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# COMBINATORIAL RESULTS FOR SEMIGROUPS OF ORDER-PRESERVING PARTIAL TRANSFORMATIONS

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**Abstract**

Let $\mathcal{PO}_n$ be the semigroup of all order-preserving partial transformations of a finite chain. It is shown that $|\mathcal{PO}_n| = c_n$ satisfies the recurrence:
$(2n-1)(n+1)c_{n+1} = 4(3n^2-1)c_n - (2n+1)(n-1)c_{n-1}$ with initial conditions $c_0 = 1$, $c_1 = 2$. It is also shown that $|E(\mathcal{PO}_n)| = e_n$ satisfies the recurrence: $e_{n+1} = 5(e_n - e_{n-1}) + 1$ with initial conditions $e_0 = 1$, $e_1 = 2$. Moreover, the cardinalities of the Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ have been computed.

## 1 Introduction

Consider a finite chain, say $X_n = \{1, 2, \ldots, n\}$ under the natural ordering and let $T_n$ and $P_n$ be the full and partial transformation semigroups on $X_n$, respectively. We shall call a partial transformation $\alpha : \text{Dom } \alpha \subseteq X_n \to X_n$ *(order)-decreasing* if $x\alpha \leq x$ for all $x$ in Dom $\alpha$, and $\alpha$ is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for $x, y$ in Dom $\alpha$. Combinatorial properties of $\mathcal{C}_n$, the semigroup of all decreasing and order-preserving full transformations on $X_n$ have been investigated by Higgins [8] and recently by Laradji and Umar [11] . These papers motivated the study of combinatorial properties of $\mathcal{PC}_n$, the semigroup of all decreasing and order-preserving partial transformations on $X_n$ by Laradji and Umar [12], where it is shown that $|\mathcal{PC}_n|$ is the double Schröder number

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and $|E(\mathcal{PC}_n)| = (3^n + 1)/2$. This paper investigates combinatorial properties of $\mathcal{PO}_n$, the (necessarily larger semigroup than $\mathcal{PC}_n$) semigroup of all order-preserving partial transformations on $X_n$, by analogy with [12].

Unlike $\mathcal{PC}_n$, the semigroup $\mathcal{PO}_n$ has been studied by Gomes and Howie [6] and Garba [4, 5], mainly from algebraic point of view. After this introductory and preliminary section, we obtain in Section 2 a recurrence satisfied by $|\mathcal{PO}_n|$ (similar to the one for $\mathcal{PC}_n$). In Section 3, we compute the total number of idempotents of $\mathcal{PO}_n$ via some natural equivalences and a linear recurrence relation. Finally, in Section 4 we compute the cardinalities of the Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$. For standard concepts in semigroup theory we refer the reader to [10] or [7].

We now recall some basic definitions from [12] that we shall need in the coming sections.

**Definition 1.1** *Consider $X_n = \{1, 2, \ldots, n\}$ and let $\alpha : X_n \to X_n$ be a partial transformation. We shall denote by $Dom\ \alpha$, the* domain *of $\alpha$ and by $Im\ \alpha$ the* image set *of $\alpha$. The* width *of $\alpha$ is $|Dom\alpha|$, the* height *of $\alpha$ is $|Im\ \alpha|$ and the* waist *of $\alpha$ is $\max(Im\ \alpha)$.*

The semigroup $P_n$, of all partial transformations of $X_n$ under the usual composition contains the subsemigroup of all decreasing and order-preserving partial transformations of $X_n$, $\mathcal{PC}_n$ which has been studied recently by Laradji and Umar [12]. However, it is $\mathcal{PO}_n$ (a larger subsemigroup of $P_n$ which also contains $\mathcal{PC}_n$) that we investigate in this paper. Formally, we define $\mathcal{PO}_n$ as

$$\mathcal{PO}_n = \{\alpha \in P_n : (\forall x, y \in \text{Dom } \alpha)x \leq y \Rightarrow x\alpha \leq y\alpha\}. \tag{1.1}$$

We also record these two results that will be needed in Section 2. The first (Lemma 1.1) known as the Vandermonde's convolution identity is in the words of Riordan [14, p. 8] perhaps the most widely used combinatorial identity, while the second (Lemma 1.2) can be obtained by repeated application of the Pascal's triangular identity.

**Lemma 1.2** [14, (3a), p. 8]. *For all natural numbers $k, m, n$ and $p$ we have*

$$\sum_{k=0}^{n} \binom{n}{m-k} \binom{p}{k} = \binom{n+p}{m}.$$

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**Lemma 1.3** *For all natural numbers $k, n$ and $r$ we have*

$$\sum_{k=0}^{n} \binom{k+r-2}{k-1} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r}.$$

# 2   The Order of $\mathcal{PO}_n$

Gomes and Howie [6] were the first to study $\mathcal{PO}_n$ (excluding the identity map) and among other things they computed the order of $\mathcal{PO}_n$, which we now record.

**Theorem 2.1** [6, Theorem 3.1]. *Let $\mathcal{PO}_n$ be as defined in (1.1). Then $|\mathcal{PO}_n|$ is the coefficient of $x^n$ in the series expansion of $(1+x)^n (1-x)^{-n}$. Equivalently,*

$$|\mathcal{PO}_n| = \sum_{r=0}^{n} \binom{n}{r} \binom{n+r-1}{r}.$$

However, from a computational point of view this result is not satisfactory if one were to compute higher orders of $\mathcal{PO}_n$. Recently, the authors in [12] computed the order of $\mathcal{PC}_n$ as $r_n$, the double Schröder number given by

$$r_n = \frac{1}{n+1} \sum_{r=0}^{n} \binom{n+1}{n-r} \binom{n+r}{r} = \frac{1}{n} \sum_{r=0}^{n} \binom{n}{r} \binom{n+r}{n-r} \tag{2.1}$$

which also satisfies the recurrence

$$(n+2)r_{n+1} = 3(2n+1)r_n - (n-1)r_{n-1} \tag{2.2}$$

for $n \geq 1$, with initial conditions $r_0 = 1, r_1 = 2$. Moreover, in the process of discovering $|\mathcal{PC}_n|$ (in [12]) the authors also found two triangular arrays of numbers which are not in Sloane's encyclopaedia of integer sequences, and it is this relative success that motivates the search for similar results for $\mathcal{PO}_n$. As in [12], we begin by defining $f(n, r, k)$ as

$$f(n, r, k) = |\{\alpha \in \mathcal{PO}_n : |\text{Dom }\alpha| = r \wedge \max(\text{Im }\alpha) = k\}|. \tag{2.3}$$

Then clearly we have

$$f(n, 0, k) = \begin{cases} 1 & (k = 0) \\ 0 & (k > 0) \end{cases}, \quad f(n, r, 0) = \begin{cases} 1 & (r = 0) \\ 0 & (r > 0) \end{cases}$$

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and perhaps less clearly, we have

$$f(n, 1, k) = n \text{ and } f(n, r, 1) = \binom{n}{r}.$$

The first corresponds to the number of all maps $\alpha$ in $\mathcal{PO}_n$ of width 1 and Im $\alpha = \{k\}$, that is, maps for which Dom $\alpha = \{x\}$ ($x \in X_n$) of which there are clearly $n$ of them, since Im $\alpha$ is fixed. The second corresponds to the number of all maps $\alpha$ in $\mathcal{PO}_n$ of width $r$ and Im $\alpha = \{1\}$, that is, the number of all subsets (of $X_n$) of size $r$, of which there are clearly $\binom{n}{r}$ of them. In general, we have

**Proposition 2.2** *Let $f(n, r, k)$ be as defined in (2.3). Then for $n \geq r, k > 0$,*

$$f(n, r, k) = \binom{n}{r}\binom{k + r - 2}{k - 1}.$$

**Proof.** First note that for all $\alpha$ in $\mathcal{PO}_n$ and $y$ in Im $\alpha$, $y\alpha^{-1}$ is convex modulo Dom $\alpha$. That is, to say, if $y_1, y_2 \in y\alpha^{-1}$ and $z \in$ Dom $\alpha$ is such that $y_1 < z < y_2$ then $z \in y\alpha^{-1}$ as well. Next note that we can choose the elements of Dom $\alpha$ (from $X_n$) in $\binom{n}{r}$ ways. However, since $|$Im $\alpha| = s$, where $1 \leq s \leq r$ and $\max($Im $\alpha) = k$, it follows that we can choose the remaining $s - 1$ elements of Im $\alpha \setminus \{k\}$ from $\{1, 2, \ldots, k - 1\}$ in $\binom{k-1}{s-1}$ ways, which can now be tied to Dom $\alpha$ in $\binom{r-1}{s-1}$ ways, by inserting $s - 1$ symbols between the $r - 1$ spaces in Dom $\alpha$, to get convex (modulo Dom $\alpha$) partitions. Thus, in all we have

$$
\begin{aligned}
f(n, r, k) &= \binom{n}{r}\sum_{s=1}^{r}\binom{k-1}{s-1}\binom{r-1}{s-1} = \binom{n}{r}\sum_{j=1}^{r-1}\binom{k-1}{k-1-j}\binom{r-1}{j} \\
&= \binom{n}{r}\binom{k+r-2}{k-1} \qquad \text{(by Lemma 1.1).}
\end{aligned}
$$

∎

**Corollary 2.3** $f(n, r, r) = \binom{n}{r}\binom{2r-2}{r-1}.$

**Corollary 2.4** *Let $\mathcal{O}_n$ be the semigroup of all order-preserving full transformations of $X_n$. Then*

$$|\{\alpha \in \mathcal{O}_n : \max(\text{Im } \alpha) = k\}| = f(n, n, k) = \binom{n+k-2}{k-1}.$$

**Lemma 2.5** *Let $G(n,0) = 1$, and for $n \geq k > 0$ let $G(n,k) = \sum_{r=0}^{n} f(n,r,k)$. Then*

$$G(n,k) = \sum_{r=0}^{n} \binom{n}{r} \binom{k+r-2}{k-1}.$$

**Lemma 2.6** *Let $G(n,k) = \sum_{r=0}^{n} f(n,r,k)$. Then $G(n,0) = 1, G(n,1) = 2^n - 1, G(n,n) = \sum_{r=0}^{n} \binom{n}{r} \binom{n+r-2}{n-1}$ and for $2 \leq k \leq n$, we have*

$$G(n,k) = 2G(n-1,k) - G(n-1,k-1) + G(n,k-1).$$

**Proof.** Let $a(k,r) = \binom{k+r-2}{k-1}$. Then it is clear that $a(k,0) = 0$ and

$$a(k,r) = a(k-1,r) + a(k,r-1). \tag{2.4}$$

Now, by Lemma 2.5

$$
\begin{aligned}
G(n,k) - G(n-1,k) &= \sum_{r=1}^{n} \binom{n}{r} a(k,r) - \sum_{r=1}^{n-1} \binom{n-1}{r} a(k,r) \\
&= \sum_{r=1}^{n-1} \binom{n-1}{r-1} a(k,r) + a(k,n) \\
&= \sum_{r=1}^{n} \binom{n-1}{r-1} a(k,r) \tag{2.5}
\end{aligned}
$$

and so

$$G(n,k-1) - G(n-1,k-1) = \sum_{r=1}^{n} \binom{n-1}{r-1} a(k-1,r). \tag{2.6}$$

From (2.5) and (2.6) we have

$$
\begin{aligned}
&G(n,k) - G(n-1,k) - G(n,k-1) + G(n-1,k-1) \\
&= \sum_{r=1}^{n} \binom{n-1}{r-1} [a(k,r) - a(k-1,r)] \\
&= \sum_{r=1}^{n} \binom{n-1}{r-1} a(k,r-1) \qquad \text{(by (2.4))} \\
&= \sum_{r=2}^{n} \binom{n-1}{r-1} a(k,r-1) \qquad \text{(since } a(k,0) = 0) \\
&= \sum_{r=1}^{n-1} \binom{n-1}{r} a(k,r) = G(n-1,k).
\end{aligned}
$$

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Hence the result follows. ■

**Corollary 2.7**

$$G(n,n) = \sum_{r=1}^{n} \binom{n}{r}\binom{n+r-2}{n-1} = \sum_{s=0}^{n-1}\binom{n}{s+1}\binom{n+s-1}{n-1}$$

$$= \sum_{s=0}^{n-1}\binom{n}{n-s-1}\binom{n+s-1}{s} = nr_{n-1}.$$

**Proposition 2.8** *Let* $F(n,r) = \sum_{k=1}^{n} f(n,r,k)$. *Then*

$$F(n,r) = \binom{n}{r}\binom{n+r-1}{n-1}.$$

**Proof.**

$$F(n,r) = \sum_{k=1}^{n} f(n,r,k) = \sum_{k=1}^{n}\binom{n}{r}\binom{k+r-2}{k-1}$$

$$= \binom{n}{r}\sum_{k=1}^{n}\binom{k+r-2}{k-1} = \binom{n}{r}\binom{n+r-1}{n-1} \qquad \text{(by Lemma 1.2)}.$$

■

**Corollary 2.9** [9, Theorem 2.1]. *Let* $\mathcal{O}_n$ *be the semigroup of all order-preserving full transformations of* $X_n$. *Then*

$$|\mathcal{O}_n| = F(n,n) = \binom{2n-1}{n-1}.$$

**Remark 2.10** *The triangular arrays of numbers* $f(n,r,r)$, $G(n,k)$ *and* $F(n,r)$ *are not listed in Sloane's encyclopaedia of integer sequences and so we believe they are new. For selected values of these numbers see Tables 1-3.*

It is now clear that we have also proved the last part of Theorem 3.1, that is

$$|\mathcal{PO}_n| = \sum_{r=0}^{n} F(n,r) = \sum_{r=0}^{n}\binom{n}{r}\binom{n+r-1}{n-1}.$$

Before we get a recurrence (similar to that for $|\mathcal{PC}_n| = r_n$ in [12]) satisfied by $|\mathcal{PO}_n| = c_n$, first we establish the following lemma linking the two cardinalities.

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**Lemma 2.11** *For all $n > 0$, we have*

$$2c_n = (n+1)r_n - (n-1)r_{n-1}.$$

**Proof.**

$$
\begin{aligned}
\text{R.H.S.} &= (n+1)r_n - (n-1)r_{n-1} \\
&= \sum_{r=0}^{n} \frac{n+1}{n}\binom{n}{r}\binom{n+r}{n-1} - \sum_{r=0}^{n-1}\binom{n-1}{r}\binom{n+r-1}{n-2} \qquad \text{(by (2.1))} \\
&= \sum_{r=0}^{n-1}\left[\frac{n+1}{n}\binom{n}{r}\binom{n+r}{n-1} - \binom{n-1}{r}\binom{n+r-1}{n-2}\right] + \frac{n+1}{n}\binom{2n}{n-1} \\
&= \sum_{r=0}^{n-1}[(n+1)(n+r) - (n-1)(n-r)]\frac{(n+r-1)!}{r!(n-r)!(r+1)!} + \frac{n+1}{n}\binom{2n}{n-1} \\
&= \sum_{r=0}^{n-1}\frac{2n(n+r-1)!}{r!(n-r)!r!} + \frac{2n(2n-1)!}{n!n!} = \sum_{r=0}^{n}\frac{2n(n+r-1)!}{r!(n-r)!r!} \\
&= 2\sum_{r=0}^{n}\binom{n}{r}\binom{n+r-1}{r} = 2c_n = \text{L.H.S.}
\end{aligned}
$$

∎

We now have

**Proposition 2.12** *Let $\mathcal{PO}_n$ be as defined in (1.1), and let $c_n = |\mathcal{PO}_n|$. Then $c_0 = 1, c_1 = 2$ and for all $n > 0$,*

$$(2n-1)(n+1)c_{n+1} = 4(3n^2 - 1)c_n - (2n+1)(n-1)c_{n-1}.$$

**Proof.** From Lemma 2.11 and (2.2) successively we have

$$
\begin{aligned}
2c_{n+1} &= (n+2)r_{n+1} - nr_n = 3(2n+1)r_n - (n-1)r_{n-1} - nr_n \\
&= (5n+3)r_n - (n-1)r_{n-1}. \tag{2.7}
\end{aligned}
$$

Eliminating $r_n$ from Lemma 2.11 and (2.7) gives

$$(n+1)c_{n+1} - (5n+3)c_n = (2n+1)(n-1)r_{n-1} \tag{2.8}$$

while eliminating $r_{n-1}$ from Lemma 2.11 and (2.7) gives

$$c_{n+1} - c_n = (2n+1)r_n$$

which in turn implies

$$c_n - c_{n-1} = (2n-1)r_{n-1}. \tag{2.9}$$

Finally, eliminating $r_{n-1}$ from (2.8) and (2.9) gives the required result. ∎

## 3 The number of idempotents in $\mathcal{PO}_n$

As many 'natural' semigroups of transformations are idempotent-generated it is not surprising that counting the number of idempotents in such semigroups has attracted the attention of Higgins [8], Howie [9], Tainiter [15] and Umar [16, 17]. Gomes and Howie [6, Theorem 3.13] showed that $\mathcal{PO}_n$ is idempotent-generated, but did not count all the idempotents in $\mathcal{PO}_n$. To investigate this number we take a slightly different approach (but essentially the same) from the previous section. First, we consider the equivalence on $E(\mathcal{PO}_n)$ given by the equality of widths and define

$$E(n,r) = |\{\alpha \in \mathcal{PO}_n : \alpha^2 = \alpha \wedge |\text{Dom } \alpha| = r\}|. \tag{3.1}$$

Then clearly we have

$$E(n,0) = 1 \text{ and } E(n,1) = n.$$

Moreover, we have from Howie [9] that

$$E(n,n) = f_{2n}$$

where $f_{2n}$ is the alternate Fibonacci number. In general, we have

**Lemma 3.1** $E(n,r) = \dfrac{n}{n-r}E(n-1,r), (n > r \geq 0)$.

**Proof.** Let $g(r,s)$ be the number of all idempotent order-preserving full transformations with domain $\{x_1, x_2, \ldots, x_r\} \subseteq X_n$ and of height $s$. To count all idempotents $\epsilon$ in $\mathcal{PO}_n$ of width $r$, we first note that we can choose the domain of $\epsilon$ (from $X_n$), say $\{x_1, x_2, \ldots, x_r\}$ in $\dbinom{n}{r}$ ways. Next we choose the elements of Im $\epsilon \subseteq \{x_1, x_2, \ldots, x_s\}$ where $s = |\text{Im } \epsilon|$ satisfies $1 \leq s \leq r$. Now since Im $\epsilon$ can be chosen in $\dbinom{r}{s}$ ways, it follows that

$$\binom{n}{r}\sum_{s=1}^{r}\binom{r}{s}g(r,s) = E(n,r) \tag{3.2}$$

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from which we deduce

$$\binom{n-1}{r} \sum_{s=1}^{r} \binom{r}{s} g(r,s) = E(n-1,r). \qquad (3.3)$$

From (3.2) and (3.3), it follows that

$$\binom{n-1}{r} E(n,r) = \binom{n}{r} E(n-1,r)$$

which in turn gives the required result. ∎

Consequently from Lemma 3.1 we deduce that

**Lemma 3.2** $E(n,r) = \binom{n}{r} E(r,r).$

Next, we consider the equivalence in $E(\mathcal{PO}_n)$ given by equality of waists and define

$$H(n,k) = |\{\alpha \in \mathcal{PO}_n : \alpha^2 = \alpha \wedge \max(\mathrm{Im}\ \alpha) = k\}|. \qquad (3.4)$$

Then clearly

$$H(n,0) = 1 \text{ and } H(n,1) = 2^{n-1}.$$

In general, we have

**Lemma 3.3** *For $0 < k < n$, $H(n,k) = 2^{n-k} H(k,k)$.*

**Proof.** Let $\epsilon$ be an idempotent in $\mathcal{PO}_n$ satisfying $\max(\mathrm{Im}\ \epsilon) = k$. Then by the order-preserving property, for all $x$ in $\{k, k+1, \ldots, n\}$ we have $x\epsilon = k$, if $x \in \mathrm{Dom}\ \epsilon$. Thus to compute $H(n,k)$, we consider all idempotents $\eta$ on $\{1, 2, \ldots, k\}$ satisfying $\max(\mathrm{Im}\ \eta) = k$, of which there are $H(k,k)$ of them. Now multiply this number by $2^{n-k}$ to get $H(n,k)$, where $2^{n-k}$ is the total number of degrees of freedom for members of $\{k+1, k+2, \ldots, n\}$, that is, for each $x$ in $\{k+1, k+2, \ldots, n\}$ either $x \in \mathrm{Dom}\ \epsilon$ (in which case $x\epsilon = k$) or $x \notin \mathrm{Dom}\ \epsilon$. ∎

Now since $H(n,k)$ depends on $H(k,k)$, we focus our attention to finding an expression for $H(n,n)$. In fact we have

**Proposition 3.4** $H(n,n) = H(n-1,n-1) + 2^{n-1} + \sum_{t=1}^{n-2}(n-t+1)2^{n-t-2}H(t,t).$

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**Proof.** Since $\max(\operatorname{Im} \epsilon) = n$, then $n \in \operatorname{Dom} \epsilon$ and $n\epsilon = n$, by idempotency. Now we consider cases:

**Case 1.** if $\min(n\epsilon^{-1}) = \{n\}$, then from the remaining $\{1, 2, \ldots, n-1\}$ elements we can construct $\sum_{t=0}^{n-1} H(n-1, t)$ idempotents to each of which we adjoin $(n\epsilon^{-1})\epsilon = n\epsilon = n$;

**Case 2.** if $\min(n\epsilon^{-1}) = \{n-1\}$, then from the remaining $\{1, 2, \ldots, n-2\}$ elements we can construct $\sum_{t=0}^{n-2} H(n-2, t)$ idempotents to each of which we adjoin $(n\epsilon^{-1})\epsilon = \{n-1, n\}\epsilon = n$.

Now, in general, if $\min(n\epsilon^{-1}) = \{n - m + 1\}$, where $2 \le m \le n$, it is clear that $\{n - m + 1, n\} \subseteq n\epsilon^{-1} \subseteq \{n - m + 1, \ldots, n\}$. However, for each of the $m - 2$ middle elements $\{n - m + 2, \ldots, n - 1\}$ there are two degrees of freedom: either $x$ (in $\{n - m + 2, \ldots, n - 1\}$) belongs to $\operatorname{Dom} \epsilon$ (in which case $x\epsilon = n$) or it does not belong to $\operatorname{Dom} \epsilon$. Thus there are $2^{m-2}$ degrees of freedom for these $m - 2$ middle elements. Next, considering the remaining elements $\{1, 2, \ldots, n - m\}$ we can construct $\sum_{t=0}^{n-m} H(n-m, t)$ idempotents, to each of which we adjoin $(n\epsilon^{-1})\epsilon \subseteq \{n-m+1, \ldots, n\}\epsilon = n$, thus giving rise to $2^{m-2} \sum_{t=0}^{n-m} H(n - m, t)$ idempotents in all. Finally, adding all the sums from all the cases we get

$$
\begin{aligned}
H(n, n) &= \sum_{t=0}^{n-1} H(n - 1, t) + \sum_{m=2}^{n} 2^{m-2} \sum_{t=0}^{n-m} H(n - m, t) \\
&= H(n - 1, n - 1) + \sum_{t=1}^{n-2} H(n - 1, t) + H(n - 1, 0) + \sum_{m=2}^{n} 2^{m-2} H(n - m, 0) \\
&\quad + \sum_{m=2}^{n-1} \sum_{t=1}^{n-m} 2^{m-2} H(n - m, t) \\
&= H(n - 1, n - 1) + 2^{n-1} + \sum_{t=1}^{n-2} H(n - 1, t) + \sum_{t=1}^{n-2} \sum_{m=t+1}^{n-1} 2^{m-2} H(n - m, t) \\
&= H(n - 1, n - 1) + 2^{n-1} + \sum_{t=1}^{n-2} 2^{n-t-1} H(t, t) + \sum_{t=1}^{n-2} \sum_{m=t+1}^{n-1} 2^{n-t-2} H(t, t) \\
&= H(n - 1, n - 1) + 2^{n-1} + \sum_{t=1}^{n-2} 2^{n-t-2} \cdot 2H(t, t) + \sum_{t=1}^{n-2} (n - t - 1) 2^{n-t-2} H(t, t)
\end{aligned}
$$

$$H(n,n) = H(n-1, n-1) + 2^{n-1} + \sum_{t=1}^{n-2}(n - t + 1)2^{n-t-2}H(t,t),$$

using Lemma 3.3 along the way. ∎

However, a simple linear recurrence satisfied by $H(n,n) = b_n$ is given by the following lemma.

**Lemma 3.5** *Let $H(n,n) = b_n$. Then $b_1 = 1, b_2 = 3$ and*

$$b_{n+1} = 5(b_n - b_{n-1}).$$

**Proof.** From Proposition 3.4, we have

$$b_n = b_{n-1} + 2^{n-1} + \sum_{t=1}^{n-2}(n - t + 1)2^{n-t-2}b_t$$

so that

$$
\begin{aligned}
b_{n+1} &= b_n + 2^n + \sum_{t=1}^{n-1}[(n - t + 1) + 1]2^{n-t-1}b_t \\
&= b_n + 2^n + \sum_{t=1}^{n-2}[(n - t + 1) + 1]2^{n-t-1}b_t + 3b_{n-1} \\
&= b_n + 2\left\{2^{n-1} + \sum_{t=-1}^{n-2}(n - t + 1)2^{n-t-2}b_t + b_{n-1}\right\} + \sum_{t=1}^{n-2}2^{n-t-1}b_t + b_{n-1} \\
&= 3b_n + b_{n-1} + \sum_{t=1}^{n-2}2^{n-t-1}b_t. \qquad (3.5)
\end{aligned}
$$

This in turn implies

$$b_n = 3b_{n-1} + b_{n-2} + \sum_{t=1}^{n-3}2^{n-t-2}b_t. \qquad (3.6)$$

However, (3.5) may be written as

$$
\begin{aligned}
b_{n+1} &= 3b_n + b_{n-1} + 2\sum_{t=1}^{n-3}2^{n-t-2}b_t + 2b_{n-2} \\
&= 3b_n + b_{n-1} + 2(b_n - 3b_{n-1} - b_{n-2}) + 2b_{n-2} \qquad \text{(by (3.6))} \\
&= 5b_n - 5b_{n-1}
\end{aligned}
$$

as required ∎

**Remark 3.6** *The triangular arrays of numbers $E(n,r)$ and $H(n,k)$ are not yet in Sloane's encyclpaedia of integer sequences. For selected values of these numbers see Tables 4 and 5.*

Now to obtain a formula for the total number of idempotents in $\mathcal{PO}_n$ we observe that

$$|\mathcal{PO}_n| = e_n = \sum_{k=0}^{n} H(n,k). \qquad (3.7)$$

Then by Lemma 3.3 and (3.7), we have

$$\begin{aligned}
e_n &= 1 + \sum_{k=1}^{n} 2^{n-k} H(k,k) = 1 + \sum_{k=1}^{n} 2^{n-k} b_k = 1 + 2\sum_{k=1}^{n-2} 2^{n-k-1} b_k + 2b_{n-1} + b_n \\
&= 1 + 2(b_{n+1} - 3b_n - b_{n-1}) + 2b_{n-1} + b_n \qquad \text{(by (3.5))} \\
&= 1 + 2b_{n+1} - 5b_n \qquad (3.8)
\end{aligned}$$

so that

$$\begin{aligned}
e_{n+1} &= 1 + 2b_{n+2} - 5b_{n+1} = 1 + 2(5b_{n+1} - 5b_n) - 5b_{n+1} \qquad \text{(by Lemma 3.5)} \\
&= 1 + 5b_{n+1} - 10b_n \qquad (3.9)
\end{aligned}$$

From (3.8) and (3.9) we deduce

$$e_{n+1} - e_n = 3b_{n+1} - 5b_n. \qquad (3.10)$$

But by (3.9) we have

$$\begin{aligned}
e_{n+2} &= 1 + 5b_{n+2} - 10b_{n+1} = 1 + 5(5b_{n+1} - 5b_n) - 10b_{n+1} \\
&= 1 + 15b_{n+1} - 25b_n = 1 + 5(3b_{n+1} - 5b_n) \\
&= 1 + 5(e_{n+1} - e_n) \qquad \text{(by (3.10))}.
\end{aligned}$$

Thus we have shown that

**Lemma 3.7** *For all $n > 0$, $e_{n+1} = 1 + 5(e_n - e_{n-1})$ with initial conditions, $e_0 = 1, e_1 = 2$.*

By the standard method of solving linear recurrence relations (see [1]) we deduce

**Theorem 3.8** *Let $e_n$ be as defined in (3.7). Then*

$$e_n = (\sqrt{5})^{n-1} \left[ \left( \frac{\sqrt{5}+1}{2} \right)^n - \left( \frac{\sqrt{5}-1}{2} \right)^n \right] + 1.$$

**Remark 3.9** *The sequence $\{b_n\}\,(n \geq 1)$ has been recorded (March 2003) as A081567 but $\{e_n\}$ is not yet in Sloane's encyclopaedia of integer sequences. For selected values see Table 5.*

The following curious result is worth recording.

**Lemma 3.10** $e_n \equiv 1 \pmod{5}, \quad (n \geq 2).$

Alternatively, we may get the formula for $e_n$ by using $E(n,r)$, since

$$
\begin{aligned}
e_n &= \sum_{r=0}^{n} E(n,r) = 1 + \sum_{r=1}^{n} E(n,r) \\
&= 1 + \sum_{r=1}^{n} \binom{n}{r} E(r,r) \qquad \text{(by Lemma 3.2)}. \\
&= 1 + \sum_{r=1}^{n} \binom{n}{r} f_{2r}
\end{aligned}
$$

where $f_{2r} = a_r$ is the alternate Fibonacci number and it satisfies the recurrence

$$a_r = 3a_{r-1} - a_{r-2},$$

from which we get

$$a_r = \frac{1}{\sqrt{5}} \left( p^r - q^r \right)$$

with

$$p = \frac{3+\sqrt{5}}{2} \text{ and } q = \frac{3-\sqrt{5}}{2}.$$

Hence

$$
\begin{aligned}
e_n &= 1 + \frac{1}{\sqrt{5}} \left\{ \sum_{r=1}^{n} \binom{n}{r} p^r - \sum_{r=1}^{n} \binom{n}{r} q^r \right\} = 1 + \frac{1}{\sqrt{5}} \left\{ (1+p)^n - (1+q)^n \right\} \\
&= 1 + \frac{1}{\sqrt{5}} \left\{ \left( \frac{5+\sqrt{5}}{2} \right)^n - \left( \frac{5-\sqrt{5}}{2} \right)^n \right\} \\
&= 1 + (\sqrt{5})^{n-1} \left\{ \left( \frac{\sqrt{5}+1}{2} \right)^n - \left( \frac{\sqrt{5}-1}{2} \right)^n \right\}.
\end{aligned}
$$

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# 4   The number of $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$-classes

It is clear that $\mathcal{PO}_n$ is a regular subsemigroup of $\mathcal{P}_n$ [6]. Hence by [10, Proposition 2.4.2] and [3, Section 2] we have

$$(\alpha, \beta) \in \mathcal{L} \text{ if and only if } \operatorname{Im} \alpha = \operatorname{Im} \beta \tag{4.1}$$

$$(\alpha, \beta) \in \mathcal{R} \text{ if and only if } \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}. \tag{4.2}$$

Moreover, it can be easily shown that

$$(\alpha, \beta) \in \mathcal{J} \text{ if and only if } |\operatorname{Im} \alpha| = |\operatorname{Im} \beta|. \tag{4.3}$$

Now let $\alpha$ in $\mathcal{PO}_n$ be such that $|\operatorname{Im} \alpha| = s$, then since $\mathcal{PO}_n$ is aperiodic [6], it follows that $|H_\alpha| = 1$, and by (4.1) we deduce that $|L_\alpha| = \begin{pmatrix} n \\ s \end{pmatrix}$. However, $|R_\alpha|$ is less clear and the next lemma provides a formula.

**Lemma 4.1** *Let $\alpha$ in $\mathcal{PO}_n$ be such that $|\operatorname{Im} \alpha| = s$. Then $|R_\alpha| = \displaystyle\sum_{r=s}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \begin{pmatrix} r-1 \\ s-1 \end{pmatrix}$.*

**Proof.**  First observe that we can choose the $r$ elements of Dom $\alpha$ in $\begin{pmatrix} n \\ r \end{pmatrix}$ ways, where $s \leq r \leq n$ and that we can partition Dom $\alpha$ into $s$ 'convex' (modulo Dom $\alpha$) subsets in $\begin{pmatrix} r-1 \\ s-1 \end{pmatrix}$ ways. Thus multiplying these two binomial coefficients and taking the sum from $r = s$ to $r = n$ yields the required result.  ∎

Next we obtain a linear recurrence satisfied by $|R_\alpha|$.

**Lemma 4.2** *Let $e(n, s) = \displaystyle\sum_{r=s}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \begin{pmatrix} r-1 \\ s-1 \end{pmatrix}$. Then $e(n, 0) = 1 = e(n, n)$ and $e(n, s) = 2e(n-1, s) + e(n-1, s-1)$.*

**Proof.**

$$2e(n-1, s) + e(n-1, s-1) - e(n, s)$$

$$= \sum_{r=s}^{n-1} 2 \binom{n-1}{r} \binom{r-1}{s-1} + \sum_{r=s-1}^{n-1} \binom{n-1}{r} \binom{r-1}{s-2}$$

$$- \sum_{rs}^{n-1} \binom{n}{r} \binom{r-1}{s-1} - \binom{n-1}{s-1}$$

$$= \sum_{r=s}^{n-1} \left\{ 2 \binom{n-1}{r} \binom{r-1}{s-1} + \binom{n-1}{r} \binom{r-1}{s-2} - \binom{n}{r} \binom{r-1}{s-1} \right\}$$

$$= \frac{1}{n(s-1)} \sum_{r=s}^{n-1} \binom{n}{r} \binom{r-1}{s-2} [(n-2r)(r-s+1) + (n-r)(s-1)]$$

$$= \frac{1}{s-1} \sum_{r=s}^{n-1} \binom{n-1}{r-1} \binom{r-1}{s-2} (n-2r+s-1)$$

$$= \frac{1}{s-1} \sum_{r=s}^{m} \binom{m}{r-1} \binom{r-1}{s-2} [(m-r) - (r-s)] \qquad (m = n-1)$$

$$= \frac{1}{(s-1)!} \sum_{r=s}^{m} \frac{m!}{(m-r+1)!(r-s+1)!} [(m-r) - (r-s)]$$

$$= \frac{m!}{(s-1)!} \sum_{r=s}^{m} \left\{ \frac{m-r+1}{(m-r+1)!(r-s+1)!} - \frac{r-s+1}{(m-r+1)!(r-s+1)!} \right\}$$

$$= \frac{m!}{(s-1)!} \sum_{r=s}^{m} \left\{ \frac{1}{(m-r)!(r-s+1)!} - \frac{1}{(m-r+1)!(r-s)!} \right\}$$

$$= \frac{m!}{(s-1)!} \sum_{r=s}^{m} (a_r - a_{r-1}) \quad \left( a_r = \frac{1}{(m-r)!(r-s+1)!} \right)$$

$$= \frac{m!}{(s-1)!} (a_m - a_{s-1}) = 0$$

as required. ∎

Two further recurrences satisfied by $e(n, s)$ are given by the next two lemmas whose proofs we omit because they are easy.

**Lemma 4.3** $e(n-1, s) + e(n-1, s-1) = 2^{n-s} \binom{n-1}{s-1}$.

**Lemma 4.4** $e(n, s) + e(n-1, s-1) = 2^{n-s+1} \binom{n-1}{s-1}$.

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Now it follows from (4.3) that

$$J(n,s) = |J_\alpha| = \binom{n}{s} e(n,s). \qquad (4.4)$$

However, a recurrence satisfied by $J(n,s)$ is given by the following lemma:

**Lemma 4.5** $J(n,0) = 1 = J(n,n)$ and for $n > s > 0$

$$\binom{n-1}{s-1} J(n,s) = \frac{2(n-s+1)}{(n-s)} \binom{n}{s-1} J(n-1,s) + \binom{n}{s} J(n-1,s-1).$$

**Proof.**

$$
\begin{aligned}
\text{L.H.S.} &= \binom{n-1}{s-1} J(n,s) \\
&= \binom{n-1}{s-1}\binom{n}{s} e(n,s) \qquad \text{(by (4.4))} \\
&= \binom{n-1}{s-1}\binom{n}{s} [2e(n-1,s) + e(n-1,s-1)] \qquad \text{(by Lemma 4.2)} \\
&= \frac{2n!}{(n-s)!(s-1)!(n-s)} \binom{n-1}{s} e(n-1,s) + \binom{n}{s}\binom{n-1}{s-1} e(n-1,s-1) \\
&= \frac{2(n-s+1)}{(n-s)} \binom{n}{s-1} J(n-1,s) + \binom{n}{s} J(n-1,s-1) \\
&= \text{R.H.S.}
\end{aligned}
$$

∎

**Remark 4.6** *The triangular array of numbers $e(n,s)$ and $J(n,s)$ are not yet listed in Sloane's encyclopaedia of integer sequences. For selected values of these numbers see Tables 6 and 7.*

We conclude the section by observing that

$$|\mathcal{PO}_n| = c_n = \sum_{s=0}^{n} J(n,s) = \sum_{s=0}^{n} \binom{n}{s} e(n,s).$$

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| k \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum f(n,r,r)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 2 | 2 | | | | | | 5 |
| 3 | 1 | 3 | 6 | 6 | | | | | 16 |
| 4 | 1 | 4 | 12 | 24 | 20 | | | | 61 |
| 5 | 1 | 5 | 20 | 60 | 100 | 70 | | | 256 |
| 6 | 1 | 6 | 30 | 120 | 300 | 420 | 252 | | 1129 |
| 7 | 1 | 7 | 42 | 210 | 700 | 1470 | 1764 | 924 | 5118 |

Table 1 *f(n, r, r)*

| k \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum G(n,k)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 3 | 4 | | | | | | 8 |
| 3 | 1 | 7 | 12 | 18 | | | | | 38 |
| 4 | 1 | 15 | 32 | 56 | 88 | | | | 192 |
| 5 | 1 | 31 | 80 | 160 | 280 | 450 | | | 1002 |
| 6 | 1 | 63 | 192 | 432 | 832 | 1452 | 2364 | | 5336 |
| 7 | 1 | 127 | 448 | 1120 | 2352 | 4244 | 7700 | 12642 | 28814 |

Table 2 *G(n, k)*

| r<br>n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum F(n,r)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 4 | 3 | | | | | | 8 |
| 3 | 1 | 9 | 18 | 10 | | | | | 38 |
| 4 | 1 | 16 | 60 | 80 | 35 | | | | 192 |
| 5 | 1 | 25 | 150 | 350 | 350 | 126 | | | 1002 |
| 6 | 1 | 36 | 315 | 1120 | 1890 | 1512 | 462 | | 5336 |
| 7 | 1 | 49 | 588 | 2940 | 7350 | 9702 | 6468 | 1716 | 28814 |

Table 3 *F(n, r)*

| r<br>n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum E(n,r)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 2 | 3 | | | | | | 6 |
| 3 | 1 | 3 | 9 | 8 | | | | | 21 |
| 4 | 1 | 4 | 18 | 32 | 21 | | | | 76 |
| 5 | 1 | 5 | 30 | 80 | 105 | 55 | | | 276 |
| 6 | 1 | 6 | 45 | 160 | 315 | 330 | 144 | | 1001 |
| 7 | 1 | 7 | 63 | 280 | 735 | 1155 | 1008 | 377 | 3626 |

Table 4 *E(n, r)*

| k \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum H(n,k)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 2 | 3 | | | | | | 6 |
| 3 | 1 | 4 | 6 | 10 | | | | | 21 |
| 4 | 1 | 8 | 12 | 20 | 35 | | | | 76 |
| 5 | 1 | 16 | 24 | 40 | 70 | 125 | | | 276 |
| 6 | 1 | 32 | 48 | 80 | 140 | 250 | 450 | | 1001 |
| 7 | 1 | 64 | 96 | 160 | 280 | 500 | 900 | 1625 | 3626 |

Table 5 *H(n, k)*

| s \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum e(n,s)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 3 | 1 | | | | | | 5 |
| 3 | 1 | 7 | 5 | 1 | | | | | 14 |
| 4 | 1 | 15 | 17 | 7 | 1 | | | | 41 |
| 5 | 1 | 31 | 49 | 31 | 9 | 1 | | | 122 |
| 6 | 1 | 63 | 129 | 111 | 49 | 11 | 1 | | 365 |
| 7 | 1 | 127 | 321 | 351 | 209 | 71 | 13 | 1 | 1094 |

Table 6 *e(n, s)*

| s / n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum J(n,s)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | 1 |
| 1 | 1 | 1 | | | | | | | 2 |
| 2 | 1 | 6 | 1 | | | | | | 8 |
| 3 | 1 | 21 | 15 | 1 | | | | | 38 |
| 4 | 1 | 60 | 102 | 28 | 1 | | | | 192 |
| 5 | 1 | 155 | 490 | 310 | 45 | 1 | | | 1002 |
| 6 | 1 | 378 | 1935 | 2220 | 735 | 66 | 1 | | 5336 |
| 7 | 1 | 889 | 6741 | 12285 | 7315 | 1491 | 91 | 1 | 28814 |

Table 7 $J(n, s)$

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