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Abstract

Let $\mathcal{PC}_n$ be the semigroup of all decreasing and order-preserving partial transformations of a finite chain. It is shown that $|\mathcal{PC}_n| = r_n$, where r_n is the large (or double) Schröder number. It is also shown that $\mathcal{PC}_n$ is a disjoint union of two subsemigroups each of order $r_n/2 = s_n$, where s_n is the (smaller) Schröder number. Moreover, the total number of idempotents of $\mathcal{PC}_n$ is shown to be $(3^n + 1)/2$.

1 Introduction

Consider a finite chain, say $X_n = \{1, 2, \ldots, n\}$ under the natural ordering and let T_n and P_n be the full transformation semigroup and the semigroup of all partial transformations on X_n, under the usual composition, respectively. We shall call a partial transformation $\alpha : X_n \to X_n$, *order-decreasing (order-increasing)* or simply *decreasing (increasing)* if $x\alpha \leq x$ $(x\alpha \geq x)$ for all x in Dom α, and α is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for x, y in Dom α. This paper investigates combinatorial properties of $\mathcal{PC}_n$, the semigroup of all decreasing and order-preserving partial transformations.

Various enumerative problems of an essentially combinatorial nature have been considered for certain classes of semigroups of transformations. For example, it is well known and indeed obvious that T_n and P_n have orders n^n and $(n+1)^n$, respectively. Only slightly less obvious are their number of idempotents given by

$$|E(T_n)| = \sum_{r=1}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) r^{n-r} \text{ and } |E(P_n)| = \sum_{r=1}^{n+1} \left(\begin{array}{c} n \\ r-1 \end{array} \right) r^{n+1-r}.$$

The first usually attributed to Tainiter [14] is actually Ex 2.2.2(a) in [1]. The second can be deduced easily via Vagner's method of representing a partial transformation by a full transformation [17], which has been used to good effect by Garba [2]. The

following list (which is by no means exhaustive) of papers and books [3, 4, 5, 6, 7, 9, 10, 13, 14 & 15] each contains some interesting combinatorial results pertaining to semigroups of transformations. Somewhat surprisingly we could find no reference on the combinatorial properties of $\mathcal{PC}_n$. In fact, the only reference we could find about $\mathcal{PC}_n$ is Higgins [8, theorem 4.2], where it is shown that any finite $\mathcal{R}$ -trivial semigroup S divides some monoid $\mathcal{PC}_n$.

In Section 2, we give the necessary definitions that we need in the paper as well as show that $\mathcal{PC}_n$ is a disjoint union of two subsemigroups of the same cardinality. In Section 3, we obtain the order of $\mathcal{PC}_n$ as the large or double Schröder number [11, 12], via a bijection with a set of certain lattice paths. In Section 4, we show that the set of all idempotents of $\mathcal{PC}_n$ is of cardinality $(3^n + 1)/2$.

2 Preliminaries

For standard terms and concepts in transformation semigroup theory see [6] or [9]. We now recall some definitions and notations to be used in the paper. Consider $X_n = \{1, 2, \ldots, n\}$ and let $\alpha : X_n \to X_n$ be a partial transformation. We shall denote by Dom α and Im α, the *domain* and *image set* of α, respectively. The semigroup P_n, of all partial transformation contains two important subsemigroups which have been studied recently. They are PD_n and $\mathcal{PO}_n$ the semigroups of all order-decreasing and order-preserving partial transformations, respectively (see [16] and [4, 5]). Now let

$$\mathcal{PC}_n = PD_n \cap \mathcal{PO}_n \tag{2.1}$$

be the semigroup of all decreasing and order-preserving partial transformations of X_n. Next let

$$Q_n = \{\alpha \in \mathcal{PC}_n : 1 \in \text{ Dom } \alpha\} \tag{2.2}$$

be the set of all maps in $\mathcal{PC}_n$ all of whose domain does contain the element 1. Then evidently we have the following result.

Lemma 2.1 *Both Q_n and Q'_n (the set complement) are subsemigroups of $\mathcal{PC}_n$. Moreover, $Q_n \cdot Q'_n = Q'_n \cdot Q_n = Q'_n$.*

Less evidently, we have

Lemma 2.2 $|Q_n| = |Q'_n|$.

Proof. Define a map ϕ from Q_n into Q'_n by

$$\phi(\alpha) = \alpha' \quad (\alpha \in Q_n, \alpha' \in Q'_n)$$

where

$$x\alpha' = x\alpha \quad (\text{for all } x \in \text{Dom } \alpha \setminus \{1\}).$$

It is clear that ϕ is a bijection since $1 \notin \text{Dom } \alpha'$ for all α' in Q'_n, and $1\alpha = 1$ for all α in Q_n.

3 The order of $\mathcal{PC}_n$

Our main objective in this section is to obtain a formula for $|\mathcal{PC}_n|$. We begin our investigation by considering lattice paths in the Cartesian plane that start at $(0,0)$, end at (n,n), contain no points above the line $y = x$, and composed only of steps $(1,0), (0,1)$ and $(1,1)$, i.e., $\rightarrow, \uparrow$ and $\nearrow$. The diagrams in Figure 1 illustrate all such paths in the 1×1 and 2×2 squares, respectively. The total number of such paths is known to be the large or double Schröeder number r_n, [11]. To establish a bijection between the set of all these paths (in an $n \times n$ square) and the set of all decreasing and order-preserving partial transformations of X_n, we make the following observation:

(OB1) In each path from $(0,0)$ to (n,n) there are exactly same number of horizontal steps as there are vertical steps.

Next, we note that it is convenient (in this section) to express α in $\mathcal{PC}_n$ (with base set $X_n = \{0, 1, 2, \ldots, n-1\}$) as

$$\alpha = \begin{pmatrix} a_1 & a_2 \cdots a_r \\ b_1 & b_2 \cdots b_r \end{pmatrix} \tag{3.1}$$

where the a_i's are distinct, but the b_i's are not necessarily distinct. Moreover, we may also assume that $0 \leq a_1 < a_2 \cdots < a_r \leq n-1$ and $0 \leq b_1 \leq b_2 \leq \cdots \leq b_r \leq n-1$. Thus

$$\text{Dom } \alpha = \{a_1, a_2, \ldots a_r\},$$

however, we refer to the sequence $(b_1, b_2, \ldots, b_r)$ as the *simage* of α, denoted by

$$\text{Sim } \alpha = (b_1, b_2, \ldots, b_r).$$

(Note that if all the b_i's are distinct then Sim α may be considered as Im α, otherwise they are not the same.) Now to each vertical step from (i,j) to $(i,j+1)$ in an arbitrary

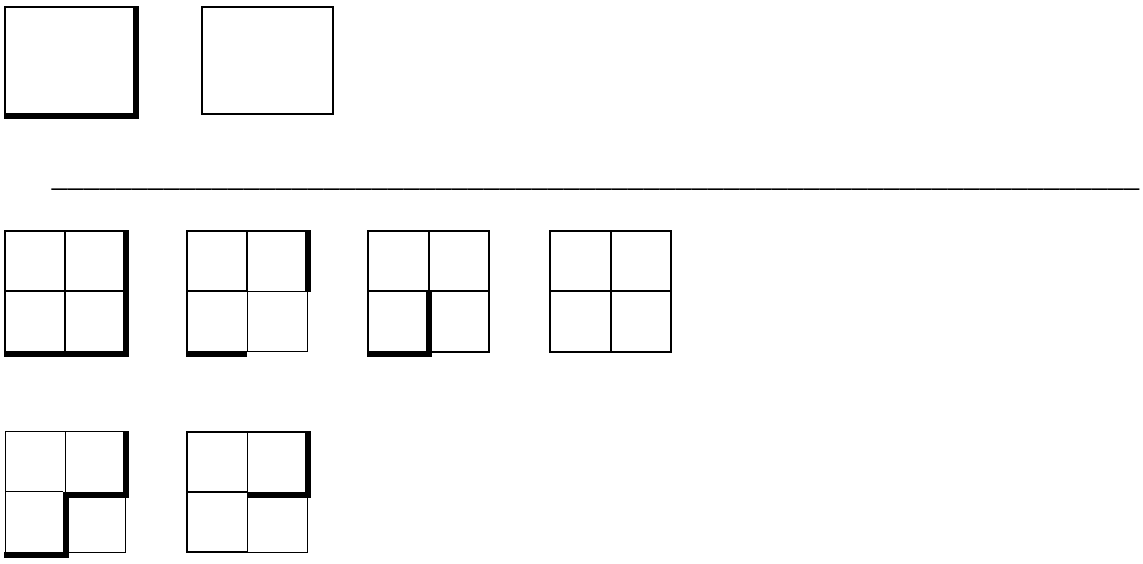

*Figure*1

path (in an $n \times n$ square) we put j in Dom α, and to each horizontal step from (i, j) to $(i+1, j)$ we put j in Sim α. The domain is then arranged in a strictly increasing order while the simage is arranged in a nondecreasing order, and by virtue of (OB1) this gives rise to a unique order-preserving map. Two examples should make these ideas more clear. The path given in Figure 2 implies Dom $\alpha = \{1, 3\}$ and Sim $\alpha = (1, 1)$. Thus the associated order-preserving map is

$$\alpha = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \in \mathcal{PC}_4$$

and the path given in Figure 3 implies Dom $\beta = \{0, 2, 3\}$ and Sim $\beta = (0, 2, 2)$. Thus the associated order-preserving map is

$$\beta = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 2 \end{pmatrix} \in \mathcal{PC}_4.$$

Moreover, the condition that our paths never cross above the diagonal line $y = x$, forces these associated order-preserving maps to be decreasing as well. In fact, at any level, a horizontal step will take us 1-unit away from the diagonal line $y = x$, and a vertical step will take us 1-unit closer to the diagonal line $y = x$. Thus, since our target is the point (n, n) on this diagonal line we must take vertical units at higher levels to compensate for each horizontal step, and so a decreasing map results. Finally, note that the unique all diagonal path corresponds to the empty map. Conversely, we show

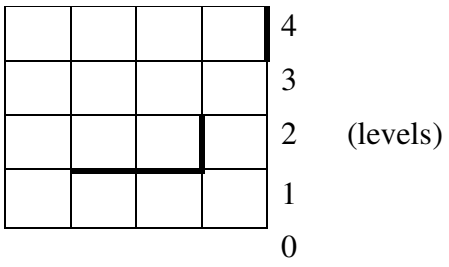

*Figure*2

that every $\alpha \in \mathcal{PC}_n$ corresponds to a path in an $n \times n$ square. For a given α in $\mathcal{PC}_n$ we first express it as in (3.1) and write

$$\text{Sim } \alpha = (b_1, b_2, \ldots, b_r), \text{ Dom } \alpha = \{a_1, a_2, \ldots, a_r\}$$

for some r in $\{1, 2, \ldots, n\}$. To construct an associated path, first note that since in general, there may be repetitions in Sim α, we shall consider it as consisting of blocks of subsequences, where an x-block consists only of x in $\{0, 1, 2, \ldots, n-1\}$, repeated say m times $(1 \leq m \leq n)$. Thus we may write

$$\text{Sim } \alpha = (x_1\text{-block}, x_2\text{-block}, \ldots, x_s\text{-block})$$

where $0 \leq x_1 < x_2 < \cdots < x_s \leq n-1$. (Note that Im $\alpha = \{x_1, x_2, \ldots, x_s\}$.) Next we label the horizontal rows of the $n \times n$ square from the bottom to the top, starting from 0 to $n-1$. Now starting from $(0, 0)$, if $x_1 = 0$ take as many horizontal steps as the length of the x_1-block, otherwise take a diagonal step to the next level (level 1). Note that the order-decreasing property guarantees that $b_1 = x_1 \leq a_1$, however there is no guarantee that $x_i \leq a_i$ (for $i > 1$), but this will not be of any disadvantage. In general, at level m $(0 \leq m \leq n-1)$, check if

(i) $m \notin$ Im α and $m \notin$ Dom α, take a diagonal step to the next level;

(ii) $m \notin$ Im α and $m \in$ Dom α, take a vertical step to the next level;

(iii) $m \in$ Im α, take as many horizontal steps as the length of the m-block followed by a vertical or diagonal step to the next level depending on whether $m \in$ Dom α or $m \notin$ Dom α, respectively.

Now since in (iii) we give priority to horizontal step(s) over vertical (and diagonal) steps, it follows by the order-decreasing property that at any level m we must have had at least as many horizontal steps as there are vertical steps (up to level $m+1$). This in turn guarantees that our paths never overshoot to cross the diagonal line $y = x$. Moreover the bijection between Dom α and Sim α guarantees that our paths always end up at (n, n) as required.

An example would be quite appropriate. Consider the map

$$\alpha = \begin{pmatrix} 0 & 2 & 3 & 5 & 6 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix} \in \mathcal{PC}_7.$$

Then

$$\text{Dom } \alpha = \{0, 2, 3, 5, 6\},\ \text{Sim } \alpha \ = \ (000, 44), \quad \text{Im } \alpha = \{0, 4\}.$$

Now start from $(0, 0)$ and take 3 horizontal steps, since the first block of Sim α is a 0-block of length 3. We are still at level 0, and we take a vertical step to level 1 since $0 \in$ Dom α. Now since $1 \notin (\text{Dom } \alpha) \cap (\text{Im } \alpha)$, we take a diagonal step to level 2. Next, since $2 \in \text{Dom } \alpha \cap (\text{Im } \alpha)'$ we take a vertical step to level 3. Then since $3 \in \text{Dom } \alpha \cap (\text{Im } \alpha)'$ we take a vertical step to level 4. Now we take 2 horizontal steps and then a diagonal step since $4 \in \text{Im } \alpha \cap (\text{Dom } \alpha)'$. We are now at level 5, from where we take a vertical step to level 6 since $5 \in \text{Dom } \alpha \cap (\text{Im } \alpha)'$. Finally, we take another vertical step to level 7, since $6 \in \text{Dom } \alpha \cap (\text{Im } \alpha)'$. Thus we have the path indicated in Figure 4.

An immediate consequence of this bijection between these paths and decreasing and order-preserving partial transformations is that it furnishes us with the order of $|\mathcal{PC}_n|$. However, before we formally state this result we first deduce from [11] and [12] that the large (or double) Schröder number denoted by r_n could be defined as

$$r_n = \frac{1}{n+1} \sum_{r=0}^{n} \begin{pmatrix} n+1 \\ n-r \end{pmatrix} \begin{pmatrix} n+r \\ r \end{pmatrix}.$$

Moreover, r_n satisfies the recurrence

$$(n+2)r_{n+1} = 3(2n+1)r_n - (n-1)r_{n-1} \tag{3.2}$$

for $n \geq 1$, with initial conditions $r_0 = 1$ and $r_1 = 2$. The (small) Schröder number is usually denoted by s_n and defined as $s_0 = 1, s_n = r_n/2\,(n \geq 1)$ and so it satisfies the same recurrence as r_n.

We now have the main result of this section.

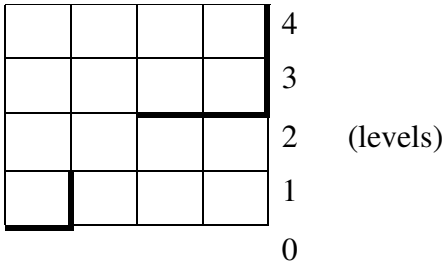

*Figure*3

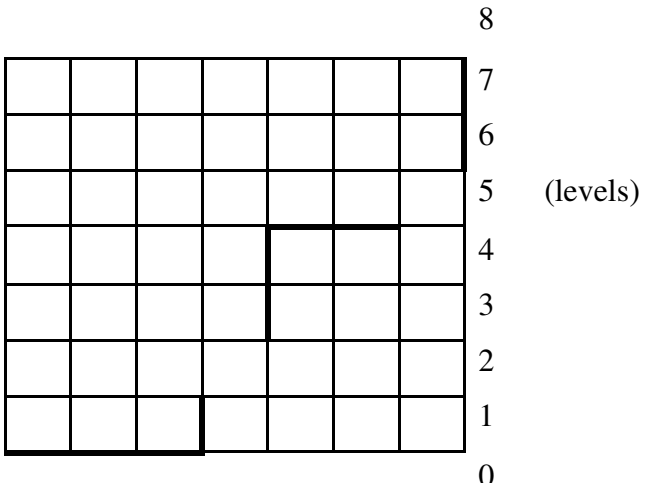

*Figure*4

Theorem 3.1 *Let $\mathcal{PC}_n$ be as defined in (2.1). Then $|\mathcal{PC}_n| = r_n$, the double Schröder number.*

For the semigroup Q_n (defined in (2.2)) and its complement we now have

Corollary 3.2 *$|Q_n| = |Q'_n| = s_n$, the (small) Schröder number.*

It is also known (from [11]) that the number of lattice paths that contain no points above the line $y = x$ and without a diagonal step is the *n-th* Catalan number: $\begin{pmatrix} 2n \\ n \end{pmatrix}/(n+1)$. However, since clearly such paths correspond to full transformations, Higgins [7, Theorem 3.1] follows immediately:

Theorem 3.3 *Let $\mathcal{C}_n$ be the semigroup of all decreasing and order-preserving full transformations of X_n. Then $|\mathcal{C}_n| = \begin{pmatrix} 2n \\ n \end{pmatrix}/(n+1)$, the n-th Catalan number.*

Moreover, composition of these kind of lattice paths is now possible via the bijection with the semigroup $\mathcal{PC}_n$. A more important consequence is we believe, the link of combinatorial questions between lattice paths and partial transformations is now firmly established.

4 The number of idempotents

As stated in the introduction the number of idempotents of various classes of semigroups of transformations has been computed. For further results see [10, 14, 15]. Our main task in this section is to compute the number of all idempotents in $\mathcal{PC}_n$. We consider

$$e(n,r) = |\{\alpha \in \mathcal{PC}_n : \alpha^2 = \alpha, |\text{Im } \alpha| = r\}|.$$

Then clearly we have

$$e(n,0) = 1 = e(n,n),$$

where the former corresponds to the empty map and the latter corresponds to the identity map. More generally, we have

Lemma 4.1 *For all $n \geq r \geq 1$, we have*

$$e(n,r) = 2e(n-1,r) + e(n-1,r-1).$$

Proof. If $n \notin \text{Dom } \alpha$ then $n \notin \text{Im } \alpha$, by idempotency and so there are $e(n-1,r)$ idempotents of this type. If on the other hand $n \in \text{Dom } \alpha$ then either $n\alpha = (n-1)\alpha < n$, of which there are again $e(n-1,r)$ idempotents of this type; or $n\alpha = n$, of which there are $e(n-1,r-1)$ idempotents of this type, by the order-decreasing property. Hence the result follows.

Now let $e_n = \sum_{r=0}^{n} e(n,r)$. Then $e_0 = 1$, and the next lemma gives a recurrence satisfied by e_n.

Lemma 4.2 *For all $n \geq 1$, we have: $e_n = 3e_{n-1} - 1$.*

Proof. By using Lemma 4.1, we have

$$\begin{aligned} e_n &= \sum_{r=0}^{n} e(n,r) = e(n,0) + e(n,1) + e(n,2) + e(n,3) + ... + e(n-1,n-1) + e(n,n) \\ &= [2e(n-1,0) - 1] + [2e(n-1,1) + e(n-1,0)] + [2e(n-1,2) + e(n-1,1)] \\ &\quad + [2e(n-1,3) + e(n-1,2)] + ... + [2e(n-1,n) + e(n-1,n-1)] \\ &= 3e(n-1,0) + 3e(n-1,1) + 3e(n-1,2) + ... + 3e(n-1,n-1) - 1 \\ &= 3\sum_{r=0}^{n-1} e(n-1,r) - 1 = 3e_{n-1} - 1. \end{aligned}$$

We now have the main result of this section.

Theorem 4.3 *Let $\mathcal{PC}_n$ be as defined in (2.1). Then $|E(\mathcal{PC}_n)| = e_n = \frac{1}{2}(3^n + 1)$.*

Proof. By the standard method of solving linear recurrence relations. See [1], for example.

Remark 4.4 It is not difficult to see that lattice paths that contain no points above the line $y = x$, in which a diagonal step never succeeds a horizontal segment, and where every length k horizontal segment is followed by exactly k vertical steps plus some (may be none) diagonal steps before another horizontal segment, correspond to idempotents in $\mathcal{PC}_n$. However, while idempotents are natural elements to study in a semigroup, their corresponding paths do not seem to have a natural description.

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References

1. I. Anderson, *A first course in combinatorial mathematics*, (Oxford University Press, 1974).

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Mathematical Surveys 7 (Providence, R. I.: American Math. Soc., 1961).

3. G.U. Garba, Idempotents in partial transformation semigroups. *Proc. Roy. Soc. Edinburgh* **116** (1990), 359-366.

4. G. U. Garba, On the idempotent ranks of certain semigroups of order-preserving transformations, *Portugaliae Mathematica* **51** (1994), 185-204.

5. G. M. S. Gomes and J. M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum*, **45** (1992), 272-282.

6. P. M. Higgins, *Techniques of semigroup theory,* (Oxford University Press, 1992).

7. P. M. Higgins, Combinatorial results for semigroups of order-preserving mappings, *Math. Proc. Camb. Phil. Soc.* **113** (1993), 281-296.

8. P. M. Higgins, Divisors of semigroups of order-preserving mappings on a finite chain, *Intern. Journal of Algebra and Computations* **5** (1995), 725-742.

9. J. M. Howie, *Fundamentals of semigroup theory* (Oxford: Clarendon Press, 1995).

10. J. M. Howie, Products of idempotents in certain semigroups of order-preserving transformations, *Proc. Edinburgh Math. Soc.* (2) **17**(1971), 223-236.

11. E. Pergola and R. A. Sulanke, Schröder Triangles, Paths and Parallelogram Polyominoes, *Journal of Integer Sequences* Vol. **1** (1998), 98.1.7.

12. R. P. Stanley, Hipparchus, Plutarch, Schröder and Hough, *Amer. Math. Monthly* **104** (1997), 344-350.

13. M. Tainiter, A characterisation of idempotents in semigroups, *J. Combin. Theory* **5** (1968), 370-373.

14. A. Umar, On the semigroups of order-decreasing finite full transformations, *Proc. Roy. Soc. Edinburgh Sect.* A **120** (1992), 129-142.

15. A. Umar, Enumeration of certain finite semigroups of transformations, *Discrete Math.* **189** (1998), 291-297.

16. A. Umar, On certain infinite semigroups of order-decreasing transformations I, *Communications in Algebra* Vol. **25** (9) (1997), 2987-2999.

17. V.V. Vagner, Representations of ordered semigroups. Math. Sb. NS. **387** (1956), 203-240, translated in Amer. Math. Soc. Trans. (2) **36** (1964), 295-336.

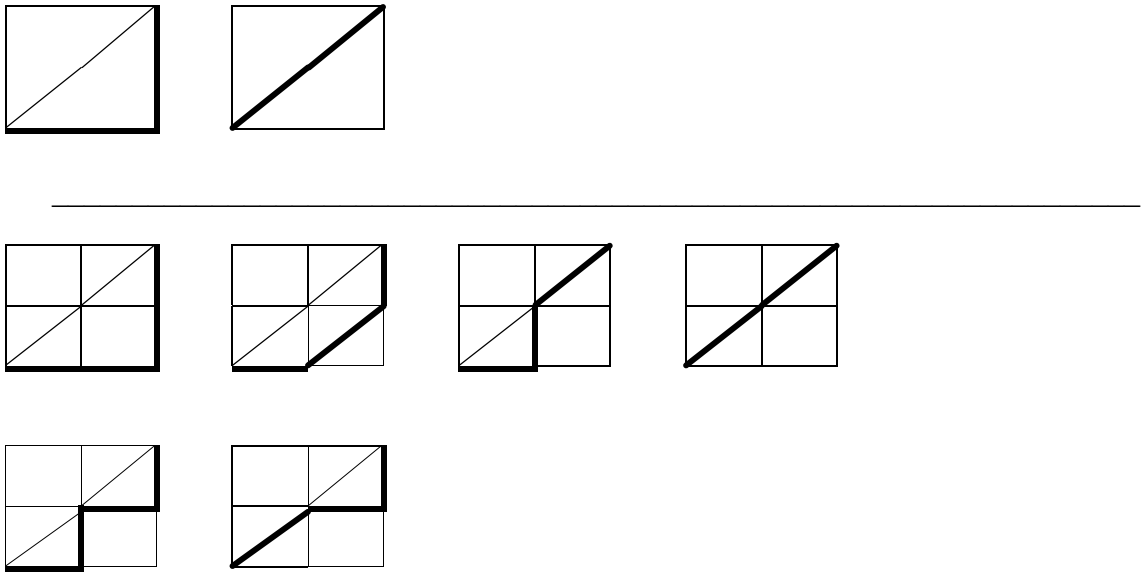

Fig. 1

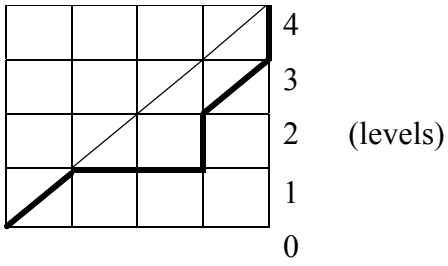

Fig. 2

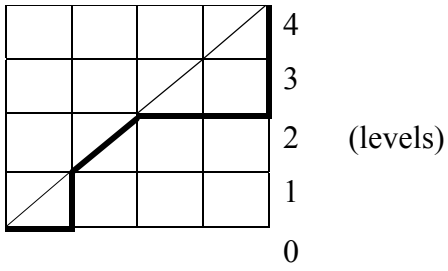

Fig. 3

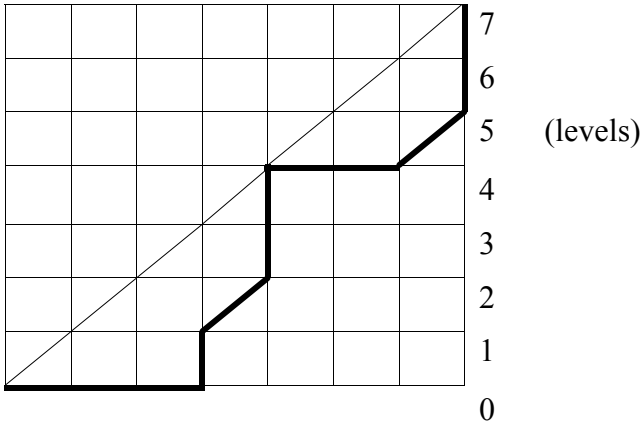

Fig. 4