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1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$ be the set of the first n natural numbers under the natural ordering and let T_n denote the full transformation semigroup, that is, the semigroup of all mappings $\alpha : X_n \to X_n$ under composition. We shall call α *order-decreasing (order-increasing)* or simply *decreasing (increasing)* if $x\alpha \leq x$ ($x\alpha \geq x$) for all x in X_n, and α is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for x, y in X_n. This paper investigates algebraic, combinatorial and rank properties of certain Rees quotient semigroups of D_n and $\mathcal{C}_n$, the semigroup of all decreasing mappings of X_n and the semigroup of all decreasing and order-preserving mappings of X_n, respectively.

Algebraic, combinatorial and rank properties of T_n have been studied over a long period and many interesting results have emerged (see, for example [1, 10, 11, 12]). A general study of D_n was initiated by Umar [16] and subsequently many results were published in [15, 18]. Higgins [6, 7] contain some delightful results concerning $\mathcal{C}_n$. As remarked by Higgins [6], both D_n and $\mathcal{C}_n$ have arisen naturally in language theory: any $\mathcal{J}$-trivial finite semigroup divides some $\mathcal{C}_n$ and any $\mathcal{R}$-trivial finite semigroup divides some D_n (see [13]). Note that Pin [13] referred to the order-preserving mappings as increasing and our order-increasing mappings as extensive. It transpires from our study that the algebraic and rank properties of $\mathcal{C}_n$ and some of its Rees quotient semigroups resemble closely those of D_n, while its combinatorial properties resemble those of O_n, the semigroup of all order-preserving mappings of X_n (see [6, 9] also).

We begin by recalling some notations and definitions that will be useful in the paper. For standard terms and concepts in semigroup theory we refer the reader to [10]. Let $\alpha : X_n \to X_n$ be a full transformation. We shall denote by Im α, the range or image set of α while $F(\alpha)$ and $S(\alpha) = X_n \setminus F(\alpha)$ will denote the set of fixed points and shifting points of α, respectively. Moreover, the cardinals of $S(\alpha)$ and $F(\alpha)$ will be denoted by $s(\alpha)$ and $f(\alpha)$, respectively.

Now for $1 \leq r \leq n$, we let

$$D(n, r) = \{\alpha \in D_n : |\text{Im } \alpha| \leq r\} \tag{1.1}$$

and

$$\mathcal{C}(n, r) = \{\alpha \in \mathcal{C}_n : |\text{Im } \alpha| \leq r\} \tag{1.2}$$

be the two-sided ideals of D_n and $\mathcal{C}_n$, respectively, consisting of all decreasing maps of height not more than r (height $\alpha = |\text{Im } \alpha|$). Note that $D(n, r)$ is in the notation of [18], $K^-(n, r)$ while $D_n = D(n, n)$ is in the notation of [15], $(S_n^-)^1$. Also, for $r \geq 2$, we let

$$DP_r(n) = D(n, r)/D(n, r-1) \tag{1.3}$$

be the Rees quotient semigroup of D_n, and

$$\mathcal{C}P_r(n) = \mathcal{C}(n, r)/\mathcal{C}(n, r-1) \tag{1.4}$$

be the Rees quotient semigroup of $\mathcal{C}_n$. The elements of $DP_r(n)(\mathcal{C}P_r(n))$ may be thought of as the elements of $D_n(\mathcal{C}_n)$ of height r precisely. The product of two elements of $DP_r(n)(\mathcal{C}P_r(n))$ is 0 whenever their product in $D_n(\mathcal{C}_n)$ is of height strictly less than r. As in the above, $DP_r(n)$ is in the notation of [18], $P_r^-(n)$. The first result in this section concerns fixed points of decreasing maps.

Lemma 1.1 *Let α, β be decreasing transformations of X_n. Then* $F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta\alpha)$.

This result (Lemma 1.1) was first proved in [16], however, Higgins [8] proved it as part of the proof of [8, Proposition 1.4] which he mistakenly attributed to [15].

We would like to introduce here a useful piece of notation from [12]. Every transformation α in T_n may be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

where $A_1, A_2, \ldots A_r$ called the *blocks* of α are all nonempty, and $A_i = a_i\alpha^{-1}(i = 1, 2, \ldots, r)$. Then every idempotent transformation α is characterized by the property that every block of α is *stationary*, that is, $a_i \in A_i$ for all i, (equivalently, if $F(\alpha) = \{a_i : i = 1, 2, \ldots, r\}$). If in addition the idempotent transformation is decreasing then $a_i = \min A_i$ for all i. It is also the case that every order-preserving transformation has all its blocks convex. The next lemma leads to the proof of Theorem 1.3 (below).

Lemma 1.2 *Let* $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$ *be an element of* $DP_r(n)$ *and let* $V(\alpha) = S(\alpha) \cap$ *Im* α *with* $v_0 = \min V(\alpha)$. *Then* $v_0 \neq t_i = \min A_i$, *for all* i.

Proof. Suppose $v_0 = t_i$ (for some i). If $v_0\alpha = v_0$ then $v_0 \in F(\alpha)$ which is a contradiction. Thus by the order-decreasing property we have $v_0 > v_0\alpha$, and either $v_0\alpha \in F(\alpha)$ or $v_0\alpha \in S(\alpha)$. The former implies that both v_0 and $v_0\alpha$ belong to same block, in fact they belong to A_i. However, this violates the minimality of $v_0 = t_i$ in A_i. The latter clearly violates the minimality of v_0 in $V(\alpha)$. In either eventuality we have a contradiction. Therefore we conclude that $v_0 \neq t_i$ for all i. ■

Theorem 1.3 *Let* $DP_r(n)$ *and* $\mathcal{C}P_r(n)$ *be as defined in (1.3) and (1.4), respectively. Then every* α *in* $DP_r(n)$ $(\mathcal{C}P_r(n))$ *is expressible as a product of idempotents in* $DP_r(n)(\mathcal{C}P_r(n))$.

Proof. First observe that proving the result for $DP_r(n)$ does not imply the proof for $\mathcal{C}P_r(n)$, and even less so the other way round. However, since in general the algebraic proofs for $\mathcal{C}P_r(n)$ are similar (but slightly more difficult) to those for $DP_r(n)$, perhaps because of the additional requirement of order-preserveness, we will only present the more difficult proof. Now suppose that $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots A_r \\ a_1 & a_2 & \cdots a_r \end{pmatrix} \in \mathcal{C}P_r(n)$. We may without loss of generality assume that $1 = a_1 < a_2 < \cdots < a_r \leq n$. Next let $V(\alpha) = S(\alpha) \cap \text{Im } \alpha$ with $v_0 = \min V(\alpha)$. Define ϵ, β by

$$\begin{aligned} A_i\epsilon &= t_i = \min A_i (i = 1, 2, \ldots, r) \\ x\beta &= \begin{cases} v_0 & (\text{if } v_0 \leq x < \max v_0\alpha^{-1}) \\ x\alpha & (\text{otherwise}). \end{cases} \end{aligned}$$

Then it is clear that ϵ is an idempotent in $\mathcal{C}P_r(n)$. Next, it is not difficult to see that β is order-decreasing and $|\text{Im } \beta| = r$. Moreover, for all $y \geq \max(v_0\alpha^{-1})$, we have $y\beta = y\alpha \geq v_0 = v_0\beta$ and for all $y < v_0$, we have $y\beta = y\alpha \leq v_0\alpha < v_0 = v_0\beta$. Thus β is order-preserving. Note also that $S(\beta) = S(\alpha) \setminus \{v_0\}$ and so $s(\beta) = s(\alpha) - 1$. Finally, observe that

$$A_i\epsilon\beta = t_i\beta = \begin{cases} v_0 = A_i\alpha & (\text{if } v_0 < t_i < \max(v_0\alpha^{-1})) \\ a_i = A_i\alpha & (\text{otherwise}) \end{cases}$$

since $v_0 \neq t_i$ for all i, by Lemma 1.2. Thus $\epsilon\beta = \alpha$, and the result now follows by induction. ■

Define a map ξ by $x\xi = 1$ (for all x in X_n). Then clearly ξ is in $\mathcal{C}(n, r)$ (and hence in $D(n, r)$) and $\alpha\xi = \xi\alpha$ for all α in $D(n, r)$. Thus ξ is the zero of $D(n, r)(\mathcal{C}(n, r))$ which we shall henceforth denote by 0. An element a in a 0-semigroup S is called *nilpotent* if $a^k = 0$ for some $k \geq 1$. Then the next two lemmas concerning nilpotents in $D(n, r)$ and $DP_r(n)$ imply the corresponding results for $\mathcal{C}(n, r)$ and $\mathcal{C}P_r(n)$, respectively.

Lemma 1.4 *Let α be an element in $D(n, r)(\mathcal{C}(n, r))$. Then α is nilpotent if and only if $f(\alpha) = 1$.*

Proof. This follows from [15, Lemma 1.5] or directly from Lemma 1.1. ■

Lemma 1.5 *Let α be an element in $DP_r(n)$ $(\mathcal{C}P_r(n))$. Then α is nilpotent if and only if $f(\alpha) < r$.*

Proof. This follows directly from Lemma 1.1. ■

2 Green's Relations and their Starred Analogues

For the definitions of Green's relations, see for example [10]. In case of ambiguity we shall denote by $\mathcal{K}_S$ for any relation $\mathcal{K}$ on S.

Theorem 2.1 *Let $DP_r(n)$ be as defined in (1.3). Then*

(1) $DP_r(n)$ is $\mathcal{R}$-trivial;

(2) for α, β in $DP_r(n)$, $(\alpha, \beta) \in \mathcal{L}$ if and only if Im α = Im β and $\min z\alpha^{-1} = \min z\beta^{-1}$ (for all z in Im α).

Proof. The proof is similar to that of [15, Lemma 2.1]. ■

Theorem 2.2 *Let $\mathcal{C}P_r(n)$ be as defined in (1.4). Then $\mathcal{C}P_r(n)$ is $\mathcal{J}$-trivial.*

Proof. Let α, β be elements in $\mathcal{C}P_r(n)$ be such that $(\alpha, \beta) \in \mathcal{J}$. Then there exist δ, γ in $\mathcal{C}P_r^1(n)$ such that $\alpha = \delta\beta\gamma$. Thus for all x in X_n, we have

$$x\alpha = x\delta\beta\gamma \leq x\delta\beta \leq x\beta$$

by the order-decreasing and order-preserving properties successively. Similarly, we can show that $x\beta \leq x\alpha$ and so $x\alpha = x\beta$ for all x in X_n. Hence $\alpha = \beta$ as required. ■

As a consequence of the two theorems above, we deduce that $\mathcal{H} = \mathcal{R}$ and $\mathcal{L} = \mathcal{D} = \mathcal{J}$ on $DP_r(n)$, and $\mathcal{H} = \mathcal{R} = \mathcal{L} = \mathcal{D} = \mathcal{J}$ on $\mathcal{C}P_r(n)$. Hence (for $r \geq 3$), the semigroups $DP_r(n)$ and $\mathcal{C}P_r(n)$ are nonregular.

By analogy with [15, Section 2] to identify the class of semigroups to which $DP_r(n)$ and $\mathcal{C}P_r(n)$ belong, we consider the starred Green's relations studied in [4, 3].

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ if and only if the elements a, b are $\mathcal{L}$-related in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually, while $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$, the lattice join of $\mathcal{L}^*$ and $\mathcal{R}^*$. A semigroup S is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. Recall also from [17] that a subsemigroup U (of a semigroup S) is called an *inverse ideal* of S if there exist u' in S such that $uu'u = u$ and both uu' and $u'u$ belong to U. (Note that an inverse ideal need not be an ideal.) Then we have

Theorem 2.3 (17, Lemma 3.1.8 & 3.1.9) . *Every inverse ideal of a semigroup S is an abundant semigroup. Moreover,*

(1) $\mathcal{L}_U^* = \mathcal{L}_S \cap (U \times U)$;

(2) $\mathcal{R}_U^* = \mathcal{R}_S \cap (U \times U)$;

(3) $\mathcal{H}_U^* = \mathcal{H}_S \cap (U \times U)$.

It is now fairly obvious that if we can show that both $DP_r(n)$ and $\mathcal{C}P_r(n)$ are inverse ideals of $P_r(n)$, the corresponding Rees quotient semigroups of T_n then the characterizations of $\mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{H}^*$ will immediately follow Theorem 2.3 and [1, Lemmas 10.55 & 10.56].

Theorem 2.4 *Let $DP_r(n)$ be as defined in (1.3) and let $\mathcal{C}P_r(n)$ be as defined in (1.4). Then both $DP_r(n)$ and $\mathcal{C}P_r(n)$ are inverse ideals of $P_r(n)$, the corresponding Rees quotient semigroup of T_n.*

Proof. Since the two proofs are similar, we only give the more 'difficult proof' that is, the proof for $\mathcal{C}P_n(r)$. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots A_r \\ a_1 & a_2 & \cdots a_r \end{pmatrix} \in \mathcal{C}P_n(r)$, where we may without loss of generality assume that $1 = a_1 < a_2 < \cdots < a_r \leq n$. To define α' in $P_r(n)$, first choose x_i to be $\min A_i$ and let

$$x\alpha' = x_i \quad (a_i \leq x < a_{i+1}),$$

where $a_{r+1} = n+1$. Then $\alpha\alpha'\alpha = \alpha$, by the construction of α' and it is not difficult to see that α' is order-preserving. Moreover, since $x\alpha = a_i$ for some i, we have

$$x\alpha\alpha' = a_i\alpha' = x_i \leq x \in A_i$$

and

$$x\alpha'\alpha = x_i\alpha = a_i \leq x \quad (a_i \leq x < a_{i+1}),$$

proving that both $\alpha\alpha'$ and $\alpha'\alpha$ are decreasing. It is also the case that $\alpha\alpha'$ and $\alpha'\alpha$ are order-preserving as both α and α' are order-preserving. Thus both $\alpha\alpha'$ and $\alpha'\alpha$ are in $\mathcal{C}P_n(r)$, as required. ■

Now as remarked after Theorem 2.3, we have

Theorem 2.5 *Let $DP_r(n)$ be as defined in (1.3) and let $\mathcal{C}P_r(n)$ be as defined in (1.4). Then for α, β in $DP_r(n)$ ($\mathcal{C}P_r(n)$), we have:*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ if and only if Im $\alpha =$ Im β,

(2) $(\alpha, \beta) \in \mathcal{R}^*$ if and only if $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$,

(3) $(\alpha, \beta) \in \mathcal{H}^*$ if and only if Im $\alpha =$ Im β and $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

And consequently, we now have

Lemma 2.6 *Every $\mathcal{R}^*$ class of $DP_r(n)$ contains a unique idempotent.*

Proof. It follows from the remarks (before Lemma 1.2) concerning idempotent transformations. ■

Lemma 2.7 *Every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class of $\mathcal{C}P_r(n)$ contains a unique idempotent.*

Proof. Uniqueness of idempotents in each $\mathcal{R}^*$-class follows from the same reasons as in the above lemma, while for $\mathcal{L}^*$-classes it follows from the remarks preceeding [6, Theorem 3.19]. ■

Remark. This is in sharp contrast to what obtains in regular semigroup theory where uniqueness of idempotents in each $\mathcal{L}^*$-class (or $\mathcal{R}^*$-class) forces the idempotents to commute and so $E(S)$ becomes a semilattice. Here every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of $\mathcal{C}P_r(n)$ contains a unique idempotent, and yet it is idempotent-generated (Theorem 1.3).

The proof of the next result is similar to the more complicated proof of [19, Lemma 2.9] on one hand, and on the other hand we have to be more careful because of the additional condition of order-preserveness, in the case of $\mathcal{C}P_r(n)$.

Lemma 2.8 *On the semigroups $DP_r(n)$ and $\mathcal{C}P_r(n)$, we have $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$.*

Proof. For the same reason stated earlier we only give the proof for $\mathcal{C}P_r(n)$. Let α, β be nonzero elements in $\mathcal{C}P_r(n)$, and so let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

where we may without loss of generality assume that $1 = a_1 < a_2 < \cdots < a_r \leq n$ and $1 = b_1 < b_2 < \cdots < b_r \leq n$. Then there is an order-preserving bijection $\theta :$ Im $\alpha \to$ Im β given by $\theta(a_i) = b_i (i = 1, 2, \ldots r)$. Now let $C = \{c_i : c_i = \max(a_i, b_i)\}$ and define δ, γ in $\mathcal{C}P_r(n)$ by

$$x\delta = a_i (c_i \leq x < c_{i+1}) \text{ and } x\gamma = b_i (c_i \leq x < c_{i+1})$$

where $c_{r+1} = n + 1$. Then clearly δ, γ are in $\mathcal{C}P_r(n)$ and by Theorem 2.5, we have that $\alpha \mathcal{L}^* \delta \mathcal{R}^* \gamma \mathcal{L}^* \beta$. Thus for any two nonzero elements α, β in $\mathcal{C}P_r(n)$ we have $(\alpha, \beta) \in \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$; equivalently, $\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$ is the universal relation on $\mathcal{C}P_r(n) \setminus \{0\}$.

On the other hand let $D = \{d_i : d_i = \min(a_i, b_i)\}$ and define δ', γ' in $\mathcal{C}P_r(n)$ by

$$A_i\delta' = d_i \text{ and } B_i\gamma' = d_i \quad (i = 1, 2, \ldots, r).$$

Then clearly δ', γ' are in $\mathcal{C}P_r(n)$, and by Theorem 2.5, we have that $\alpha \mathcal{R}^* \delta' \mathcal{L}^* \gamma' \mathcal{R}^* \beta$. Thus for any two nonzero elements α, β in $\mathcal{C}P_r(n)$ we have that $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$; equivalently, $\mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$ is the universal relation on $CP_r(n) \setminus \{0\}$. Hence the result follows. ■

A semigroup S is called $[0]-*bisimple$ if it has a unique nonzero $\mathcal{D}^*$-class. Thus, we have now shown the following result.

Theorem 2.9 *Let $DP_r(n)$ and $\mathcal{C}P_r(n)$ be as defined in (1.3) and (1.4), respectively. Then both $DP_r(n)$ and $\mathcal{C}P_r(n)$ are idempotent-generated 0-*bisimple primitive abundant semigroups.*

Proof. It is only primitiveness that we have not shown, however, it follows from the primitiveness of $P_r(n)$. ■

3 Some combinatorial results

As observed in the introduction, combinatorial properties of various classes of semigroups have been investigated. We draw particular attention to [6] and [15]. Our main aim in this section is to find a formula for $|\mathcal{C}P_r(n)|$, since $|DP_r(n)|$ could be deduced from [15, Lemma 4.4]. In fact $|DP_r(n)| = 1 + e(n,r)$, where $e(n,r)$ is the Eulerian number given by

$$e(n,1) = 1 = e(n,n) \text{ and } e(n,r) = re(n-1,r) + (n-r+1)e(n-1,r-1).$$

It turns out that from our investigation of $|\mathcal{C}P_r(n)|$ we get as a corollary [6, Theorem 3.1] which states that $|\mathcal{C}(n,n)| = |\mathcal{C}_n| = \frac{1}{n}\begin{pmatrix} 2n \\ n-1 \end{pmatrix}$, the n-th Catalan number.

First for $n \geq k \geq r \geq 1$, we consider

$$J(n,r,k) = |\{\alpha \in \mathcal{C}(n,r) : |\text{Im } \alpha| = r \wedge \max(\text{Im } \alpha) = k\}|. \tag{3.1}$$

Then it is evident that

$$J(n,n,n) = 1, \;\; J(n,1,k) = \begin{cases} 1 & (\text{if } k = 1) \\ 0 & (\text{if } k > 1) \end{cases}$$

and

$$J(n,r,k) = 0 \quad \text{if } \; k < r \text{ or } \; k > n.$$

Less evidently is the following recurrence relation satisfied by $J(n,r,k)$:

Lemma 3.1 $J(n,r,k) = J(n-1,r,k) + \sum_{t=r-1}^{n-1} J(n-1,r-1,t)$.

Proof. Maps α (in $\mathcal{C}(n,r)$) for which $|\mathrm{Im}\ \alpha| = r$ and $\max(\mathrm{Im}\ \alpha) = k$ divide naturally into two classes: $n\alpha = (n-1)\alpha = k$ or $(n-1)\alpha < n\alpha = k$. Then it is not difficult to see that there are $J(n-1,r,k)$ maps of the former type, and there are $\sum_{t=r-1}^{k-1} J(n-1,r-1,t)$ maps of the latter type. Adding the two numbers yields the required result. ■

A closed formula for $J(n,r,k)$ is possible, but before we propose this formula we would like to state these two results from [14]. The first (Lemma 3.2) known as the *Vandermonde convolution identity* is in the words of Riordan [14, p. 8] perhaps the most widely used combinatorial identity, while the second (Lemma 3.3) is a combination of equations (3) and (3b) from [14, p. 8].

Lemma 3.2 (14, Equation (3a) p. 8)

$$\sum_{k=0}^{n} \binom{n}{m-k}\binom{p}{k} = \binom{n+p}{m}.$$

Lemma 3.3 *For any $c \in \mathbb{R}$, and $q, m \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{j=0}^{m} (c-j)\binom{q+j}{j} = (c-m-1)\binom{m+q+1}{m} + \binom{m+q+2}{m}.$$

Proposition 3.4 *Let $J(n,r,k)$ be as defined in (3.1). Then*

$$J(n,r,k) = \frac{n-k+1}{n-r+1}\binom{n-1}{r-1}\binom{k-2}{r-2}.$$

The proof of Proposition 3.4 is by induction, however, we would like to anchor the induction by this lemma:

Lemma 3.5 $J(n,r,r) = \binom{n-1}{r-1}$.

Proof. Since $J(n,1,1) = 1 = \binom{n-1}{0}$ is true, we now suppose that the result is true for all $r < n$. Then by Lemma 3.1, we have

$$\begin{aligned} J(n,r,r) &= J(n-1,r,r) + J(n-1,r-1,r-1) \\ &= \binom{n-2}{r-1} + \binom{n-2}{r-2} \qquad \text{(by Induction Hypothesis)} \\ &= \binom{n-1}{r-1} \end{aligned}$$

as required.

Now coming back to the proof of Proposition 3.4, we suppose that the result is true for all $s \le h < n+1$, and so, by Lemma 3.1,

$$J(n+1,s,h) = J(n,s,h) + \sum_{t=s-1}^{h-1} J(n,s-1,t)$$

$$= \frac{n-h+1}{n-s+1}\binom{n-1}{s-1}\binom{h-2}{s-2} + \sum_{t=s-1}^{h-1} \frac{n-t+1}{n-s+2}\binom{n-1}{s-2}\binom{t-2}{s-3}.$$

Put $j = t-s+1$, so that when $t = s-1$, $j = 0$ and when $t = h-1$, $j = h-s$. Thus

$$\begin{aligned} &\sum_{t=s-1}^{h-1} \frac{n-t+1}{n-s+2}\binom{n-1}{s-2}\binom{t-2}{s-3} \\ &= \sum_{j=0}^{h-s} \frac{n-(j+s-1)+1}{n-s+2}\binom{n-1}{s-2}\binom{j+s-3}{s-3} \\ &= \sum_{j=0}^{h-s} \frac{(n-s+2)-j}{n-s+2}\binom{n-1}{s-2}\binom{s-3+j}{s-3} \\ &= \frac{1}{n-s+2}\binom{n-1}{s-2}\sum_{j=0}^{h-s}\{(n-s+2)-j\}\binom{s-3+j}{j} \end{aligned}$$

(Using Lemma 3.3, with $c = n-s+2, m = k-s$ and $q = s-3$)

$$\begin{aligned} &= \frac{1}{n-s+2}\binom{n-1}{s-2}\left\{(n-h+1)\binom{h-2}{h-s} + \binom{h-1}{h-s}\right\} \\ &= \frac{1}{n-s+2}\binom{n-1}{s-2}\binom{h-2}{s-2}\left\{(n-h+1) + \frac{h-1}{s-1}\right\} \qquad (3.2) \end{aligned}$$

Thus

$$\begin{aligned} J(n+1,s,h) &= \frac{n-h+1}{n-s+1}\binom{n-1}{s-1}\binom{n-2}{s-2} \\ &\quad + \frac{1}{n-s+2}\binom{n-1}{s-2}\left\{(n-h+1)\binom{h-2}{h-s}+\binom{h-1}{h-s}\right\} \\ &= \frac{n-h+2}{n-s+2}\binom{n}{s-1}\binom{h-2}{s-2} \end{aligned}$$

after some algebraic manipulations.

To complete the induction step we still need to verify $J(n+1,s,n+1)$. By Lemma 3.1

$$\begin{aligned} J(n+1,s,n+1) &= \sum_{t=s-1}^{n} J(n,s-1,t) \\ &= \sum_{t=s+1}^{n} \frac{n-t+1}{n-s+2}\binom{n-1}{s-2}\binom{t-2}{s-2} \quad \text{(by Induction Hypothesis)} \\ &= \frac{1}{n-s+2}\binom{n}{s-1}\binom{n-1}{s-2} \quad \text{(using (3.2) with } h-1=n\text{)} \end{aligned}$$

as required. Hence the proof of Proposition 3.4 is complete. ■

Proposition 3.6 *Let* $J(n,r)=\sum_{k=r}^{n} J(n,r,k)$. *Then* $J(n,r)=\frac{1}{n-r+1}\binom{n-1}{r-1}\binom{n}{r}$.

Proof. $J(n,r)=\sum_{k=r}^{n} J(n,r,k)$

$$\begin{aligned} &= \sum_{k=r}^{n} \frac{n-k+1}{n-r+1}\binom{n-1}{r-1}\binom{k-2}{r-2} \quad \text{(by Proposition 3.4)} \\ &= \frac{1}{n-r+1}\binom{n-1}{r-1}\sum_{k=r}^{n}[(n+1)-k]\binom{k-2}{k-r} \\ &= \frac{1}{n-r+1}\binom{n-1}{r-1}\sum_{j=0}^{n-r}[(n+1-r)-j]\binom{r-2+j}{j} \quad \text{(with } k-r=j\text{)} \end{aligned}$$

(using Lemma 3.3 with $c = n+1-r$, $m = n-r$ and $q = r-2$)

$$= \frac{1}{n-r+1}\binom{n-1}{r-1}\left\{(n+1-r-n+r-1)\binom{n-r+r-2+1}{n-r} + \binom{n-r+r-2+2}{n-r}\right\}$$
$$= \frac{1}{n-r+1}\binom{n-1}{r-1}\binom{n}{n-r}$$
$$= \frac{1}{n-r+1}\binom{n-1}{r-1}\binom{n}{r}.$$

■

Hence we deduce that

Theorem 3.7 *Let $\mathcal{C}P_r(n)$ be as defined in (1.4). Then*
$|\mathcal{C}P_r(n)| = \frac{1}{n-r+1}\binom{n-1}{r-1}\binom{n}{r} + 1.$

Proof. The extra 1 added to the result in Proposition 3.6 accounts for the 0 element.

■

Theorem 3.8 *Let $\mathcal{C}(n,r)$ be as defined in (1.2). Then*

$$|\mathcal{C}(n,r)| = \sum_{t=1}^{r} \frac{1}{n-t+1}\binom{n-1}{t-1}\binom{n}{t}.$$

A useful corollary to Theorem 3.8 is

Corollary 3.9 (6, Theorem 3.1)

$$|\mathcal{C}_n| = |\mathcal{C}(n,n)| = \sum_{t=1}^{n} \frac{1}{n-t+1}\binom{n-1}{t-1}\binom{n}{t} = \frac{1}{n}\binom{2n}{n-1}.$$

Proof. It remains to show the last equality only, which is established as follows:

$$\sum_{t=1}^{n} \frac{1}{n-t+1}\binom{n-1}{t-1}\binom{n}{t} = \sum_{t=1}^{n} \frac{n!(n-1)!}{(n-t+1)(n-t)!(t-1)!(n-t)!t!}$$
$$= \frac{1}{n}\sum_{t=1}^{n}\binom{n}{t-1}\binom{n}{n-t}$$
$$= \frac{1}{n}\binom{2n}{n-1}$$

by Lemma 3.2. ■

Another result of independent interest is

Proposition 3.10 *Let* $F(n,k) = \sum_{r=1}^{k} J(n,r,k)$. *Then* $F(n,k) = \frac{n-k+1}{n} \binom{n+k-2}{n-1}$.

Proof. From Proposition 3.4, we have

$$\begin{aligned} F(n,k) = \sum_{r=1}^{k} J(n,r,k) &= \sum_{r=1}^{k} \frac{n-k+1}{n-r+1} \binom{n-1}{r-1} \binom{k-2}{r-2} \\ &= (n-k+1) \sum_{r=1}^{k} \frac{1}{n-r+1} \binom{n-1}{r-1} \binom{k-2}{r-2} \\ &= \frac{(n-k+1)}{n} \sum_{r=1}^{k} \binom{n}{r-1} \binom{k-2}{r-2} \\ &= \frac{(n-k+1)}{n} \sum_{r=1}^{k} \binom{n}{n-r+1} \binom{k-2}{r-2} \\ &= \frac{n-k+1}{n} \binom{n+k-2}{n-1}, \end{aligned}$$

by Lemma 3.2. ■

Corollary 3.11 $|\mathcal{C}_n| = |\mathcal{C}(n,n)| = \sum_{k=1}^{n} F(n,k) = \frac{1}{n} \binom{2n}{n-1}$.

Proof.

$$\begin{aligned} |\mathcal{C}_n| &= \sum_{k=1}^{n} F(n,k) \\ &= \sum_{k=1}^{n} \frac{n-k+1}{n} \binom{n+k-2}{n-1} \\ &= \frac{1}{n} \sum_{j=0}^{n-1} (n-j) \binom{(n-1)+j}{j} \quad \text{(with } j = k-1\text{)} \\ &= \frac{1}{n} \left[(n-(n-1)-1) \binom{2n-1}{n-1} + \binom{2n}{n-1} \right] \\ &= \frac{1}{n} \binom{2n}{n-1} \end{aligned}$$

using Lemma 3.3, with $c = n, q = n-1 = m$, to get the step before the last. ■

Finally, from Lemma 2.7 we deduce that

Lemma 3.12 $|E(\mathcal{C}P_r(n))| = \binom{n-1}{r-1} + 1.$

Proof. It follows from the fact that there are $\binom{n-1}{r-1}$ $\mathcal{L}^*$-classes in $\mathcal{C}P_r(n)$ plus the zero element. ■

4 Rank Properties

As in [11], the rank of a finite semigroup S is defined by

$$\text{rank } S = \min\{|A| : A \subseteq S, \langle A \rangle = S\}.$$

If S is generated by its set of idempotents E, then the idempotent rank of S is denoted and defined by

$$\text{idrank } S = \min\{|A| : A \subseteq E, \langle A \rangle = S\}.$$

In this section we investigate the rank and idempotent rank of $\mathcal{C}(n, r)$ and $\mathcal{C}P_r(n)$. Related questions on various classes of semigroups of transformations have been considered in recent years. In particular, Howie and McFadden [11], considered the semigroup $P_r(n)$ $(2 \leq r \leq n-1)$ and showed that both the rank and idempotent rank are equal to $S(n, r)$, the Stirling number of the second kind. Garba [5] obtained analogous results for the semigroup of order-preserving transformations. Similarly, in [18] it was shown that both the rank and idempotent rank of $DP_r(n)$ are equal to $S(n, r)$ as in [11] while in Higgins [6, Section 2.1] it was shown that $\mathcal{C}_n$ admits a unique minimal generating system.

The main result of this section is

Proposition 4.1 *Let $\mathcal{C}(n, r)$ and $\mathcal{C}P_r(n)$ be as defined in (1.2) and (1.4), respectively. Then for $1 \leq r \leq n-1$,*

$$\begin{aligned} \textit{rank } \mathcal{C}(n, r) &= \textit{idrank } \mathcal{C}(n, r) = \textit{rank } \mathcal{C}P_r(n) = \textit{idrank } \mathcal{C}P_r(n) \\ &= \binom{n-1}{r-1}. \end{aligned}$$

We are going to prove this proposition for $\mathcal{C}(n,r)$ and deduce the result for $\mathcal{C}P_r(n)$. As a first step towards the proof of Proposition 4.1, we establish the following lemma:

Lemma 4.2 *Let* $\epsilon = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}$ *be an idempotent element in* $\mathcal{C}(n,r)$*. Then there exist idempotents* η_1, η_2 *in* $\mathcal{C}(n,r)$ *for which* $|Im\ \eta_1| = |Im\ \eta_2| = k+1$ *and* $\epsilon = \eta_1\eta_2$*.*

Proof. Suppose that

$$\epsilon = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}$$

is an idempotent in $\mathcal{C}(n,r)$. We may without loss of generality assume that $1 = a_1 < a_2 < \cdots < a_k \leq n$ and $2 \leq k < r \leq n$. Notice also that by the convexity of the block A_i and idempotency we have $a_{i+1} > \max A_i$ for all $i = 1, 2, \ldots, k-1$. This observation will guarantee that the mappings η_1, η_2 we define below are order-preserving. However, before defining the required idempotent mappings η_1, η_2, we note that essentially we can either have $|A_i| \geq 2$ and $|A_j| \geq 2$; or $|A_i| \geq 3$ for some $i, j \in \{1, 2, \ldots, k\}$. In the former case we choose an element $a_i' \neq a_i$ in A_i and $a_j' \neq a_j$ in A_j; in the latter case we choose two distinct elements a_i', a_i'' in $A_i \setminus \{a_i\}$. Then in the former we define

$$\begin{array}{ll} a_i'\eta_1 = a_i', \quad x\eta_1 = x\epsilon \quad (x \neq a_i') \\ a_j'\eta_2 = a_j', \quad y\eta_2 = y\epsilon \quad (y \neq a_j') \end{array}$$

in the latter we define

$$\begin{array}{ll} a_i'\eta_1 = a_i', \quad x\eta_1 = x\epsilon \quad (x \neq a_i') \\ a_j''\eta_2 = a_i'', \quad y\eta_2 = y\epsilon \quad (y \neq a_i''). \end{array}$$

In both cases it is clear that η_1, η_2 are idempotents, and η_1, η_2 are both decreasing and order-preserving so that $\eta_1, \eta_2 \in \mathcal{C}(n,r)$. ■

An immediate consequence of the above lemma and Theorem 1.3 is that $\mathcal{C}(n,r)$ is generated by its idempotents of height r. Thus, by Lemma 3.12 we deduce that for $1 \leq r \leq n-1$,

$$\text{idrank}\ \mathcal{C}(n,r) \leq \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}.$$

To show the reverse inequality, we show that E_r, the set of idempotents of $\mathcal{C}(n,r)$ of height exactly r, is a minimal generating set for $\mathcal{C}(n,r)$ and equivalently, for $\mathcal{C}P_r(n)$. We achieve this by showing that the product of an idempotent of height r and any other element (idempotent or nonidempotent), is not an idempotent of height r. Let ϵ be an idempotent of height r and let $\eta\,(\neq \epsilon)$ be an arbitrary element in $\mathcal{C}(n,r)$. If η is an idempotent then we have

$$F(\eta) = \text{Im } \eta \neq \text{Im } \epsilon = F(\epsilon)$$

since Im $\epsilon =$ Im η implies $\epsilon = \eta$ for any two idempotents in $\mathcal{C}(n,r)$. Hence by Lemma 1.1

$$F(\epsilon\eta) = F(\epsilon) \cap F(\eta) \subset F(\epsilon)$$

which implies that $f(\epsilon\eta) < r$. If η is not an idempotent then $f(\eta) < r$ and so $f(\epsilon\eta) < r$. Thus, we have

$$\text{idrank } \mathcal{C}(n,r) \geq \binom{n-1}{r-1}.$$

Moreover, by a theorem of Doyen [2], $\mathcal{C}(n,r)$ being $\mathcal{J}$-trivial has a unique minimum generating set which must be E_r in this case. Hence the rank and idempotent rank of $\mathcal{C}(n,r)$ are equal. Therefore the proof of Proposition 4.1 is complete.

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