King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 309

October 2003

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ON A PROBLEM OF FUCHS AND SALCE

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In [1, Problem 33 p. 336], Fuchs and Salce pose the problem of characterizing integral domains such that all epic images of their quotient fields are absolutely pure (or FP-injective). In this note, we solve a generalized version of this problem for arbitrary (associative) rings and prove in particular that the domains with the above property are precisely the Prüfer domains. First, it is clear that if for a ring R there exists an injective left R-module Q containing R all of whose epic images are absolutely pure, then the FP-injective dimension of every left ideal I of R (FP-i.d.$_R I$) is at most 1. Throughout R is a ring with 1, which is not necessarily commutative and possibly with zero divisors, and all modules are unital left R-modules. As in [1], the projective, flat, injective dimensions of an R-module M are denoted by p.d.$_R M$, w.d.$_R M$, i.d.$_R M$, respectively.

Lemma 1. *Let A, B be finitely generated submodules of a projective left R-module F. If FP-i.d.$_R(A \cap B) \leq 1$, then the exact canonical sequence $0 \longrightarrow A \cap B \longrightarrow A \oplus B \longrightarrow A + B \longrightarrow 0$ splits.*

Proof. Let $I = A + B$, $J = A \cap B$. We have an FP-injective resolution $0 \longrightarrow J \longrightarrow Q \longrightarrow Q/J \longrightarrow 0$ by hypothesis, that induces the exact sequence $0 = Ext^1(P, Q/J) \longrightarrow Ext^2(P, J) \longrightarrow Ext^2(P, Q) = 0$ for each finitely presented module P, i.e. $Ext^2(P, J) = 0$. Let F' be a finitely generated projective submodule of F (F may evidently be assumed to be free) containing A and B. Then F'/I is finitely presented, and therefore the first and third terms of the sequence $Ext^1(F', J) \longrightarrow Ext^1(I, J) \longrightarrow Ext^2(F'/I, J)$, induced by the exact sequence $0 \longrightarrow I \longrightarrow F' \longrightarrow F'/I \longrightarrow 0$, vanish. Hence $Ext^1(I, J) = 0$ and $0 \longrightarrow J \longrightarrow A \oplus B \longrightarrow I \longrightarrow 0$ splits. $\square$

Using induction and Lemma 1, one can prove

Lemma 2. *Let F be a projective left R-module. Suppose that FP-i.d.$_R$ $M \leq 1$ for every submodule M of F. Then every finitely generated submodule of F is a direct summand of a direct sum of cyclic submodules of F.* $\square$

Recall that a left PF (respectively PP) ring is one in which every principal left ideal is flat (respectively projective). Let us call R a left PFP ring if every principal left ideal is finitely presented. It is clear that R is PP if and only if it is both PF and PFP. One can easily show that R is

(a) left PF if and only if $ab = 0$ for $a, b \in R$ implies that there exists $b' \in R$ such that $a = ab'$ and $b'b = 0$.

(b) left PP if and only if for each $c \in R$, there exists $b \in R$ with $bc = c$ such that $ac = 0$ implies that $ab = 0$.

Clearly, if R has no zero divisors then it is a PP ring. Using Lemma 2 (with $F = R$) and the fact that left semihereditary (respectively, left coherent) rings are exactly those over which quotients (repectively, pure quotients) of absolutely pure left modules are absolutely pure, (see [3] and [4]), we obtain

Proposition 3. (i) *Let R be a ring such that FP-i.d.$_R I \leq 1$ for every left ideal I. Then R is left PF (respectively, PFP) if and only if R has weak dimension at most 1 (respectively, R is left coherent).*

(ii) Let R be a left PP ring. Then R is left semihereditary if and only if FP-i.d.$_R I \leq 1$ for every left ideal I of R if and only if every finitely generated left ideal is a direct summand of a direct sum of principal ideals. In particular, when R is a commutative domain with quotient field Q, then R is a Prüfer domain if and only if all epic images of Q are absolutely pure. □

Remark. Suppose R is a commutative PF ring. It was shown by Jøndrup [2] that all ideals of R are flat if every 2-generated ideal is flat. Proposition 3 and the proof of Lemma 1 then imply that R is semihereditary (i.e. a Prüfer ring) if and only if FP-i.d.$_R(Ra \cap Rb) \leq 1$ for all $a, b \in R$ if and only if, for any $a, b \in R$, there exists an injective module E such that $E/(Ra \cap Rb)$ has the injective property with respect to all exact sequences $0 \longrightarrow I \longrightarrow R$ where I is a 2-generated left ideal of R.

REFERENCES

1. L. Fuchs and L. Salce, Modules over Non-Noetherian Domains, Mathematical Surveys and Monographs , vol. 84 (American Mathematical Society, 2000).

2. S. Jøndrup, PP rings and finitely generated flat ideals, Proc. Amer. Math. Soc. 28 (1971), 431-435.

3. C. Megibben, Absolutely pure modules, Proc. Amer. Math. Soc. 26 (1970), 561-566.

4. B. Stenstrom, Coherent rings and FP-injective modules, J. London Math. Soc. (2) 2 (1970), 323-329.