a *flat* R-coalgebra (hence $\mathcal{M}^C$ and ${}^C\mathcal{M}$ are *abelian* categories). A right (respectively a left) C-comodule M is called *coflat*, if the functor $M\Box_C - : {}^C\mathcal{M} \to \mathcal{M}_R$ (respectively $-\Box_C M : \mathcal{M}^C \to \mathcal{M}_R$) is exact.

Lemma 2.3. (Compare [29, Page 127], [12, 10.6]) *Let C be an R-coalgebra and $M \in \mathcal{M}^C$, $N \in {}^C\mathcal{M}$. If W_R is flat, then there are isomorphisms of R-modules*

$$W\otimes_R (M\Box_C N) \simeq (W\otimes_R M)\Box_C N \ \text{ and } \ (M\Box_C N)\otimes_R W \simeq M\Box_C(N\otimes_R W). \qquad (14)$$

The following result can easily be derived with the help of Lemma 2.3:

Corollary 2.4. *Let C, D be R-coalgebras and $(M, \varrho_M^C, \varrho_M^D) \in {}^C\mathcal{M}^D$.*

1. *Assume C_R to be flat. For every $N \in {}^D\mathcal{M}$, $M\Box_D N$ is a left C-comodule through*

$$\varrho_M^C \Box_D id_N : M\Box_D N \to (C\otimes_R M)\Box_D N \simeq C\otimes_R (M\Box_D N).$$

2. *Assume ${}_RD$ to be flat. For every $L \in \mathcal{M}^C$, $L\Box_C M$ is a right D-comodule through*

$$id_L \Box_C \varrho_M^D : L\Box_C M \mapsto L\Box_C(M\otimes_R D) \simeq (L\Box_C M)\otimes_R D.$$

Remark 2.5. ([7, Lemma II.2.5, Folgerung II.2.6]) Let C be a *flat* R-coalgebra. For every $M \in \mathcal{M}^C$, the mapping $\varrho_M : M \to M\Box_C C$ is an isomorphism in $\mathcal{M}^C$ with inverse $\lambda_M : m\otimes c \mapsto m\varepsilon(c)$ and moreover we have

$$M\otimes_R - \simeq M\Box_C(C\otimes_R -) : \mathcal{M}_R \to \mathcal{M}_{\mathbb{Z}}.$$

If M is coflat in $\mathcal{M}^C$, then M_R is flat.

The Associativity of the cotensor products is not valid in general (see [18]). However we have it in special cases, e.g. :

Lemma 2.6. ([7, Folgerung II.3.4.]) *Let C, D be* flat *R-coalgebras, $N \in {}^D\mathcal{M}$, $M \in {}^C\mathcal{M}^D$ and $L \in \mathcal{M}^C$. If $L \in \mathcal{M}^C$ (or $N \in {}^D\mathcal{M}$) is coflat, then we have an isomorphism of R-modules*

$$(L\Box_C M)\Box_D N \simeq L\Box_C(M\Box_D N). \tag{15}$$

Notation. For an R-algebra A we denote with $A^e := A \otimes_R A^{op}$ the *enveloping R-algebra* of A.

Lemma 2.7. *Let A be an R-algebra, $M, N \in {}_A\mathcal{M}$ and consider A, $\mathrm{Hom}_R(N, M)$ with the canonical left A^e-module structures. Then we have a functorial isomorphism*

$$\mathrm{Hom}_{A^e-}(A, \mathrm{Hom}_R(N, M)) \simeq \mathrm{Hom}_{A-}(N, M).$$

Proof. The isomorphism is given by

$$\Phi_{N,M} : \mathrm{Hom}_{A^e-}(A, \mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_{A-}(N, M),\ f \mapsto f(1_A)$$

with inverse $\Psi_{N,M} : g \mapsto [a \mapsto ag(-)]]$. One can easily show that $\Phi_{N,M}$ and $\Psi_{N,M}$ are functorial in M and N.■

In the case of a base field, the cotensor functor is equivalent to a suitable Hom-functor (e.g. [9, Proposition 3.1]). Over arbitrary ground rings we have

Proposition 2.8. *Let $P = (A, C) \in \mathcal{P}_m$, $(M, \varrho_M) \in \mathcal{M}^C$, $(N, \varrho_N) \in {}^C\mathcal{M}$ and consider A, $M \otimes_R N$ with the* canonical *left A^e-module structures.*

1. *If $\alpha^P_{M\otimes_R N}$ is injective, then we have for $\sum m_i \otimes n_i \in M \otimes_R N$:*

$$\sum m_i \otimes n_i \in M\Box_C N \Leftrightarrow \sum am_i \otimes n_i = \sum m_i \otimes n_i a \text{ for all } a \in A.$$

2. *If $P \in \mathcal{P}^\alpha_m$, then we have a functorial isomorphism*

$$M\Box_C N \simeq \mathrm{Hom}_{A^e-}(A, M \otimes_R N).$$

Proof. 1. Let $\alpha^P_{M\otimes_R N}$ be injective and set $\psi := \alpha^P_{M\otimes_R N} \circ \tau_{(23)}$. Then

$$\begin{array}{lrcl} & \sum m_i \otimes n_i \in M\Box_C N & & \\ \Leftrightarrow & \sum m_{i<0>} \otimes m_{i<1>} \otimes n_i & = & \sum m_i \otimes n_{i<-1>} \otimes n_{i<0>}, \\ \Leftrightarrow & \psi(\sum m_{i<0>} \otimes m_{i<1>} \otimes n_i)(a) & = & \psi(\sum m_i \otimes n_{i<-1>} \otimes n_{i<0>})(a),\ \forall\ a \in A \\ \Leftrightarrow & \sum m_{i<0>} < a, m_{i<1>} > \otimes n_i & = & \sum m_i \otimes < a, n_{i<-1>} > n_{i<0>},\ \forall\ a \in A \\ \Leftrightarrow & \sum am_i \otimes n_i & = & \sum m_i \otimes n_i a,\ \forall\ a \in A. \end{array}$$

2. The isomorphism is given through

$$\gamma_{M,N} : M\Box_C N \to \mathrm{Hom}_{A^e-}(A, M \otimes_R N),\ m \otimes n \mapsto [a \mapsto am \otimes n\ (= m \otimes na)]$$

with inverse $\beta_{M,N} : f \mapsto f(1_A)$. It is easy to see that $\gamma_{M,N}$ and $\beta_{M,N}$ are functorial in M and N.■

Lemma 2.9. ([37, 15.7], [11, II, 4.2, Proposition 2]) *Let A be an R-algebra, K, K' be left A-modules, L be an R-module and consider the R-linear mapping*

$$\upsilon : \mathrm{Hom}_{A-}(K, K') \otimes_R L \to \mathrm{Hom}_{A-}(K, K' \otimes_R L),\ h \otimes l \mapsto h(-) \otimes l. \tag{16}$$

1. *If ${}_RL$ is flat and ${}_AK$ is finitely generated (respectively finitely presented), then υ is injective (respectively bijective).*
2. *If ${}_AK$ be K'-projective and ${}_AK$ is finitely generated, then υ is bijective.*
3. *If ${}_AK$ be K'-projective and ${}_RL$ is finitely presented, then υ is bijective.*
4. *If ${}_RL$ is projective (respectively finitely generated projective), then υ is injective (respectively bijective).*

2.10. ([12, 3.11]) Let ${}_RC$ be a flat R-coalgebra. Let M be a left C-comodule and consider the R-linear mapping

$$\gamma : M^* \to \mathrm{Hom}_R(M, C),\ f \mapsto [m \mapsto \sum f(m_{<-1>})m_{<0>}]. \tag{17}$$

If ${}_RM$ is finitely presented, then $\mathrm{Hom}_R(M, C) \simeq M^* \otimes_R C$ (see Lemma 2.9) and M^* is a right C-comodule through

$$\varrho_{M^*} : M^* \xrightarrow{\gamma} \mathrm{Hom}_R(M, C) \simeq M^* \otimes_R C. \tag{18}$$

If M is a right C-comodule and M_R is finitely presented, then M^* becomes analogously a left C-comodule.

With the help of Lemmas 2.7 and 2.9, the following result can be derived directly from Proposition 2.8:

Corollary 2.11. *Let $P = (A, C) \in \mathcal{P}_m^{\alpha}$.*

1. *Let $M, N \in \mathcal{M}^C$. If M_R is flat and ${}_RN$ is finitely presented, or N_R is finitely generated projective, then we have functorial isomorphisms*

$$\begin{array}{rcl} M\Box_C N^* & \simeq & \mathrm{Hom}_{A^e-}(A, M \otimes_R N^*) \quad \simeq \quad \mathrm{Hom}_{A^e-}(A, \mathrm{Hom}_R(N, M)) \\ & \simeq & \mathrm{Hom}_{A-}(N, M) \qquad\qquad\ \ = \quad \mathrm{Hom}^C(N, M). \end{array}$$

2. *Let $M \in \mathcal{M}^C$, N be a C-bicomodule and consider N with the induced left A^e-module structure. Then we have isomorphisms of R-modules*

$$M\Box_C N \simeq \mathrm{Hom}_{A^e-}(A, M \otimes_R N) \simeq\ M \otimes_R \mathrm{Hom}_{A^e-}(A, N),$$

if any one of the following conditions is satisfied:

   (a) *M_R is flat and ${}_{A^e}A$ is finitely presented (e.g. A is an affine R-algebra [37, 23.6]);*

(b) $_{A^e}A$ *is* N*-projective and finitely generated;*

(c) $_{A^e}A$ *is* N*-projective and* M_R *is finitely presented;*

(d) M_R *is finitely generated projective.*

3. *Let* $N \in {}^C\mathcal{M}$*,* M *be a* C*-bicomodule and consider* M *with the induced left* A^e*-module structure. Then we have an isomorphism of* R*-modules*

$$M\Box_C N \simeq \mathrm{Hom}_{A^e-}(A, M \otimes_R N) \simeq \mathrm{Hom}_{A^e-}(A, M) \otimes_R N,$$

if any one of the following conditions is satisfied:

(a) $_RN$ *is flat and* $_{A^e}A$ *is finitely presented (e.g.* A *is an affine* R*-algebra* [37, 23.6]*);*

(b) $_{A^e}A$ *is* M*-projective and finitely generated;*

(c) $_{A^e}A$ *is* M*-projective and* $_RN$ *is finitely presented;*

(d) $_RN$ *is finitely generated projective.*

Injective comodules

For $P = (A, C) \in \mathcal{P}^{\alpha}_{m}$ we get from [38, 16.3] the following characterizations of the injective objects in $\mathcal{M}^C \simeq \mathrm{Rat}^C({}_A\mathcal{M}) = \sigma[{}_AC]$:

Lemma 2.12. *Let* $P = (A, C) \in \mathcal{P}^{\alpha}_{m}$*. For every* $U \in \mathrm{Rat}^C({}_A\mathcal{M})$ *the following are equivalent:*

1. U *is injective in* $\mathrm{Rat}^C({}_A\mathcal{M})$*;*
2. $\mathrm{Hom}^C(-, U) \simeq \mathrm{Hom}_{A-}(-, U) : \mathrm{Rat}^C({}_A\mathcal{M}) \to \mathcal{M}_R$ *is exact;*
3. U *is* C*-injective in* $\mathrm{Rat}^C({}_A\mathcal{M})$*;*
4. U *is* K*-injective for every (finitely generated, cyclic) left* A*-submodule* $K \subset C$*;*
5. *every exact sequence* $0 \to U \to L \to N \to 0$ *in* $\mathrm{Rat}^C({}_A\mathcal{M})$ *splits.*
6. *every exact sequence* $0 \to U \to L \to N \to 0$ *in* $\mathrm{Rat}^C({}_A\mathcal{M})$*, in which* N *is a factor module of* C *(or* A*) splits.*

The following Lemma plays an important role in the study of injective objects in the category $\mathrm{Rat}^C({}_A\mathcal{M})$, where $(A, C) \in \mathcal{P}^{\alpha}_{m}$:

Lemma 2.13. *Let* $(A, C) \in \mathcal{P}^{\alpha}_{m}$*. If* R *is a QF ring then a* C*-rational left* A*-module* M*, with* $_RM$ *flat, is injective in* $\mathrm{Rat}^C({}_A\mathcal{M})$ *if and only if* M *is coflat in* $\mathcal{M}^C$*.*

Proof. By Theorem 1.15 we have the isomorphism of categories

$$\sigma[{}_AC] = \mathrm{Rat}^C({}_A\mathcal{M}) \simeq \mathcal{M}^C$$

and we get the result by [12, 10.12].■

Lemma 2.14. *If $P = (A, C) \in \mathcal{P}_m^\alpha$, then $- \otimes_R C : \mathcal{M}_R \to \mathrm{Rat}^C({}_A\mathcal{M})$ respects injective objects.*

Proof. By Theorem 1.15 $\mathcal{M}^C \simeq \mathrm{Rat}^C({}_A\mathcal{M}) = \sigma[{}_AC]$, i.e. $\mathrm{Rat}^C({}_A\mathcal{M}) \subset {}_A\mathcal{M}$ is a closed subcategory. The *exact* forgetful functor $\digamma : \mathcal{M}^C \to \mathcal{M}_R$ is left adjoint to $-\otimes_R C : \mathcal{M}_R \to \mathcal{M}^C$ and the result follows then by [38, 45.6].■

Proposition 2.15. *Let $(A, C) \in \mathcal{P}_m^\alpha$ and $M \in \mathrm{Rat}^C({}_A\mathcal{M})$.*

1. *M is an A-submodule of an injective C-rational left A-module.*
2. *Every injective object in $\mathrm{Rat}^C({}_A\mathcal{M})$ is C-generated.*
3. *M is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ if and only if there exists an injective R-module X for which ${}_AM$ is a direct summand of $X \otimes_R C$.*
4. *Let M be injective in $\mathcal{M}_R$. Then M is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ if and only if $\varrho_M : M \to M \otimes_R C$ splits in ${}_A\mathcal{M}$.*
5. *Let R be Noetherian. Then M is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ if and only if $M^{(\Lambda)}$ is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ for every index set Λ. Moreover, direct limits of injectives in $\mathrm{Rat}^C({}_A\mathcal{M})$ are injective.*
6. *Let A be separable (i.e. ${}_{A^e}A$ is projective). Then $M \in \mathcal{M}^C$ is coflat if and only if M_R is flat.*

Proof.

1. Let $M \in \mathrm{Rat}^C({}_A\mathcal{M})$ and denote with $E(M)$ the injective hull of M in $\mathcal{M}_R$. By Lemma 2.14 $E(M) \otimes_R C$ is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$. Obviously $(\iota_M \otimes_R id_C) \circ \varrho_M : M \hookrightarrow E(M) \otimes_R C$ is A-linear and the result follows.
2. Let $(M, \varrho_M) \in \mathrm{Rat}^C({}_A\mathcal{M})$ be injective. By Lemma 2.12, there exists an epimorphism of left A-modules $\beta : M \otimes_R C \to M$, such that $\beta \circ \varrho_M = id_M$. If $R^{(\Lambda)} \xrightarrow{\pi} M \to 0$ is a free representation of M in $\mathcal{M}_R$, then we get the following exact sequence in $\mathrm{Rat}^C({}_A\mathcal{M})$:
$$C^{(\Lambda)} \simeq R^{(\Lambda)} \otimes_R C \xrightarrow{\beta\circ(\pi\otimes id_C)} M \longrightarrow 0\,.$$
3. Let X be an injective R-module, such that ${}_AM$ is a direct summand of $X \otimes_R C$. By Lemma 2.14 $X \otimes_R C$ is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ and consequently M is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$. On the other hand, let M be injective in $\mathrm{Rat}^C({}_A\mathcal{M})$ and denote with $E(M)$ the injective hull of M in $\mathcal{M}_R$. Then we get an exact sequence in $\mathrm{Rat}^C({}_A\mathcal{M})$
$$0 \longrightarrow M \xrightarrow{(\iota\otimes id_C)\circ\varrho_M} E(M) \otimes_R C\,. \tag{19}$$
Now (19) splits in $\mathrm{Rat}^C({}_A\mathcal{M})$ by Lemma 2.12 and the result follows.

4. Follows from Lemmata 2.12 and 2.14.

5. By [4, Folgerung 2.2.24] ${}_AC$ is locally Noetherian. The result follows then from the isomorphism of categories $\mathrm{Rat}^C({}_A\mathcal{M}) \simeq \sigma[{}_AC]$ and [38, 27.3].

6. If $M \in \mathcal{M}^C$ is coflat, then M_R is flat (by Remark 2.5). Assume now that ${}_{A^e}A$ is projective. If M_R is flat, then by Proposition 2.8

$$M\Box_C- \simeq \mathrm{Hom}_{A^e-}(A,-)\circ(M\otimes_R -)$$

is exact, i.e. M is coflat.■

Corollary 2.16. *Let $(A,C)\in\mathcal{P}_m^\alpha$. If R is semisimple (e.g. a field), then:*

1. *$M\in\mathrm{Rat}^C({}_A\mathcal{M})$ is injective if and only if ${}_AM$ is a direct summand of ${}_AC^{(\Lambda)}$ for some index set Λ.*

2. *If A is separable, then C is right semisimple (i.e. every right C-comodules is injective).*

3 Coinduction Functors in $\mathcal{P}_m^\alpha$

By his study of the induced representations of quantum groups, Z. Lin ([24, 3.2], [23]) considered *induction functors* for *admissible Hopf R-pairings* over Dedekind rings. His aspect was inspired by the induction functors in the theory of affine algebraic groups and quantum groups. We generalize his results to the coinduction functor for the category of *measuring α-pairings* $\mathcal{P}_m^\alpha\subset\mathcal{P}_m$ and show that it is isomorphic to a coinduction functor between categories of Type $\sigma[M]$. Moreover we get as nice description of it as a composition of a suitable Hom-functor and a Trace-functor.

3.1. Let A,B be R-algebras and $\xi : A\to B$ be an R-algebra morphism. Then every left B-module becomes a left A-module in a canonical way and we get the so called *restriction functor* $(-)_\xi : {}_B\mathcal{M}\to{}_A\mathcal{M}$. Considering B with the canonical A-bimodule structure, we have the functor $\mathrm{Hom}_{A-}(B,-):{}_A\mathcal{M}\to{}_B\mathcal{M}$. Moreover $((-)_\xi,\mathrm{Hom}_{A-}(B,-))$ is an adjoint pair of *covariant* functors through the functorial canonical isomorphisms

$$\mathrm{Hom}_{A-}(M_\xi,N)\simeq\mathrm{Hom}_{A-}(B\otimes_B M,N)\simeq\mathrm{Hom}_{B-}(M,\mathrm{Hom}_{A-}(B,N)).$$

If we consider the *induction functor* $B\otimes_A - :\ {}_A\mathcal{M}\to{}_B\mathcal{M}$, then $(B\otimes_A -,(-)_\xi)$ is an adjoint pair of *covariant* functors through the functorial canonical isomorphisms

$$\mathrm{Hom}_{B-}(B\otimes_A N,M)\simeq\mathrm{Hom}_{A-}(N,\mathrm{Hom}_{B-}(B,M))\simeq\mathrm{Hom}_{A-}(N,M_\xi).$$

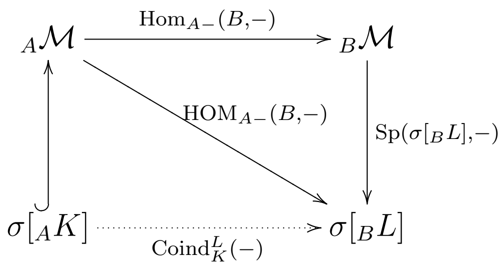
3.2. The general coinduction functor. Let A, B be R-Algebras and $\xi : A \to B$ be an R-algebra morphism. If L is a left B-module, then we get the covariant functor

$$\mathrm{HOM}_{A-}(B, -) := \mathrm{Sp}(\sigma[_BL], \mathrm{Hom}_{A-}(B, -)) : {}_A\mathcal{M} \to \sigma[_BL]. \tag{20}$$

For every left A-module K (20) restricts to the covariant *coinduction functor*

$$\mathrm{Coind}_K^L(-) := \mathrm{Sp}(\sigma[_BL], \mathrm{Hom}_{A-}(B, -)) : \sigma[_AK] \to \sigma[_BL], \tag{21}$$

i.e. $\mathrm{Coind}_K^L(-)$ is defined through the commutativity of the following diagram:



If $(L)_\xi$ is K-subgenerated as a left A-module, then $(-)_\xi : {}_B\mathcal{M} \to {}_A\mathcal{M}$ restricts to $(-)_\xi : \sigma[_BL] \to \sigma[_AK]$ and $((-)_\xi, \mathrm{Coind}_K^L(-))$ turns to be an adjoint pair of covariant functors.

3.3. The ad-corestriction functor. Let C, D be R-coalgebras and $\theta : D \to C$ be an R-coalgebra morphism. Then we get the covariant *corestriction functor*

$$(-)^\theta : \mathcal{M}^D \to \mathcal{M}^C, \ (M, \varrho_M) \mapsto (M, (id_M \otimes \theta) \circ \varrho_M). \tag{22}$$

On the other hand, consider D as a left C-comodule through

$$\varrho_D^C : D \xrightarrow{\Delta_D} D \otimes_R D \xrightarrow{\theta \otimes id} C \otimes_R D.$$

If ${}_RD$ is *flat*, then for every $M \in \mathcal{M}^C$ the cotensor product $M \Box_C D$ becomes a right D-comodule through

$$M \Box_C D \overset{id \Box_C \Delta_D}{\longrightarrow} M \Box_C (D \otimes_R D) \simeq (M \Box_C D) \otimes_R D$$

and we get the *ad-corestriction functor*

$$- \Box_C D : \mathcal{M}^C \to \mathcal{M}^D, \ M \mapsto M \Box_C D.$$

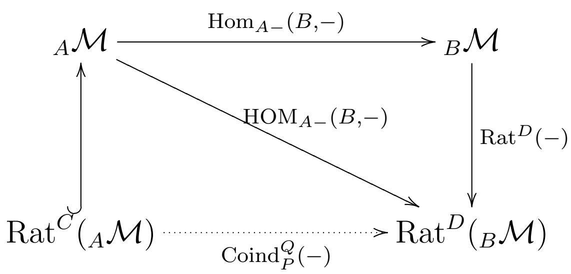
3.4. Let $Q = (B, D) \in \mathcal{P}_m^\alpha$. For every R-algebra A with R-algebra morphism $\xi : A \to B$ we have the covariant functor

$$\mathrm{HOM}_{A-}(B, -) := \mathrm{Rat}^D(-) \circ \mathrm{Hom}_{A-}(B, -) : \ {}_A\mathcal{M} \to \mathrm{Rat}^D({}_B\mathcal{M}).$$

If $P = (A, C) \in \mathcal{P}_m^\alpha$, then $\mathrm{HOM}_{A-}(B, -)$ restricts to the *coinduction functor* from P to Q :

$$\mathrm{Coind}_P^Q(-) : \mathrm{Rat}^C({}_A\mathcal{M}) \to \mathrm{Rat}^D({}_B\mathcal{M}), \ M \mapsto \mathrm{Rat}^D(\mathrm{Hom}_{A-}(B, M)),$$

i.e. $\mathrm{Coind}_P^Q(-)$ is defined through the commutativity of the following diagram:



Proposition 3.5. *Let $P = (A, C)$, $Q = (B, D) \in \mathcal{P}_m$ and $(\xi, \theta) : (B, D) \to (A, C)$ be a morphism in $\mathcal{P}_m$.*

1. *If $_RD$ is flat, then $((-)^\theta, -\Box_C D)$ is an adjoint pair of covariant functors.*

2. *If $P, Q \in \mathcal{P}_m^\alpha$ and B is* commutative*, then we have for every $N \in {}_A\mathcal{M}$:*

$$\mathrm{HOM}_{A-}(B, N) = \mathrm{HOM}_{A-}(B, \mathrm{Rat}^C(_AN)).$$

Proof. 1. One can show easily that the mapping

$$\Phi_{N,L} : \mathrm{Hom}^D(N, L\Box_C D) \to \mathrm{Hom}^C(N^\theta, L),\ f \mapsto (id_L \Box_C \theta) \circ f$$

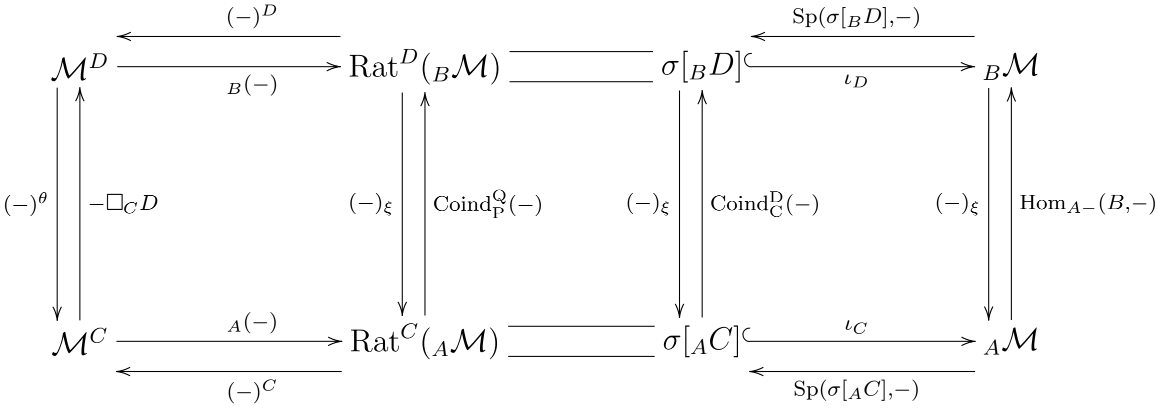
is an isomorphism with inverse $g \mapsto (g\Box_C id_D) \circ \varrho_N$ and moreover that it is functorial in $N \in \mathcal{M}^D$ and $L \in \mathcal{M}^C$.

2. If $g \in \mathrm{HOM}_{A-}(B, N)$, then we have for all $a \in A$ and $b \in B$:

$$\begin{array}{rclcl} a(g(b)) & = & g(a \rightharpoondown b) & = & g(\xi(a)b) \\ & = & g(b\xi(a)) & = & (\xi(a)g)(b) \\ & = & \sum g_{<0>}(b) < \xi(a), g_{<1>} > & = & \sum g_{<0>}(b) < a, \theta(g_{<1>}) > . \end{array}$$

Consequently $g(B) \subseteq \mathrm{Rat}^C(_AN)$ and the result follows.■

3.6. Let $P = (A, C)$, $Q = (B, D) \in \mathcal{P}_m^\alpha$, $(\xi, \theta) : (B, D) \to (A, C)$ be a morphism in $\mathcal{P}_m^\alpha$ and denote the restriction of $(-)_\xi : {}_B\mathcal{M} \to {}_A\mathcal{M}$ on $\mathrm{Rat}^D(_B\mathcal{M}) = \sigma[_BD]$ also with $(-)_\xi$. Through the isomorphism of categories $\mathcal{M}^C \simeq \mathrm{Rat}^C(_A\mathcal{M}) = \sigma[_AC]$ and $\mathcal{M}^D \simeq \mathrm{Rat}^D(_B\mathcal{M}) = \sigma[_BD]$ (compare Theorem 1.15) we get an equivalence of functors $(-)^\theta \approx (-)_\xi$. Considering the covariant functors (10) we get a commutative diagram of pairwise adjoint covariant functors



(23)

Theorem 3.7. *Let* $P = (A, C)$, $Q = (B, D) \in \mathcal{P}_m^\alpha$ *(so that in particular* ${}_RC$ *and* ${}_RD$ *are flat) and* $(\xi, \theta) : (B, D) \to (A, C)$ *be a morphism in* $\mathcal{P}_m^\alpha$*. Through the isomorphisms of categories* $\mathcal{M}^C \simeq \mathrm{Rat}^C({}_A\mathcal{M}) = \sigma[{}_AC]$ *and* $\mathcal{M}^D \simeq \mathrm{Rat}^D({}_B\mathcal{M}) = \sigma[{}_BD]$ *(compare Theorem 1.14) the following functors are equivalent*

$$\begin{array}{llcl} -\square_C D & : \mathcal{M}^C & \to & \mathcal{M}^D, \\ \mathrm{Coind}_P^Q(-) & : \mathrm{Rat}^C({}_A\mathcal{M}) & \to & \mathrm{Rat}^D({}_B\mathcal{M}), \\ \mathrm{Hom}_{A^e-}(A, - \otimes_R D) & : \mathrm{Rat}^C({}_A\mathcal{M}) & \to & \mathrm{Rat}^D({}_B\mathcal{M}), \\ \mathrm{Coind}_C^D(-) & : \sigma[{}_AC] & \to & \sigma[{}_BD]. \end{array}$$

Proof. Consider for every $N \in \mathcal{M}^C$ the *injective* R-linear mapping

$$\gamma_N := (\alpha_N^Q)|_{N\square_C D} : N\square_C D \to \mathrm{Hom}_R(B, N), \;\; \sum n_i \otimes d_i \mapsto [b \mapsto \sum n_i < b, d_i >].$$

Then we have for all $a \in A$ and $b \in B$:

$$\begin{array}{rcll} \gamma_N(\sum n_i \otimes d_i)(a \rightharpoondown b) & = & \sum n_i < a \rightharpoondown b, d_i > & \\ & = & \sum n_i < b, d_i \leftharpoonup a > & \\ & = & \gamma_N(\sum n_i \otimes d_i \leftharpoonup a)(b) & \\ & = & \gamma_N(\sum an_i \otimes d_i)(b) & \text{(compare Lemma 2.8 (1))} \\ & = & \sum an_i < b, d_i > & \\ & = & a(\gamma_N(n_i \otimes d_i)(b)), & \end{array}$$

i.e. $\gamma_N(N\square_C D) \subset \mathrm{Hom}_{A-}(B, N)$. Moreover we have for arbitrary $\sum n_i \otimes d_i \in N\square_C D$ and $b, \widetilde{b} \in B$:

$$\begin{array}{rclcl} \gamma_N(\widetilde{b}(\sum n_i \otimes d_i))(b) & = & \gamma_N(\sum n_i \otimes \widetilde{b} \rightharpoonup d_i)(b) & = & \sum n_i < b, \widetilde{b} \rightharpoonup d_i > \\ & = & \sum n_i < b\widetilde{b}, d_i > & = & \gamma_N(\sum n_i \otimes d_i)(b\widetilde{b}) \\ & = & (\widetilde{b}\gamma_N(\sum n_i \otimes d_i))(b), & & \end{array}$$

i.e. γ_N is B-linear. But ${}_RD$ is flat, so $N\Box_C D \in \mathcal{M}^D$ by Corollary 2.4 and it follows by Lemma 1.9 that $\gamma_N(N\Box_C D) \subset \mathrm{HOM}_{A-}(B,N)$. Now we show that the following R-linear mapping is well defined

$$\beta_N : \mathrm{HOM}_{A-}(B,N) \to N\Box_C D,\ f \mapsto \sum f_{<0>}(1_B) \otimes f_{<1>}.$$

For all $f \in \mathrm{HOM}_{A-}(B,N)$, $a \in A$ and $b \in B$ we have

$$\begin{array}{rcl}
\gamma_N(\sum a(f_{<0>}(1_B)) \otimes f_{<1>})(b) & = & \sum a(f_{<0>}(1_B)) < b, f_{<1>} > \\
& = & \sum f_{<0>}(a \rightharpoonup 1_B)) < b, f_{<1>} > \\
& = & \sum f_{<0>}(\xi(a)) < b, f_{<1>} > \\
& = & \sum(\xi(a) f_{<0>})(1_B) < b, f_{<1>} > \\
& = & \sum f_{<0><0>}(1_B) < \xi(a), f_{<0><1>} >< b, f_{<1>} > \\
& = & \sum f_{<0>}(1_B) < \xi(a), f_{<1>1} >< b, f_{<1>2} > \\
& = & \sum f_{<0>}(1_B) < \xi(a)b, f_{<1>} > \\
& = & \sum f_{<0>}(1_B) < a \rightharpoonup b, f_{<1>} > \\
& = & \sum f_{<0>}(1_B) < b, f_{<1>} \leftharpoonup a > \\
& = & \gamma_N(\sum f_{<0>}(1_B) \otimes f_{<1>} \leftharpoonup a)(b),
\end{array}$$

i.e. $\sum a(f_{<0>}(1_B)) \otimes f_{<1>} = \sum f_{<0>}(1_B) \otimes f_{<1>} \leftharpoonup a$ (since γ_N is injective). It follows then by Proposition 2.8 (1) that $\sum f_{<0>}(1_B) \otimes f_{<1>} \in N\Box_C D$, i.e. β_N is well defined. Moreover, we have for all $f \in \mathrm{HOM}_{A-}(B,N)$ and $b \in B$:

$$\begin{array}{rcl}
(\gamma_N \circ \beta_N)(f)(b) & = & \gamma_N(\sum f_{<0>}(1_B) \otimes f_{<1>})(b) \\
& = & \sum f_{<0>}(1_B) < b, f_{<1>} > \\
& = & (bf)(1_B) = f(b),
\end{array}$$

hence $\gamma_N \circ \beta_N = id$. Obviously $\beta_N \circ \gamma_N = id$. Consequently γ_N and β_N are isomorphisms. It is easy to show that γ_N and β_N are functorial in N, hence $-\Box_C D \approx \mathrm{Coind}_P^Q(-)$. The equivalences $\mathrm{Coind}_P^Q(-) \approx \mathrm{Coind}_C^D(-)$ and $-\Box_C D \approx \mathrm{Hom}_{A^e-}(A, - \otimes_R D)$ follow now by Theorem 1.15 and Proposition 2.8 (2), respectively.■

3.8. Let $Q = (B,D) \in \mathcal{P}_m^\alpha$ and consider the trivial R-pairing $P = (R,R) \in \mathcal{P}_m^\alpha$ with the morphism of measuring R-pairings $(\eta_B, \varepsilon_D) : (B,D) \to (R,R)$. Then we have for every $M \in \mathcal{M}^R \simeq \mathcal{M}_R$

$$\mathrm{Coind}_P^Q(-) := \mathrm{HOM}_R(B,-) \simeq - \otimes_R D.$$

Notice that $\digamma \simeq (-)^\varepsilon : \mathcal{M}^D \to \mathcal{M}_R$, where $\digamma$ is the forgetful functor, hence $(\digamma, \mathrm{Coind}_P^Q(-))$ is an adjoint pair of covariant functors.

3.9. Universal Property. Let $P = (A,C)$, $Q = (B,D) \in \mathcal{P}_m^\alpha$ and $(\xi,\theta) : (B,D) \to (A,C)$ be a morphism in $\mathcal{P}_m^\alpha$. Then $\mathrm{Coind}_P^Q(-)$ has the following universal property: if $N \in \mathcal{M}^D$, $M \in \mathcal{M}^C$ and $\phi \in \mathrm{Hom}^C(N^\theta, M)$, then there exists a unique $\widetilde{\phi} \in \mathrm{Hom}^D(N, \mathrm{Coind}_P^Q(M))$, such that $\phi(n) = \widetilde{\phi}(n)(1_B)$ for every $n \in N$.

In what follows we list some properties of the coinduction functor:

3.10. Let $P=(A,C)$, $Q=(B,D)\in\mathcal{P}_m^\alpha$ and $(\xi,\theta):(B,D)\to(A,C)$ be a morphism in $\mathcal{P}_m^\alpha$.

1. $\mathrm{Coind}_P^Q(-)$ respects direct limits: if $\{N_\lambda\}_\Lambda$ is a directed system in $\mathrm{Rat}^C({}_A\mathcal{M})$, then

$$\mathrm{Coind}_P^Q(\underrightarrow{lim}N_\lambda)\simeq\underrightarrow{lim}N_\lambda\Box_C D\simeq\underrightarrow{lim}(N_\lambda\Box_C D)=\underrightarrow{lim}\mathrm{Coind}_P^Q(N_\lambda).$$

2. $\mathrm{Rat}^D(-)$ & $\mathrm{Hom}_{A-}(B,-)$ are left-exact, hence

$$\mathrm{Coind}_P^Q(-):=\mathrm{Rat}^D(-)\circ\mathrm{Hom}_{A-}(B,-)$$

   is left-exact. If moreover ${}_AB$ is projective (hence $\mathrm{Hom}_{A-}(B,-)$ is exact) and $\mathrm{Rat}^D(-)$ is exact, then $\mathrm{Coind}_P^Q(-)$ is exact.

3. $\mathrm{Coind}_P^Q(-)\simeq-\Box_C D$ is exact if and only if D is coflat in ${}^C\mathcal{M}$. If R is a QF ring, then $\mathrm{Coind}_P^Q(-)$ is exact if and only if D is injective in ${}^C\mathrm{Rat}(\mathcal{M}_A)$.

4. By Lemma 3.5 (1) $((-)^\theta,-\Box_C D)$ is an adjoint pair of covariant functors, hence $\mathrm{Coind}_P^Q(-)\simeq-\Box_C D$ respects inverse Limit, i.e. direct products, kernels and injective objects (since $(-)^\theta:\mathcal{M}^D\to\mathcal{M}^C$ is exact). In particular, if C is injective in $\mathrm{Rat}^C({}_A\mathcal{M})$, then $D\simeq C\Box_C D\simeq\mathrm{Coind}_P^Q(C)$ is injective in $\mathrm{Rat}^D({}_B\mathcal{M})$.

5. Let A be *separable*. Then $-\Box_C D\simeq\mathrm{Coind}_P^Q(-)\simeq\mathrm{Hom}_{A^e-}(A,-\otimes_R D)$ is exact, i.e. D is coflat in ${}^C\mathcal{M}$. If moreover R is a QF ring, then D is injective in ${}^C\mathcal{M}$.

A version of the following result was obtained by Y. Doi [14, Proposition 5] in the case of a base field:

Proposition 3.11. *Let $P=(A,C)$, $Q=(B,D)\in\mathcal{P}_m^\alpha$ and $(\xi,\theta):(B,D)\to(A,C)$ be a morphism in $\mathcal{P}_m^\alpha$. If R is a QF ring, then the following are equivalent:*

1. *The functor* $\mathrm{Coind}_P^Q(-):\mathrm{Rat}^C({}_A\mathcal{M})\to\mathrm{Rat}^D({}_B\mathcal{M})$ *is exact;*
2. *D is coflat in ${}^C\mathcal{M}$;*
3. *D is injective in ${}^C\mathrm{Rat}(\mathcal{M}_A)$;*
4. *If M is an injective left D-comodule that is flat in $\mathcal{M}_R$, then M is injective in ${}^C\mathrm{Rat}(\mathcal{M}_A)$.*

Proof. (1) $\Leftrightarrow$ (2) Follows from the isomorphism of functors $\mathrm{Coind}_P^Q(-)\simeq-\Box_C D:\mathcal{M}^C\to\mathcal{M}^D$.

(2) $\Leftrightarrow$ (3) By Remark 1.5 ${}_RD$ is flat, so the equivalence follows from Lemma 2.13.

(2) $\Rightarrow$ (4) Let M be a left D-comodule and assume that M is injective in ${}^D\mathrm{Rat}(\mathcal{M}_B)$ and flat in $\mathcal{M}_R$. Then M is coflat in ${}^D\mathcal{M}$ (by Lemma 2.13) and we have (by Lemma 2.6) an isomorphism of functors

$$-\Box_C M\simeq(-\Box_C D)\Box_D M:\mathcal{M}^C\to\mathcal{M}_R$$

By assumption $-\Box_C D : \mathcal{M}^C \to \mathcal{M}^D$ and $-\Box_D M : \mathcal{M}^D \to \mathcal{M}_R$ are exact and so $-\Box_C M$ is exact. By Lemma 2.13 M is injective in ${}^C\mathrm{Rat}(\mathcal{M}_A)$.

(4) ⇒ (3) Since R is injective, D is injective in ${}^D\mathrm{Rat}(\mathcal{M}_B) \simeq {}^D\mathcal{M}$. It follows then from the assumption that D is injective in ${}^C\mathrm{Rat}(\mathcal{M}_A)$.■

As a consequence of Theorem 1.15 and [3, Proposition 3.23] we get:

Corollary 3.12. *Let R be Noetherian, A, B be R-algebras and $\xi : A \to B$ be an R-algebra morphism.*

1. *Let A, B be cofinitely R-cogenerated α-algebras, $P := (A, A^\circ)$, $Q := (B, B^\circ)$ and consider the morphism of measuring α-pairings $(\xi, \xi^\circ) : (B, B^\circ) \to (A, A^\circ)$. Then we have for every right A°-comodule N :*

$$\mathrm{Coind}_P^Q(N) = \{f \in \mathrm{Hom}_{A-}(B, N)|\ Bf \textit{ is finitely generated in } \mathcal{M}_R\}.$$

2. *Let $\mathfrak{F}_A$, $\mathfrak{F}_B$ be cofinitely R-cogenerated α-filters of R-cofinite A-ideals, B-ideals respectively and consider A, B as a left linear topological R-algebra with the induced left linear topologies $\mathfrak{T}(\mathfrak{F}_A)$, $\mathfrak{T}(\mathfrak{F}_B)$ respectively. If $\xi : A \to B$ be an R-algebra morphism that is continuous with respect to $(\mathfrak{T}(\mathfrak{F}_A), \mathfrak{T}(\mathfrak{F}_B))$, $P := (A, A^\circ_{\mathfrak{F}_A})$ and $Q := (B, B^\circ_{\mathfrak{F}_B})$, then we have for every $N \in \mathcal{M}^{A^\circ_{\mathfrak{F}_A}}$:*

$$\mathrm{Coind}_P^Q(N) = \{f \in \mathrm{Hom}_{A-}(B, N)|\ (0 : f) \supset \widetilde{I} \textit{ for some } \widetilde{I} \in \mathfrak{F}_B\}.$$

4 Hopf R-pairings

Definition 4.1. Let H be an R-bialgebra. An H-ideal, which is also an H-coideal, is called a *bi-ideal*. If H is a Hopf R-algebra with antipode S_H and $J \subset H$ is an H-bi-ideal with $S_H(J) \subset J$, then J is called a *Hopf ideal*.

4.2. The category $\mathcal{P}_{Big}$. A *bialgebra R-pairing* is an R-pairing $P = (H, K)$, where H, K are R-bialgebras and $\kappa_P : H \to K^*$, $\chi_P : K \to H^*$ are R-algebra morphisms. For bialgebra R-pairings $(H, K), (Y, K)$ a morphism of R-pairings $(\xi, \theta) : (Y, Z) \to (H, K)$ is said to be a *morphism of bialgebra R-pairings,* if $\xi : H \to Y$ and $\theta : Z \to K$ are R-bialgebra morphisms. With $\mathcal{P}_{Big} \subset \mathcal{P}_m$ we denote the subcategory of bialgebra R-pairings and with $\mathcal{P}^\alpha_{Big} \subset \mathcal{P}_{Big}$ the *full* subcategory, whose objects satisfy the α-condition.

If $P = (H, K) \in \mathcal{P}_{Big}$, $Z \subset K$ is a (pure) R-subbialgebra and $J \subset H$ is an *H-bi-ideal* with $< J, Z >= 0$, then $Q = (H/J, Z)$ is a bialgebra R-pairing, $(\pi_J, \iota_Z) : (H/J, Z) \to (H, K)$ is a morphism in $\mathcal{P}_{Big}$ and $Q \subset P$ is called a (pure) *bialgebra R-subpairing.* Obviously $\mathcal{P}^\alpha_{Big} \subset \mathcal{P}_{Big}$ is closed under pure bialgebra R-subpairings.

4.3. The category $\mathcal{P}_{Hopf}$. A *Hopf R-pairing* $P = (H, K)$ is a bialgebra R-pairing with H, K Hopf R-algebras. With $\mathcal{P}_{Hopf} \subset \mathcal{P}_{Big}$ we denote the *full* subcategory of Hopf

R-pairings and with $\mathcal{P}^{\alpha}_{Hopf} \subset \mathcal{P}_{Hopf}$ the *full* subcategory, whose objects satisfy the α-condition. If $P = (H, K)$ is a Hopf R-pairing, $Z \subset K$ a (pure) Hopf R-subalgebra and $J \subset H$ a *Hopf ideal* with $< J, Z >= 0$, then $Q := (H/J, Z)$ is a Hopf R-pairing, $(\pi_J, \iota_Z) : (H/J, Z) \rightarrow (H, K)$ is a morphism in $\mathcal{P}_{Hopf}$ and $Q \subset P$ is called a (pure) *Hopf R-subpairing.* Obviously $\mathcal{P}^{\alpha}_{Hopf} \subset \mathcal{P}_{Hopf}$ is closed under pure Hopf R-subpairings.

Example 4.4. Let R be Noetherian and H be an α-bialgebra (respectively a Hopf α-algebra). Then H° is by ([6, Theorem 2.8]) an R-bialgebra (respectively a Hopf R-algebra). Moreover it is easy to see that (H, H°) is a bialgebra α-pairing (respectively a Hopf α-pairing).

Remarks 4.5.

1. (Compare [33]) If $P = (H, K)$ is a Hopf R-pairing, then

$$< S_H(h), k >=< h, S_K(k) > \text{ for all } h \in H \text{ and } k \in K.$$

2. Let R be Noetherian. If $P = (H, K)$ is a bialgebra R-pairing (respectively a Hopf R-pairing), then $\kappa_P(H) \subset K^{\circ}$ and $\chi_P(K) \subset H^{\circ}$. If $(H, K) \in \mathcal{P}_{Big}$ and $H \in \mathbf{Big}^{\alpha}_{R}$ (respectively $K \in \mathbf{Big}^{\alpha}_{R}$), then $\chi_P : K \rightarrow H^{\circ}$ (respectively $\kappa_P : H \rightarrow K^{\circ}$) is an R-bialgebra morphism.

Quasi-Admissible filters.

By the study of induced representations of quantum groups, Z. Lin [24] and M. Takeuchi [32] studied the so called *admissible filters* of ideals of a Hopf R-algebra over arbitrary (Dedekind) rings. In what follows we introduce what we call the *quasi-admissible filters* and generalize some of their results to the class of (not necessarily cofinitary) *quasi-admissible α-filters.*

4.6. Let A, B be R-algebras and $\mathfrak{F}_A$, $\mathfrak{F}_B$ be filters consisting of R-cofinite A-ideals, B-ideals respectively. Then the filter basis

$$\mathfrak{F}_A \times \mathfrak{F}_B := \{\mathrm{Im}(\iota_I \otimes id_B) + \mathrm{Im}(id_A \otimes \iota_J) |\ I \in \mathfrak{F}_A,\ J \in \mathfrak{F}_B\} \tag{24}$$

induces on $A \otimes_R B$ a topology $\mathfrak{T}(\mathfrak{F}_A \times \mathfrak{F}_B)$, such that $(A \otimes_R B, \mathfrak{T}(\mathfrak{F}_A \times \mathfrak{F}_B))$ is a linear topological R-algebra and $\mathfrak{F}_A \times \mathfrak{F}_B$ is a neighbourhood basis of $0_{A \otimes_R B}$.

4.7. Let H be an R-bialgebra (that is not a Hopf R-algebra), $\mathfrak{F} \subset \mathcal{K}_H$ be a filter and consider the induced linear topological R-algebras $(H, \mathfrak{T}(\mathfrak{F}))$ and $(H \otimes_R H, \mathfrak{T}(\mathfrak{F} \times \mathfrak{F}))$. We call $\mathfrak{F}$ *quasi-admissible,* if $\Delta_H : H \rightarrow H \otimes_R H$ and $\varepsilon_H : H \rightarrow R$ are continuous, i.e. if $\mathfrak{F}$ satisfies the following axioms:

$$\text{(A1)} \quad \forall\ I, J \in \mathfrak{F} \text{ there exists } L \in \mathfrak{F}, \text{ such that } \Delta_H(L) \subseteq \mathrm{Im}(\iota_I \otimes id_H) + \mathrm{Im}(id_H \otimes \iota_J) \tag{25}$$

and

$$\text{(A2)} \quad \exists\ I \in \mathfrak{F}, \text{ such that } \mathrm{Ker}(\varepsilon_H) \supset I. \tag{26}$$

If H is a Hopf R-algebra, then we call a filter $\mathfrak{F} \subset \mathcal{K}_H$ *quasi-admissible*, if it satisfies (A1), (A2) as well as

$$\text{(A3)} \quad \text{for every } I \in \mathfrak{F} \text{ there exists } J \in \mathfrak{F}, \text{ such that } S_H(J) \subseteq I \tag{27}$$

(i.e. if Δ_H, ε_H and S_H are continuous). In [24] and [32], a cofinitary quasi-admissible filter of R-cofinite H-ideals (for a Hopf R-algebra H) is called *admissible*.

Definition 4.8. We call an R-bialgebra (respectively Hopf R-algebra) H a *quasi-admissible R-bialgebra* (respectively a *quasi-admissible Hopf R-algebra*), if the class of R-cofinite H-ideals $\mathcal{K}_H$ is a *quasi-admissible* filter.

Lemma 4.9. *If the ground ring R is Noetherian, then every R-bialgebra (Hopf R-algebra) is quasi-admissible.*

Proof. Let H be an R-bialgebra. Since R is Noetherian, $\mathcal{K}_H$ is a filter. Moreover, $H \simeq R \oplus \operatorname{Ker}(\varepsilon_H)$, hence $\operatorname{Ker}(\varepsilon_H) \in \mathcal{K}_H$. Let $I, J \in \mathcal{K}_H$ and set $L := \operatorname{Im}(\iota_I \otimes id_H) + \operatorname{Im}(id_H \otimes \iota_J)$. Notice that $(H \otimes_R H)/L \simeq H/I \otimes_R H/J$ (e.g. [11, II-3.6, III-4.2]), hence $L \in \mathcal{K}_{H \otimes_R H}$. By definition $\Delta : H \to H \otimes_R H$ is an R-algebra morphism and it follows, by the assumption R is Noetherian, that $\Delta^{-1}(L) \lhd H$ is an R-cofinite ideal. Consequently H is a quasi-admissible R-bialgebra.

If H is moreover a Hopf R-algebra, then $S_H : H \to H$ is an R-algebra antimorphism and it follows, from the assumption R is Noetherian, that for every R-cofinite ideal $I \lhd H$ the H-ideal $S_H^{-1}(I) \lhd H$ is R-cofinite. Consequently H is a quasi-admissible Hopf R-algebra.■

Definition 4.10. ([34]) An R-coalgebra C is called *infinitesimal flat,* if $C = \underrightarrow{lim} C_\lambda$ for a directed system of *finitely generated projective* R-subcoalgebras $\{C_\lambda\}_\Lambda$.

Proposition 4.11. *Let H be an R-bialgebra (respectively a Hopf R-algebra) and $\mathfrak{F} \subset \mathcal{K}_H$ be a quasi-admissible filter.*

1. *If R is Noetherian and $\mathfrak{F}$ is an α-filter, then $H^\circ_{\mathfrak{F}}$ is an R-bialgebra (respectively a Hopf R-algebra) and $(H, H^\circ_{\mathfrak{F}})$ is a bialgebra α-pairing (respectively a Hopf α-pairing).*

2. *If $\mathfrak{F}$ is moreover cofinitary, then $H^\circ_{\mathfrak{F}}$ is an* infinitesimal flat *R-bialgebra (Hopf R-algebra) and $(H, H^\circ_{\mathfrak{F}})$ is a bialgebra α-pairing (a Hopf α-pairing).*

Proof. 1. Let H be an R-bialgebra. Obviously $H^\circ_{\mathfrak{F}} \subset H^\circ$ is an H-subbimodule under the regular left and the right H-actions (12) and so an R-coalgebra by Theorem 1.20. If $f(I) = 0$ and $g(J) = 0$ for $I, J \in \mathfrak{F}$, then there exists by (25) some $L \in \mathfrak{F}$, such that $\Delta(L) \subseteq \operatorname{Im}(\iota_I \otimes id_H) + \operatorname{Im}(id_H \otimes \iota_J)$. Consequently $\Delta^\circ(f \otimes g)(L) = (f \underline{\otimes} g)(\Delta(L)) = 0$, i.e. $f \star g \in \operatorname{An}(L) \subset H^\circ_{\mathfrak{F}}$. By (26) $\varepsilon_H \in H^\circ_{\mathfrak{F}}$ and so $H^\circ_{\mathfrak{F}} \subset H^*$ is an R-subalgebra. It is easy to see that $\Delta^\circ : H^\circ_{\mathfrak{F}} \otimes_R H^\circ_{\mathfrak{F}} \to H^\circ_{\mathfrak{F}}$ and $\varepsilon^\circ : R \to H^\circ_{\mathfrak{F}}$ are coalgebra morphisms, i.e. $H^\circ_{\mathfrak{F}}$ is an R-bialgebra. If H is a Hopf R-algebra with Antipode S, then it follows from (27) that $S^\circ(H^\circ_{\mathfrak{F}}) \subseteq H^\circ_{\mathfrak{F}}$, hence $H^\circ_{\mathfrak{F}}$ is a Hopf R-algebra with antipode S°.

2. See [32].■

As a consequence of Lemma 4.9 and Proposition 4.11 we get

Corollary 4.12. *Let R be Noetherian. If H is an α-bialgebra (respectively a Hopf α-algebra), then H° is an R-bialgebra (respectively a Hopf R-algebra). If H is cofinitary, then H° is an infinitesimal flat R-bialgebra (respectively Hopf R-algebra).*

Proposition 4.13. *Let H be an R-bialgebra, $\mathfrak{F}$ be a quasi-admissible filter of R-cofinite H-ideals and consider H as a left linear topological R-algebra with the induced left linear topology $\mathfrak{T}(\mathfrak{F})$. If R is an injective cogenerator, then the following are equivalent:*

1. *$\mathfrak{T}(\mathfrak{F})$ is Hausdorff;*
2. *the canonical R-linear mapping $\lambda : H \to H_{\mathfrak{F}}^{\circ *}$ is injective;*
3. *$H_{\mathfrak{F}}^{\circ} \subset H^{*}$ is dense;*
4. *$\sigma[{}_{H_{\mathfrak{F}}^{\circ}}H] = \sigma[{}_{H^{*}}H]$.*

Proof. By assumption H/I is R-cogenerated for every $I \in \mathfrak{F}$ (hence $I = \mathrm{KeAn}(I)$ by [38, 28.1]) and so

$$\overline{0_A} := \bigcap_{I \in \mathfrak{F}} I = \bigcap_{I \in \mathfrak{F}} \mathrm{KeAn}(I) = \mathrm{Ke}(\sum_{I \in \mathfrak{F}} \mathrm{An}(I)) = \mathrm{Ke}(H_{\mathfrak{F}}^{\circ}) = \mathrm{Ker}(\lambda).$$

Since R is an injective cogenerator, the equivalence (2) $\Leftrightarrow$ (3) follows from [2, Theorem 1.8 (2)]. By assumption $\mathfrak{F}$ is quasi-admissible, hence $H_{\mathfrak{F}}^{\circ} \subset H^{*}$ is an R-subalgebra and the equivalence (3) $\Leftrightarrow$ (4) follows by Lemma 1.2.■

The proof of the following Proposition is along the lines of the proof of [5, Theorem 4.10]:

Proposition 4.14. *Let H, K be R-bialgebras (Hopf R-algebras) with quasi-admissible filters $\mathfrak{F}_H$, $\mathfrak{F}_K$ and consider the canonical R-linear mapping $\delta : H^{*} \otimes_R K^{*} \to (H \otimes_R K)^{*}$ and the filter $\mathfrak{F}$ of R-cofinite $H \otimes_R K$-ideals generated by $\mathfrak{F}_H \times \mathfrak{F}_K$.*

1. *If $\mathfrak{F}_H$ and $\mathfrak{F}_K$ are moreover cofinitary (i.e. admissible filters), then $(H \otimes_R K)_{\mathfrak{F}}^{\circ}$ is an R-bialgebra (respectively a Hopf R-algebra). If R is Noetherian, then δ induces an isomorphism of R-bialgebras (respectively Hopf R-algebras) $H_{\mathfrak{F}_H}^{\circ} \otimes_R K_{\mathfrak{F}_K}^{\circ} \simeq (H \otimes_R K)_{\mathfrak{F}}^{\circ}$.*
2. *Let R be Noetherian. If $\mathfrak{F}_K$ is an α-filter and $\mathfrak{F}_H$ is cofinitary, then $(H \otimes_R K)_{\mathfrak{F}}^{\circ}$ is an R-bialgebra (a Hopf R-algebra) and δ induces an isomorphism of R-bialgebras (Hopf R-algebras) $H_{\mathfrak{F}_H}^{\circ} \otimes_R K_{\mathfrak{F}_K}^{\circ} \simeq (H \otimes_R K)_{\mathfrak{F}}^{\circ}$.*

Definition 4.15. The ring R is called *hereditary*, if every ideal $I \lhd R$ is projective.

Theorem 4.16. *Let R be Noetherian.*

1. *If H is an α-bialgebra (respectively a Hopf α-algebra), then $(H, H^{\circ}) \in \mathcal{P}_{Big}^{\alpha}$ (respectively $(H, H^{\circ}) \in \mathcal{P}_{Hopf}^{\alpha}$). If moreover H is commutative (cocommutative), then H° is cocommutative (commutative).*

2. *If R is hereditary, then there are* self-adjoint *contravariant functors*

$$\begin{array}{rrcl} (-)^{\circ} : & \mathbf{Big}_R & \to & \mathbf{Big}_R, \\ : & \mathbf{CBig}_R & \to & \mathbf{CCBig}_R, \\ : & \mathbf{CCBig}_R & \to & \mathbf{CBig}_R, \end{array} \quad \begin{array}{rrcl} (-)^{\circ} : & \mathbf{Hopf}_R & \to & \mathbf{Hopf}_R. \\ : & \mathbf{CBig}_R & \to & \mathbf{CCBig}_R \\ : & \mathbf{CCHopf}_R & \to & \mathbf{CHopf}_R \end{array}$$

Proof. 1. If H is an α-bialgebra (a Hopf α-algebra), then H° is by corollary 4.12 an R-bialgebra (a Hopf R-algebra) and $(H, H^{\circ}) \in \mathcal{P}_{Big}^{\alpha}$ (respectively $(H, H^{\circ}) \in \mathcal{P}_{Hopf}^{\alpha}$). The duality between the commutativity and the cocommutativity follows now from [3, Lemma 2.2].

2. Let R be hereditary. Then for every R-bialgebra (respectively Hopf R-algebra) H the continuous dual R-module $H^{\circ} \subset R^H$ is pure, [6, Proposition 2.11], hence every R-bialgebra (respectively Hopf R-algebra) is an α-bialgebra (respectively a Hopf α-algebra) and H° is an R-bialgebra (respectively a Hopf R-algebra). Moreover

$$\Upsilon_{H,K} : \mathrm{Big}_R(H, K^{\circ}) \to \mathrm{Big}_R(K, H^{\circ}),\ f \mapsto [k \mapsto f(-)(k)]$$

is an isomorphism with inverse

$$\Psi_{H,K} : \mathrm{Big}_R(K, H^{\circ}) \to \mathrm{Big}_R(H, K^{\circ}),\ g \mapsto [h \mapsto g(-)(h)].$$

It is easy to show that $\Upsilon_{H,K}$ and $\Psi_{H,K}$ are functorial in H and K.■

5 Coinduction functors in $\mathcal{P}_{Hopf}^{\alpha}$

In this section we consider the coinduction functors for the category of Hopf α-pairings respectively bialgebra α-pairings that unify several important situations (e.g. [15], [8], [24, 3.2]).

Definition 5.1. Let H be an R-bialgebra. For every left H-module M we call the R-submodule

$$M^H := \{m \in M|\ hm = \varepsilon(h)m \text{ for all } h \in H\}$$

the *submodule of H-invariants* of M. For every right H-comodule M we call

$$M^{coH} := \{m \in M|\ \varrho_M(m) = m \otimes 1_H\}$$

the *submodule of H-coinvariants* of M.

5.2. Let H be an R-bialgebra. If M, N are right (respectively left) H-modules, then $M \otimes_R N$ is a right (respectively a left) H-module with the *canonical* H-module structure

$$(m \otimes n)h := \sum mh_1 \otimes nh_2 \text{ (respectively } h(m \otimes n) := \sum h_1 m \otimes h_2 n). \tag{28}$$

In particular the ground ring R is an H-bimodule through

$$h \rightharpoonup r := \varepsilon(h)r =: r \leftharpoonup h \text{ for all } h \in H \text{ and } r \in R.$$

5.3. Let K be an R-bialgebra. If M, N are right (respectively left) K-comodules, then $M \otimes_R N$ is a right (respectively a left) K-comodule through the *canonical* right (respectively left) K-comodule structure

$$m \otimes n \mapsto \sum m_{<0>} \otimes n_{<0>} \otimes m_{<1>} n_{<1>} \text{ (resp. } m \otimes n \mapsto \sum m_{<-1>} n_{<-1>} \otimes m_{<0>} n_{<0>}). \tag{29}$$

In particular the ground ring R is a K-bicomodule throughout

$$R \to R \otimes_R K,\ r \mapsto r \otimes 1_K \text{ and } R \to K \otimes_R R,\ r \mapsto 1_K \otimes r.$$

Lemma 5.4. *Let $P = (H, K) \in \mathcal{P}_{Big}$, (M, ϱ_M) be a right K-comodule and consider M with the induced left H-module structure. If $\alpha_M^P : M \otimes_R K \to \mathrm{Hom}_R(H, M)$ is injective, then $M^H = M^{coK}$.*

Proof. We have for all $m \in M^{coK}$ and $h \in H$:

$$hm = m < h, 1_K >= m\varepsilon_H(h) \text{ for every } h \in H,$$

i.e. $m \in M^H$. On the other hand, we have for all $m \in M^H$ and $h \in H$:

$$\begin{array}{rclcl} \alpha_M^P(\sum m_{<0>} \otimes m_{<1>})(h) & = & \sum m_{<0>} < h, m_{<1>} > & = & hm \\ & = & m\varepsilon_H(h) & = & m < h, 1_K > \\ & = & \alpha_M^P(m \otimes 1_K)(h). & & \end{array}$$

If α_M^P is injective, then $\varrho_M(m) = \sum m_{<0>} \otimes m_{<1>} = m \otimes 1_K$, i.e. $m \in M^{coK}$ and consequently $M^H = M^{coK}$.■

Lemma 5.5. *Let H be a Hopf R-algebra and $M, N \in {}_H\mathcal{M}$. Then $\mathrm{Hom}_R(M, N)$ is a left H-module through*

$$(hf)(m) = \sum h_1 f(S_H(h_2)m) \text{ for all } h \in H, m \in M \text{ and } f \in \mathrm{Hom}_R(M, N). \tag{30}$$

Moreover $\mathrm{Hom}_{H-}(M, N) = \mathrm{Hom}_R(M, N)^H$.

Proof. For all $h, \widetilde{h} \in H$, $f \in \mathrm{Hom}_R(M,N)$ and $m \in M$ we have

$$\begin{array}{lllll}((h\widetilde{h})f)(m) & := & \sum(h\widetilde{h})_1 f(S_H((h\widetilde{h})_2)m) & = & \sum h_1\widetilde{h}_1 f(S_H(h_2\widetilde{h}_2)m) \\ & = & \sum h_1\widetilde{h}_1 f(S_H(\widetilde{h}_2)S_H(h_2)m) & = & \sum h_1((\widetilde{h}f)(S_H(h_2)m)) \\ & = & (h(\widetilde{h}f))(m), & & \end{array}$$

i.e. $\mathrm{Hom}_R(M,N)$ is a left H-module with the left H-action (30).

For all $f \in \mathrm{Hom}_{H-}(M,N)$, $h \in H$ and $m \in M$ we have

$$(hf)(m) := \sum h_1 f(S_H(h_2)m) = \sum h_1 S_H(h_2) f(m) = (\varepsilon(h)1_H)f(m) = (\varepsilon(h)f)(m),$$

i.e. $f \in \mathrm{Hom}_R(M,N)^H$. On the other hand, if $g \in \mathrm{Hom}_R(M,N)^H$, then we have for all $h \in H$ and $m \in M$:

$$\begin{array}{lllll}g(hm) & = & g(\sum \varepsilon(h_1)h_2 m) & = & \sum(\varepsilon(h_1)g)(h_2 m) \\ & = & \sum(h_1 g)(h_2 m) & = & \sum h_{11}(g(S_H(h_{12})h_2 m)) \\ & = & \sum h_1(g(S_H(h_{21})h_{22}m)) & = & \sum h_1 g(\varepsilon(h_2)1_H m) \\ & = & (\sum h_1\varepsilon(h_2))g(m) & = & hg(m),\end{array}$$

i.e. $g \in \mathrm{Hom}_{H-}(M,N)$.■

The following lemma generalizes the corresponding results [24, Page 165] and [23, Page 103]:

Lemma 5.6. *Let $P = (H,K)$, $Q = (Y,Z) \in \mathcal{P}^{\alpha}_{Hopf}$, $(\xi,\theta) : (Y,Z) \to (H,K)$ be a morphism in $\mathcal{P}^{\alpha}_{Hopf}$ and $N \in {}_H\mathcal{M}$.*

1. $\mathrm{Hom}_R(Y,N)$ *is a left H-module through*

$$(hf)(y) = \sum h_1 f(S_Y(\xi(h_2)y)) \text{ for all } h \in H, f \in \mathrm{Hom}_R(Y,N) \text{ and } y \in Y. \quad (31)$$

2. *If we consider* $\mathrm{Hom}_R(Y,N)$ *with the canonical left Y-module structure, then*

$$h(yf) = y(hf) \text{ for all } h \in H, y \in Y \text{ and } f \in \mathrm{Hom}_R(Y,N).$$

   So $\mathrm{Hom}_R(Y,N)^H \subseteq \mathrm{Hom}_R(Y,N)$ *is a left Y-submodule.*

3. *If ${}_HN$ is K-rational, then $N \otimes_R Z$ is a right K-comodule through*

$$\psi : N \otimes_R Z \to N \otimes_R Z \otimes_R K,\ n \otimes z \mapsto \sum n_{<0>} \otimes z_2 \otimes n_{<1>} S_K(\theta(z_1)). \quad (32)$$

Proof. 1. By assumption $\xi : H \to Y$ is a Hopf R-algebra morphism and so $\xi(S_H(h)) = S_Y(\xi(h))$ for every $h \in H$. If we consider the left H-module Y_ξ, then the left H-action on $\mathrm{Hom}_R(Y_\xi, N)$ in (30) coincides with that in (31), hence $\mathrm{Hom}_R(Y,N)$ is a left H-module by Lemma 5.5.

2. Trivial.

3. Z is obviously a right K-comodule through

$$\varrho_Z : Z \to Z \otimes_R K,\ z \mapsto \sum z_2 \otimes S_K(\theta(z_1)) \text{ for every } z \in Z.$$

By assumption and Theorem 1.14 N is a right K-comodule and so $(N \otimes_R Z, \psi)$ is, by 5.3, a right K-comodule.■

5.7. Let $P = (H, K)$, $Q = (Z, Y) \in \mathcal{P}^{\alpha}_{Hopf}$ and $(\xi, \theta) : (Y, Z) \to (H, K)$ be a morphism in $\mathcal{P}^{\alpha}_{Hopf}$. For every $N \in \mathrm{Rat}^K({}_H\mathcal{M})$ consider $N \otimes_R Z$ with the right K-comodule structure (32). If we consider the coinduction functor

$$\mathrm{Coind}^Q_P(-) : \mathrm{Rat}^K({}_H\mathcal{M}) \to \mathrm{Rat}^Z({}_Y\mathcal{M}),\ N \mapsto \mathrm{HOM}_{H-}(Y, N) := \mathrm{Rat}^Z({}_Y(\mathrm{Hom}_{H-}(Y, N))),$$

then we have functorial isomorphisms

$$\begin{array}{rcll} (N \otimes_R Z)^{coK} & \simeq & (N \otimes_R Z)^H & \text{(Lemma 5.4);} \\ & \simeq & \mathrm{HOM}_R(Y, N)^H & \text{(3.8);} \\ & = & \mathrm{Rat}^Z({}_Y(\mathrm{Hom}_R(Y, N)^H)) & \\ & = & \mathrm{HOM}_{H-}(Y, N) := \mathrm{Coind}^Q_P(N) & \text{(Lemma 5.5);} \\ & \simeq & N \Box_K Z & \text{(Theorem 3.7);} \\ & \simeq & \mathrm{Hom}_{H^e}(H, N \otimes_R Z). & \text{(Proposition 2.8).} \end{array}$$

Corollary 5.8. *Let $P = (H, K)$, $Q = (Y, Z) \in \mathcal{P}^{\alpha}_{Hopf}$ and $(\xi, \theta) : (Y, Z) \to (H, K)$ be a morphism in $\mathcal{P}^{\alpha}_{Hopf}$. Let $M \in \mathcal{M}^Z$, $N \in \mathcal{M}^K$ and consider $M^{\theta} \otimes_R N$ with the canonical right K-comodule structure. If M_R is flat, then there is an isomorphism of Z-comodules*

$$\mathrm{Coind}^Q_P(M^{\theta} \otimes_R N) \simeq (M^{\theta} \otimes_R N)\Box_K Z \simeq M \otimes_R (N \Box_K Z) \simeq M \otimes_R \mathrm{Coind}^Q_P(N).$$

6 Classical Duality

Over a commutative base field one has a *duality* between the groups and the commutative Hopf algebras (e.g. [28, 9.3], [30]). In this section we show that such a duality is valid over *hereditary Noetherian* ground rings.

Definition 6.1. Let (C, Δ, ε) be an R-coalgebra. With

$$\mathcal{G}(C) := \{0 \neq x \in C |\ \Delta(x) = x \otimes x \text{ and } \varepsilon(x) = 1_R\}$$

we denote the set of *group-like elements* of C. If $x, y \in \mathcal{G}(C)$, then we denote with

$$P_{(x,y)}(c) := \{c \in C |\ \Delta(c) = x \otimes c + c \otimes y\}$$

the set of (x, y)*-primitive elements* in C. For an R-bialgebra B we call the $(1_B, 1_B)$-primitive elements of B *primitive elements.*

The following result is easy to prove

Lemma 6.2. *Let C be an R-coalgebra.*

1. *If D is an R-coalgebra and $f : D \to C$ is an R-coalgebra morphism, then $f(\mathcal{G}(D)) \subseteq \mathcal{G}(C)$.*

2. *If $\{0_R, 1_R\}$ are the only idempotents in R (e.g. R is a domain) and $\Delta_C(x) = x \otimes x$ for some $0 \neq x \in C$, then $\varepsilon_C(x) = 1_R$, i.e. $x \in \mathcal{G}(C)$.*

3. *If $x, y \in \mathcal{G}(C)$ and $c \in P_{(x,y)}(C)$, then $\varepsilon_C(c) = 0$.*

4. *For every R-coalgebra C we have a bijection*

$$\mathrm{Cog}_R(R, C) \leftrightarrow \mathcal{G}(C),\ \ f \mapsto f(1_R) \text{ and } x \mapsto [1_R \mapsto x]\ \forall\ f \in \mathrm{Cog}_R(R, C),\ \ x \in \mathcal{G}(C).$$

5. *If R is Noetherian and A is an α-algebra, then $\mathrm{Alg}_R(A, R) = \mathcal{G}(A^{\circ}) = \mathrm{Cog}_R(R, A^{\circ})$.*

6.3. For every set G the free R-module RG becomes a *cocommutative* R-coalgebra $\mathcal{K}(G) := (RG, \Delta_g, \varepsilon_g)$, where the comultiplication Δ_g and the counit ε_g are given by the linear extension of their images on the elements of G :

$$\Delta_g(x) = x \otimes x \text{ and } \varepsilon_g(x) = 1 \text{ for every } x \in G.$$

If (G, μ_G, e_G) is a monoid, then μ_G respectively e_G induce on RG a multiplication μ respectively a unity η, such that $\mathcal{K}(G) = (RG, \mu, \eta, \Delta_g, \varepsilon_g)$ is an R-bialgebra. If G is moreover a group, then RG is a Hopf R-algebra with antipode defined on the basis elements as $S_g : RG \to RG,\ x \mapsto x^{-1}$ for every $x \in G$. On the other hand, let H be an R-bialgebra. Then $\Delta_H(1_H) = 1_H \otimes 1_H$ and we have for all $x, y \in \mathcal{G}(H)$:

$$\Delta_H(xy) = \Delta_H(x)\Delta_H(y) = (x \otimes x)(y \otimes y) = xy \otimes xy,$$

i.e. xy is a group-like element in H and $\mathcal{G}(H)$ is a monoid. If H is moreover a Hopf R-algebra and $x \in \mathcal{G}(H)$, then $x^{-1} := S_H(x) \in \mathcal{G}(H)$, i.e. $\mathcal{G}(H)$ is a group.

Proposition 6.4. ([19]) *Denote with* **Ens**, **Mon** *and* **Gr** *the categories of sets, monoids and groups respectively. Then we have adjoint pairs of covariant functors $(\mathcal{K}(-), \mathcal{G}(-))$:*

$$\begin{array}{lllllllll} \mathcal{K}(-) & : & \mathbf{Ens} & \to & \mathbf{CCog}_R, & \mathcal{G}(-) & : & \mathbf{CCog}_R & \to & \mathbf{Ens} \\ & : & \mathbf{Mon} & \to & \mathbf{CCBialg}_R, & & : & \mathbf{CCBialg}_R & \to & \mathbf{Mon} \\ & : & \mathbf{Gr} & \to & \mathbf{CCHopf}_R, & & : & \mathbf{CCHopf}_R & \to & \mathbf{Gr}. \end{array}$$

If R is moreover an integral domain, then we have a natural isomorphism $\mathcal{G}(-) \circ \mathcal{K}(-) \simeq id$.

Representative mappings

6.5. Let R be Noetherian, (G,μ,e) be a monoid (respectively a group) and denote with

$$\mathcal{R}(G) := \{f \in R^G \mid GfG \text{ is finitely generated in } \mathcal{M}_R\} \simeq (RG)^\circ$$

the set of *representative mappings* on G. We call G an *α-monoid* (respectively an *α-group*), if $(RG,\mathcal{R}(G))$ is an α-pairing, or equivalently if $\mathcal{R}(G) \subset R^G$ is pure.

As a consequence of Lemma 1.19 and Corollary 4.12 we get

Corollary 6.6. *Let R be Noetherian. If G is an α-monoid, then $\mathcal{R}(G)$ is an R-bialgebra. If G is moreover an α-group, then $\mathcal{R}(G)$ is a Hopf R-algebra with antipode*

$$S:\mathcal{R}(G)\to\mathcal{R}(G),\ S(f)(x)=f(x^{-1}) \text{ for } f\in\mathcal{R}(G) \text{ and } x\in G.$$

Notation. Let G be a monoid. The category of unital left (respectively right) G-modules is denoted by ${}_G\mathcal{M}$ (respectively $\mathcal{M}_G$).

As a consequence of Theorem 1.20 we get

Corollary 6.7. *Let R be Noetherian, G be a monoid and $C \subseteq \mathcal{R}(G)$ be a G-subbimodule. If $P=(RG,C)$ is an α-pairing, then C is an R-coalgebra and we have category isomorphisms*

$$\begin{array}{rcl} \mathcal{M}^C & \simeq & \mathrm{Rat}^C({}_G\mathcal{M}) = \sigma[{}_{RG}C] \\ & \simeq & \mathrm{Rat}^C({}_{C^*}\mathcal{M}) = \sigma[{}_{C^*}C] \end{array} \ \& \ \begin{array}{rcl} {}^C\mathcal{M} & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_G) = \sigma[C_{RG}] \\ & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_{C^*}) = \sigma[C_{C^*}]. \end{array}$$

6.8. Let G be a monoid. A left (respectively right) G-module will be called *locally finite*, if $(RG)m$ (respectively $m(RG)$) is finitely generated in $\mathcal{M}_R$ for every $m \in M$. For every monoid G denote with $\mathrm{Loc}({}_G\mathcal{M}) \subset {}_G\mathcal{M}$ (respectively $\mathrm{Loc}(\mathcal{M}_G) \subseteq \mathcal{M}_G$) the full subcategory of locally finite left (respectively right) G-modules.

As a consequence of [3, Proposition 3.23] we get

Proposition 6.9. *Let R be Noetherian and G be a monoid.*

1. *Every $\mathcal{R}(G)$-subgenerated left (respectively right) G-module is locally finite.*
2. *If RG is cofinitely R-cogenerated, then $\sigma[{}_G\mathcal{R}(G)] = \mathrm{Loc}({}_G\mathcal{M})$ and $\sigma[\mathcal{R}(G)_G] = \mathrm{Loc}(\mathcal{M}_G)$. If G is moreover an α-monoid, then we have category isomorphisms*

$$\begin{array}{rcl} \mathcal{M}^{\mathcal{R}(G)} & \simeq & \mathrm{Rat}^{\mathcal{R}(G)}({}_G\mathcal{M}) = \sigma[{}_G\mathcal{R}(G)] = \mathrm{Loc}({}_G\mathcal{M}); \\ {}^{\mathcal{R}(G)}\mathcal{M} & \simeq & {}^{\mathcal{R}(G)}\mathrm{Rat}(\mathcal{M}_G) = \sigma[\mathcal{R}(G)_G] = \mathrm{Loc}(\mathcal{M}_G). \end{array}$$

The following result generalizes the classical duality between monoids (groups) and *commutative* R-bialgebras (Hopf R-algebras), e.g. [28, 9.3], from the case of base fields to the case of arbitrary hereditary Noetherian rings.

Theorem 6.10. *If R is Noetherian and hereditary, then there is a* duality *between monoids (respectively groups) and commutative R-bialgebras (respectively Hopf R-algebras) through the right-adjoint contravariant functors*

$$\begin{array}{rcccl} \mathcal{R}(-) & : & \mathbf{Mon} & \to & \mathbf{CBig}_R, \\ & : & \mathbf{Gr} & \to & \mathbf{CHopf}_R, \end{array} \quad \begin{array}{rcccl} \mathrm{Alg}_R(-,R) & : & \mathbf{CBig}_R & \to & \mathbf{Mon}, \\ & : & \mathbf{CHopf}_R & \to & \mathbf{Gr}. \end{array}$$

Proof. Let R be Noetherian and hereditary. Then for every R-algebra A, the character module $A^\circ \subset R^A$ is pure (e.g. [6, Proposition 2.11]). If G is a monoid (respectively a group), then $\mathcal{K}(G) = (RG, \mu, \eta, \Delta_g, \varepsilon_g)$ is by 6.3 a *cocommutative* R-bialgebra (respectively Hopf R-algebra) and so $\mathcal{R}(G) = (RG)^\circ$ is by Theorem 4.16 a *commutative* R-bialgebra (respectively Hopf R-algebra). If H is an R-bialgebra (respectively a Hopf R-algebra), then H° is by Theorem 4.16 an R-bialgebra (respectively a Hopf R-algebra), hence $\mathrm{Alg}_R(H,R) = \mathcal{G}(H^\circ)$ is a monoid (respectively a group). It is easy to see that we have isomorphisms of functors

$$\mathcal{R}(-) \simeq (-)^\circ \circ \mathcal{K}(-) \text{ and } \mathrm{Alg}_R(-,R) \simeq \mathcal{G}(-) \circ (-)^\circ.$$

The result follows now from Theorems 4.16 and 6.4.■

7 Affine group schemes

Affine groups schemes over arbitrary commutative ground rings were presented by J. Jantzen [21]. If $\mathfrak{G}$ is an affine group scheme with *coordinate ring* $R(\mathfrak{G})$, then the category of *left $\mathfrak{G}$-modules* ${}_{\mathfrak{G}}\mathcal{M}$ and the category of right $R(\mathfrak{G})$-comodules $\mathcal{M}^{R(\mathfrak{G})}$ are equivalent. In the case $R(\mathfrak{G})$ is locally projective as an R-module we extend this equivalence to the category of $R(\mathfrak{G})$-rational left $R(\mathfrak{G})^*$-modules $\mathrm{Rat}^{R(\mathfrak{G})}({}_{R(\mathfrak{G})^*}\mathcal{M})$ which turns to be equal to the category of $R(\mathfrak{G})$-subgenerated left $R(\mathfrak{G})^*$-modules $\sigma[{}_{R(\mathfrak{G})^*}R(\mathfrak{G})]$. It follows that in this case ${}_{\mathfrak{G}}\mathcal{M}$ is a Grothendieck category of type $\sigma[M]$ and one can use the well developed theory of such categories (e.g. [38], [37]) to study the category ${}_{\mathfrak{G}}\mathcal{M}$.

7.1. With an *R-functor* (respectively a *monoid R-functor,* a *group R-functor*) we understand a functor from the category of *commutative* R-algebras $\mathbf{CAlg}_R$ to $\mathbf{Ens}$ (respectively to $\mathbf{Mon}$, $\mathbf{Gr}$). An *affine scheme* (respectively an *affine monoid scheme*, an *affine group scheme*) over R is a *representable* R-functor (respectively monoid R-functor, group R-functor)

$$\begin{array}{rcccl} \mathfrak{G} = \mathrm{Alg}_R(H,-) & : & \mathbf{CAlg}_R & \to & \mathbf{Ens}, \\ & : & \mathbf{CBig}_R & \to & \mathbf{Mon}, \\ & : & \mathbf{CHopf_R} & \to & \mathbf{Gr}. \end{array}$$

The *commutative* R-algebra H is called the *coordinate ring* of $\mathfrak{G}$ and is denoted with $R(\mathfrak{G})$. With $\mathbf{Aff}_R$ (respectively $\mathbf{AffMon}_R$, $\mathbf{AffGr}_R$) we denote the category of affine schemes (respectively affine monoid schemes, affine group schemes) with morphisms the *natural transformations.*

7.2. $\mathfrak{G}$-modules. ([21, 2.7]) Let $\mathfrak{G} = \mathrm{Alg}_R(H, -)$ be an affine group scheme. An R-module M is said to be a *left* (respectively *a right*) *$\mathfrak{G}$-module*, if there is a $\mathfrak{G}(A)$ module structure on $M \otimes_R A$ (respectively on $A \otimes_R M$), functorial in A, for every commutative R-algebra A. The category of left (respectively right) $\mathfrak{G}$-modules and $\mathfrak{G}$-linear mappings will be denoted by ${}_{\mathfrak{G}}\mathcal{M}$ (respectively by $\mathcal{M}_{\mathfrak{G}}$).

7.3. Yoneda Lemma. ([38, 44.3]) Let $\mathfrak{C}$ be a category, $F : \mathfrak{C} \to \mathbf{Ens}$ be a covariant functor and denote for $A \in \mathfrak{C}$ the class of functorial morphisms between $Mor_{\mathfrak{C}}(A, -)$ and F with $\underline{\underline{Nat}}(Mor_{\mathfrak{C}}(A, -), F)$. Then the following *Yoneda-mapping* is bijective:

$$\underline{\underline{Nat}}(Mor_{\mathfrak{C}}(A, -), F) \to F(A), \ \phi \mapsto \phi_A(id_A).$$

With the help of Yoneda-Lemma (Compare [21, Chapter 2]) one obtains:

Proposition 7.4. *Let R be an arbitrary commutative ring.*

1. *If $\mathfrak{G} = \mathrm{Alg}_R(H, -)$ is an affine monoid scheme (respectively an affine group scheme), then the coordinate ring $H = R(\mathfrak{G})$ is an R-bialgebra (respectively a Hopf R-algerba) and we have equivalences of categories*

$$\mathbf{AffMon}_R \approx (\mathbf{CBig}_R)^{op} \text{ and } \mathbf{AffGr}_R \approx (\mathbf{CHopf}_R)^{op}.$$

2. *For every affine group scheme $\mathfrak{G}$ with coordinate ring $R(\mathfrak{G})$, the category of left $\mathfrak{G}$-modules ${}_{\mathfrak{G}}\mathcal{M}$ and the category of right $R(\mathfrak{G})$-comodules $\mathcal{M}^{R(\mathfrak{G})}$ are equivalent.*

7.5. Let $\mathfrak{G}$ be an affine group scheme with coordinate ring $R(\mathfrak{G})$, $\omega := \mathrm{Ker}(\varepsilon_{R(\mathfrak{G})})$, $\mathfrak{F}_\omega := \{\omega^n | \ n \geq 1\}$ and consider $R(\mathfrak{G})^*$ with the finite topology and $R(\mathfrak{G})$ with the induced left linear topology $\mathfrak{T}(\mathfrak{F}_\omega)$. By [21, 7.7]

$$hy(\mathfrak{G}) := \{f \in R(\mathfrak{G})^* | \ f(\omega^n) = 0 \text{ for some } n \geq 1\} \tag{33}$$

is an R-subalgebra of $R(\mathfrak{G})^*$, the so called *hyperalgebra* of $\mathfrak{G}$, and we get a measuring R-pairing $(hy(\mathfrak{G}), R(\mathfrak{G}))$. If $hy(\mathfrak{G}) \subset R(\mathfrak{G})^*$ is dense, then we call $\mathfrak{G}$ *connected.* If $R(\mathfrak{G})/\omega^n$ is finitely generated projective in $\mathcal{M}_R$ for every $n \geq 1$, then $\mathfrak{G}$ is called *infinitesimal flat.* We say $\mathfrak{G}$ satisfies the *α-condition* (or $\mathfrak{G}$ is an *affine α-group scheme*), if $(hy(\mathfrak{G}), R(\mathfrak{G}))$ satisfies the α-condition. We call $\mathfrak{G}$ *locally projective*, if $R(\mathfrak{G})$ is locally projective as an R-module.

Theorem 7.6. *Let $\mathfrak{G}$ be an affine group scheme with coordinate ring $R(\mathfrak{G})$.*

1. *If $\mathfrak{G}$ is locally projective, then there are equivalences of categories*

$${}_{\mathfrak{G}}\mathcal{M} \approx \mathcal{M}^{R(\mathfrak{G})} \simeq \mathrm{Rat}^{R(\mathfrak{G})}({}_{R(\mathfrak{G})^*}\mathcal{M}) = \sigma[{}_{R(\mathfrak{G})^*}R(\mathfrak{G})].$$

2. *$\mathfrak{G}$ is an affine α-group scheme if and only if $\mathfrak{G}$ is locally projective and connected. If these equivalent conditions are satisfied, then we have equivalences of categories*

$$\begin{array}{lllllll} {}_{\mathfrak{G}}\mathcal{M} & \approx & \mathcal{M}^{R(\mathfrak{G})} & \simeq & \mathrm{Rat}^{R(\mathfrak{G})}({}_{R(\mathfrak{G})^*}\mathcal{M}) & = & \sigma[{}_{R(\mathfrak{G})^*}R(\mathfrak{G})] \\ & & & \simeq & \mathrm{Rat}^{R(\mathfrak{G})}({}_{hy(\mathfrak{G})}\mathcal{M}) & = & \sigma[{}_{hy(\mathfrak{G})}R(\mathfrak{G})]. \end{array}$$

3. *The following are equivalent:*

   (i) $\mathfrak{G}$ is connected (i.e. $hy(\mathfrak{G}) \subset R(\mathfrak{G})^*$ is dense);

   (ii) $\sigma[{}_{hy(\mathfrak{G})}R(\mathfrak{G})] = \sigma[{}_{R(\mathfrak{G})^*}R(\mathfrak{G})]$.

   If R is a injective cogenerator, then (i), (ii) are moreover equivalent to:

   (iii) $R(\mathfrak{G}) \hookrightarrow hy(\mathfrak{G})^*$;

   (iv) $\mathfrak{T}(\mathfrak{F}_\omega)$ is Hausdorff.

Proof. 1. The equivalence ${}_{\mathfrak{G}}\mathcal{M} \approx \mathcal{M}^{R(\mathfrak{G})}$ follows from Proposition 7.4. The remaining category isomorphisms follow from Theorem 1.14.

2. Follows from Theorem 1.14.

3. $hy(\mathfrak{G}) \subset R(\mathfrak{G})^*$ is an R-subalgebra and so the equivalence *(i)* $\Leftrightarrow$ *(ii)* follows by Lemma 1.2.

   Let R be an injective cogenerator.

   The equivalence *(i)* $\Leftrightarrow$ *(iii)* follows from [2, Theorem 1.8 (2)]. Consider now the measuring R-pairings $\mathfrak{G} := (hy(\mathfrak{G}), R(\mathfrak{G}))$. Then we have

$$\overline{0_{R(\mathfrak{G})}} = \bigcap_{n=1}^{\infty} \omega^n = \bigcap_{n=1}^{\infty} \mathrm{KeAn}(\omega^n) = \mathrm{Ke}(\sum_{n=1}^{\infty} \mathrm{An}(\omega^n)) = \mathrm{Ke}(hy(\mathfrak{G})) = \mathrm{Ker}(\chi_{\mathfrak{G}}).$$

   Consequently $\mathfrak{T}(\mathfrak{F}_\omega)$ is Hausdorff if and only if $R(\mathfrak{G}) \overset{\chi_{\mathfrak{G}}}{\hookrightarrow} hy(\mathfrak{G})^*$ and we are done.■

Coinduction functors for affine α-schemes

7.7. Let $\mathfrak{G}$, $\mathfrak{H}$ be affine α-group schemes and $\varphi : \mathfrak{H} \to \mathfrak{G}$ be a morphism in $\mathbf{AffGr}_R$. Then φ induces a Hopf R-algebra morphism $\varphi_\# : R(\mathfrak{G}) \to R(\mathfrak{H})$ (called a *comorphism*) and we get a morphism in $\mathcal{P}_m^\alpha$

$$(\varphi_\#^*, \varphi_\#) : (R(\mathfrak{G})^*, R(\mathfrak{G})) \to (R(\mathfrak{H})^*, R(\mathfrak{H})).$$

By Theorem 7.6 ${}_{\mathfrak{H}}\mathcal{M} \approx \sigma[{}_{R(\mathfrak{H})^*}R(\mathfrak{H})]$, ${}_{\mathfrak{G}}\mathcal{M} \approx \sigma[{}_{R(\mathfrak{G})^*}R(\mathfrak{G})]$ and so we have the *coinduction functor*

$$\mathrm{Coind}_{\mathfrak{H}}^{\mathfrak{G}}(-) := \mathrm{Rat}^{R(\mathfrak{G})}(\mathrm{Hom}_{R(\mathfrak{H})^*-}(R(\mathfrak{G})^*, -) : \ {}_{\mathfrak{H}}\mathcal{M} \to {}_{\mathfrak{G}}\mathcal{M}.$$

Lemma 7.8. ([30, Lemma 6.1.1, Corollary 6.1.2]) *Let $I \lhd A$ be an ideal. If ${}_A I$ (respectively I_A) is finitely generated, then ${}_A I^n$ (respectively I_A^n) is finitely generated for every $n \geq 1$. If moreover $I \subset A$ is R-cofinite, then $I^n \subset A$ is R-cofinite.*

Corollary 7.9. *Let $\mathfrak{G}$ be an affine monoid scheme (respectively an affine group scheme) with coordinate ring $R(\mathfrak{G})$.*

1. *If R is Noetherian, ${}_{R(\mathfrak{G})}\omega$ is finitely generated and $hy(\mathfrak{G}) \subset R^{R(\mathfrak{G})}$ is pure, then $hy(\mathfrak{G})$ is an R-bialgebra (respectively a Hopf R-algebra) and $(R(\mathfrak{G}), hy(\mathfrak{G})) \in \mathcal{P}^{\alpha}_{Big}$ (respectively $(R(\mathfrak{G}), hy(\mathfrak{G})) \in \mathcal{P}^{\alpha}_{Hopf}$).*

2. *If $\mathfrak{G}$ is* infinitesimal flat, *then $hy(\mathfrak{G})$ is an* infinitesimal flat *R-bialgebra (respectively Hopf R-algebra) and $(R(\mathfrak{G}), hy(\mathfrak{G})) \in \mathcal{P}^{\alpha}_{Big}$ (respectively $(R(\mathfrak{G}), hy(\mathfrak{G})) \in \mathcal{P}^{\alpha}_{Hopf}$).*

Proof. 1. If ${}_{R(\mathfrak{G})}\omega$ is finitely generated, then $\mathfrak{F}_\omega \subset \mathcal{K}_{R(\mathfrak{G})}$ by Lemma 7.8 and so $hy(\mathfrak{G}) \subset R(\mathfrak{G})^{\circ}$ is an $R(\mathfrak{G})$-subbimodule. The result follows then from Proposition 4.11 (1).

2. The result follows from [28, Lemma 9.2.1] and Proposition 4.11 (2).■

Acknowledgment. This paper (up to a few changes) includes parts of my doctoral thesis at the Heinrich-Heine Universität, Düsseldorf (Germany). I am so grateful to my advisor Prof. Robert Wisbauer for his wonderful supervision and the continuous encouragement and support. The author thanks also the referee for his or her useful suggestions.

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