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A STOCHASTIC VERSION OF FAN'S BEST APPROXIMATION THEOREM

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Abstract

We establish some deterministic and random approximation results with the help of two continuous maps in the context of a metrizable topological vector space. These results generalize some well known results in approximation theory.

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1. Introduction

A lot of work has been done on the existence of best approximation for continuous and nonexpansive mappings on Hilbert spaces, Banach spaces and locally convex topological vector spaces. These results include both single and multivalued maps. In general, fixed point theorems and the related techniques have been used to prove the results about best approximation. We refer to [4, 6, 7, 13, 14] and references therein.

In 1969, Ky Fan [6, Theorem 1] proved the following best approximation result:

THEOREM A. *Let C be a compact convex set in a locally convex Hausdorff topological vector space X. If $f : C \to X$ is continuous, then either f has a fixed point or there*

exist an $x \in C$ and a continuous seminorm p on X such that

$$p(x - fx) = d_p(fx, C)$$

where $d_p(fx, C) = inf\{p(fx) - y : y \in C\}$.

This theorem has been of great importance in nonlinear analysis, game theory and minimax theorems and it has been extended in various directions by many authors (e.g. see [11] and [14]). Prolla [13] has generalized it for a pair of continuous functions on the subset C of a normed space.

The purpose of this paper is to generalize Prolla's main result by considering a continuous function and the other one being a continuous almost quasi-convex onto function on a suitable subset of a metrizable topological vector space, using Ky Fan's intersection lemma [6] as a main tool. Stochastic versions of our results are established as well. As a consequence, a stochastic generalization of the celebrated Fan's best approximation theorem (Theorem A) follows.

In Section 3, we prove some approximation results for single-valued continuous quasi-convex mappings on a compact as well as on a noncompact subset of a metrizable topological vector space.

In Section 4, we present random versions of the results in Section 3. Section 2 deals with certain technical preliminaries and establishes notational conventions. Even though some of the concepts are standard, they are included here to facilitate reading.

2. Preliminaries

Let X denote a topological vector space (TVS, for short). Throughout, we assume that its topology is tacitly generated by an F-norm on it; that is, there is a real-valued

map, say, q on X such that (i) $q(x) \geq 0$ and $q(x) = 0$ iff $x = 0$; (ii) $q(x+y) \leq q(x) + q(y)$; (iii) $q(\lambda x) \leq q(x)$ for all $x, y \in X$ and for all scalars λ with $|\lambda| \leq 1$; (iv) if $q(x_n) \to 0$, then $q(\lambda x_n) \to 0$ for all scalars λ; (v) if $\lambda_n \to 0$, then $q(\lambda_n x) \to 0$ for all $x \in X$, where (λ_n) is a sequence of scalars. The formula $d(x, y) = q(x - y)$ defines a metric on X.

We denote by 2^X, $C(X)$ and $CK(X)$ the families of all nonempty, nonempty closed and nonempty convex compact subsets of X.

Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω. Let $P(Z)$ be a collection of subsets of a set Z. Denote by $\hat{N}$ the set of all infinite sequences of positive integers and by $\hat{N}_0$, the set of all finite sequences of positive integers. A subset A of Z is said to be obtained from $P(Z)$ by *Souslin operation* if there is a map $k : \hat{N}_0 \to P(Z)$ such that $A = \bigcup_{x \in \hat{N}} \bigcap_{n=1}^{\infty} (k(r|_n)$, where $r|_n$ denotes the first n elements of the finite sequence $r \in \hat{N}$. Note that the union in the Souslin operation is uncountable. So, if $P(Z)$ is a σ-algebra, then A may be outside $P(Z)$. If $P(Z)$ is closed under the Souslin operation, then it is called a *Souslin family*. For more details about Souslin family we refer to Shahzad [15] and Wagner [17].

Let $T : \Omega \to 2^X$ be a multivalued mapping. The set

$$Gr(T) = \{(\omega, x) \in \Omega \times X : x \in T(\omega)\}$$

is called the *the graph* of T.

A mapping $T : \Omega \to 2^X$ is said to be *measurable* (respectively, *weakly measurable*) if $T^{-1}(B) \in \Sigma$ for each closed (respectively, open) subset B of X, where

$$T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \phi\}.$$

It is known that the measurability of $T : \Omega \to 2^X$ implies the weak measurability but not conversely, in general.

A mapping $f : \Omega \to X$ is said to be a *selector* of a mapping $T : \Omega \to 2^X$ if $f(\omega) \in T(\omega)$ for all $\omega \in \Omega$.

Let Y, Z be two metric spaces. A function $f : \Omega \times Y \to Z$ is said to be a *Caratheodory function* if for each $y \in Y$, $T(\cdot, y)$ is measurable and for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous.

Random operators with stochastic domain have been studied by Engl [5] and Shahzad [15].

Following Engl [5] and Papageorgiou [10], we say that a mapping $T : \Omega \to 2^X$ is *separable* if there exists a countable set $D \subseteq X$ such that for all $\omega \in \Omega$, $cl(D \cap T(\Omega)) = T(\omega)$. For instance, if T has closed, convex and solid (that is, nonempty interior) values, then T is separable. Further, it is clear from the definition of separability that T has closed values.

Let $F : \Omega \to C(X)$ be a weakly measurable mapping. A mapping $T : Gr(F) \to 2^X$ is called a *multivalued random operator* with stochastic domain $F(\cdot)$ if for all $x \in X$ and all $U \subseteq X$ open, $\{\omega \in \Omega : T(\omega, x) \cap U \neq \phi, \ \ x \in F(\omega)\} \in \Sigma$.

Let $F : \Omega \to C(X)$ be a weakly measurable mapping. A random operator $T : Gr(F) \to X$ with stochastic domain $F(\cdot)$ is called a *random contraction* if $T(\omega, \cdot)$ is a contraction on $F(\omega)$ for all $\omega \in \Omega$.

For a finite subset $\{x_1, \ldots, x_n\}$ of a TVS X, we write the *convex hull* of $\{x_1, \ldots, x_n\}$ as

$$Co\{x_1, \ldots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \quad 0 \leq \alpha_i \leq 1, \quad \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We shall need the following result known as Ky Fan's intersection lemma [6].

THEOREM B. *Let C be a subset of a TVS X and $F : C \to 2^X$ a closed-valued map such that $Co(x_1, \ldots, x_n) \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1, \ldots, x_n\}$ of C. If $F(x_0)$*

is compact for at least one x_0 in C, then $\bigcap_{x \in C} F(x) \neq \phi$.

THEOREM C. ([9], Theorem 1). Let *Let C be a nonempty convex subset of a Hausdorff TVS X and $A \subseteq C \times C$ such that*

(a) for each $x \in C$, the set $\{y \in C : (x, y) \in A\}$ is closed in C;

(b) for each $y \in C$, the set $\{x \in C : (x, y) \notin A\}$ is convex or empty;

(c) $(x, x) \in A$ for each $x \in C$;

(d) C has a nonempty compact convex subset X_0 such that the set $B = \{y \in C : (x, y) \in A$ for all $x \in X_0\}$ is compact.

Then there exists a point $y_0 \in B$ such that $C \times \{y_0\} \subset A$.

Let X be a metrizable TVS with a metric d on it, C a convex subset of X and $g : C \to C$ a continuous map. Then g is said to be (cf. [12])

(i) almost affine if

$$d\Big(g(rx_1 + (1-r)x_2), y\Big) \leq rd(gx_1, y) + (1-r)d(gx_2, y),$$

(ii) almost quasi-convex if

$$d\Big(g(rx_1 + (1-r)x_2), y\Big) \leq \max\{d(gx_1, y), d(gx_2, y)\},$$

where $x_1, x_2 \in C$, $y \in X$ and $0 < r < 1$.

It is easy to see that (i) implies (ii) but not conversely, in general (see also [16] for related concepts). A random operator $f : \Omega \times C \to X$ is continuous (almost affine, almost quasi-convex) if for each $\omega \in \Omega$, the map $f(\omega, \cdot) : C \to X$ is so.

3. Approximation in Metrizable Topological Vector Spaces

We begin with the following theorem which generalizes the main result of Prolla [13] to a wider class of functions defined on a subset of a metrizable TVS with its proof based on Ky Fan's intersection lemma (Theorem B). This result also extends Theorem 1 of Carbone [4] and partially Theorem 2.1 in [11] (see also [12]).

THEOREM 3.1. *Let C be a nonempty compact convex subset of a metrizable TVS X and $g : C \to C$ a continuous almost quasi-convex onto function. If $f : C \to X$ is a continuous function, then there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$.*

Proof. For each $z \in C$, define

$$F(z) = \{y \in C : d(gy, fy) \leq d(gz, fy)\}.$$

Since f and g are continuous, therefore for each $z \in C$, $F(z)$ is a closed set and hence a compact subset of C.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of C. Then, $Co(x_1, \ldots, x_n) \subseteq \bigcup_{i=1}^{n} F(x_i)$. If this is not the case, then there is some u in $Co(x_1, \ldots, x_n)$ such that $u \notin \bigcup_{i=1}^{n} F(x_i)$. Now $u = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$ and as $u \notin \bigcup_{i=1}^{n} F(x_i)$, so $d(gx_i, fu) < d(gu, fu)$ for all $i = 1, 2, \ldots, n$. Since g is almost quasi-convex, therefore

$$d(gu, fu) = d\left(g\left(\sum_{i=1}^{n} \alpha_i x_i\right), fu\right) \leq \max_i d(gx_i, fu) < d(gu, fu)$$

which is impossible. Then, by Theorem A, $\bigcap_{x \in C} F(x) \neq \phi$ and hence there is $y \in \bigcap_{x \in C} F(x)$ so that, for all $x \in C$,

$$d(gy, fy) \leq d(gx, fy).$$

Since g is onto, we get $d(gy, fy) \leq d(z, fy)$ for all $z \in C$ and hence $d(gy, fy) = d(fy, C)$. ∎

The compactness of C in Theorem 3.1 can be replaced by a weaker condition to

obtain the following generalization of Theorem 2 of Carbone [4].

THEOREM 3.2. *Let C be a nonempty convex subset of a metrizable TVS X and $g : C \to C$ a continuous almost quasi-convex onto function. Suppose $f : C \to X$ is a continuous function. If C has a nonempty compact convex subset B such that the set*

$$D = \{y \in C : d(fy, gy) \leq d(fy, gx) \text{ for all } x \in B\}$$

is compact, then there exists an element $y \in D$ such that $d(fy, gy) = d(fy, C)$.

Proof. Let $A = \{(x, y) \in C \times C : d(fy, gy) \leq d(fy, gx)\}$. Obviously, $(x, x) \in A$ for all $x \in C$. By the continuity of f and g, the set $\{y \in C : (x, y) \in A\}$ is closed in C for each $x \in C$. The set

$$K = \{x \in C : (x, y) \notin A\} = \{x \in C : d(fy, gy) > d(fy, gx)\}$$

is convex. Indeed, suppose $x_1, x_2 \in K$. Then $d(gx_1, fy) < d(fy, gy)$ and $d(gx_2, fy) < d(fy, gy)$. Since g is almost quasi-convex, we have for $0 < \lambda < 1$,

$$\begin{aligned} d(g(\lambda x_1 + (1-\lambda)x_2), fy) &\leq \max\{d(gx_1, fy), d(gx_2, fy)\} \\ &< d(fy, gy). \end{aligned}$$

This implies that $\lambda x_1 + (1-\lambda)x_2 \in K$.

By Theorem C, there exists $y \in D$ such that $C \times \{y\} \subset A$. That is, $d(fy, gy) \leq d(fy, gx)$ for all $x \in C$. As g is onto, so $d(fy, gy) = d(fy, C)$ for some $y \in B$. ∎

REMARKS 3.3 (i) If we consider $f : C \to C$ in Theorems 3.1 and 3.2, then y becomes a coincidence point of f and g (that is, $fy = gy$).

(ii) All the results obtained so far trivially hold when X is a Fréchet space.

4. Random Approximation

In this section we establish the random versions of Theorems 3.1 and 3.2 which in turn extend Theorem 5 of [3] and Theorem 5 of [15] to the general framework of metrizable topological vector spaces.

THEOREM 4.1. *Let C be a compact and convex subset of a complete metrizable TVS X and $g : \Omega \times C \to C$ a continuous almost quasi-convex and onto random operator. If $T : \Omega \times C \to X$ is a continuous random operator, then there exists a measurable map $\xi : \Omega \to C$ satisfying*

$$d(g(\omega, \xi(\omega)), T(\omega, \xi(\omega))) = d(T(\omega, \xi(\omega)), C)$$

for each $\omega \in \Omega$.

Proof. Let $F : \Omega \to 2^C$ be defined by

$$F(\omega) = \{x \in C : d(g(\omega, x), T(\omega, x)) = d(T(\omega, x), C)\}.$$

By Theorem 3.1, $F(\omega) \neq \phi$ for all $\omega \in \Omega$. Also, $F(\omega)$ is compact for each $\omega \in \Omega$. Let G be a closed subset of C. Put

$$L(G) = \bigcap_{n=1}^{\infty} \bigcup_{x \in D_n} \left\{\omega \in \Omega : d(g(\omega, x), T(\omega, x)) < d(T(\omega, x), C) + \frac{1}{n}\right\},$$

where $D_n = \left\{x \in D : d(x, G) < \frac{1}{n}\right\}$.

Note that the functions $p : \Omega \times C \to \mathbb{R}^+$ and $q : \Omega \times C \to \mathbb{R}^+$ defined by $p(\omega, x) = d(g(\omega, x), T(\omega, x))$ and $q(\omega, x) = d(T(\omega, x), C)$ are measurable in ω and continuous in x (see [15, Theorem 5]). Following arguments similar to those in the proof of Theorem 5 of [3], we can show that F is measurable. Applying a selection theorem due to Kuratowski and Nardzewski [8] we get a measurable map $\xi : \Omega \to C$ such that $\xi(\omega) \in F(\omega)$ for all $\omega \in \Omega$. The result now follows from the definition of $F(\omega)$.

DEFINITION 4.2. Let (X, d_1) and (Y, d_2) be two metric spaces. The pair of metric

spaces (X, Y) is said to have the *Kirzbraun property* or *property* (K) according to Shahzad [15] if for all choices $x_i \in X$, $y_i \in Y$ and $\gamma_i > 0$, $i \in I$ (I an arbitrary index set) such that the intersection of the balls $B(y_i, \gamma_i)$ in X is nonempty and $d_2(y_i, y_j) \leq d_1(x_i, x_j)$, $i, j \in I$, then the intersection of the balls $B(y_i, \gamma_i)$ in Y is also nonempty.

We need the following result of Shahzad [15, Theorem 1].

THEOREM 4.3. *Let (Ω, Σ) be a measurable space with Σ a Souslin family. Let X and Y be separable complete metric spaces such that the pair (X, Y) has property (K) and $F : \Omega \to 2^X$ a separable weakly measurable function. Then every random contraction $f : Gr(F) \to Y$ with stochastic domain $F(\cdot)$ can be extended to a random contraction defined on $\Omega \times X$.*

REMARK 4.4. The conclusion of Lemma 6 of Engl [5] remains valid for separable complete metric spaces (cf. [15], p. 442).

THEOREM 4.5. *Let (Ω, Σ) be a measurable space with Σ a Souslin family and X a separable complete metrizable TVS. Assume that $F : \Omega \to 2^X$ is a separable weakly measurable convex-valued multifunction and $f : Gr(F) \to X$ is a random contraction with stochastic domain $F(\cdot)$. If $g : Gr(F) \to X$ is a continuous almost quasi-convex onto random operator with stochastic domain $F(\cdot)$ such that $g(\omega, x) \in F(\omega)$ for all $(\omega, x) \in Gr(F)$. Suppose that $G_0 : \Omega \to CK(X)$ is a measurable multifunction with $G_0(\omega) \subseteq F(\omega)$ for all $\omega \in \Omega$ such that for a weakly measurable multifunction D,*

$$D(\omega) = \{y \in F(\omega) : d(f(\omega, y), g(\omega, y)) \leq d(f(\omega, y), g(\omega, x)) \text{ for all } x \in G_0(\omega)\}$$

is compact for each $\omega \in \Omega$. If the pair (X, X) has property (K), then there exists a measurable map $\xi : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi(\omega) \in D(\omega)$ and

$$d(f(\omega), \xi(\omega)), g(\omega, \xi(\omega))) = d(f(\omega, \xi(\omega)), F(\omega)).$$

Proof. By Theorem 4.3, we get a random contraction $\hat{f} : \Omega \times X \to X$. Let $H : \Omega \to 2^X$ be defined by

$$H(\omega) = \left\{ x \in D(\omega) : d(g(\omega, x), \hat{f}(\omega, x)) = d(\hat{f}(\omega, x), F(\omega)) \right\}.$$

By Theorem 3.2, $H(\omega) \neq \phi$ for each $\omega \in \Omega$. Define maps $h, k : \Omega \times X \to \mathbb{R}^+$ by $h(\omega, x) = d(\hat{f}(\omega, x), F(\omega))$ and $k(\omega, x) = d(\hat{f}(\omega, x), g(\omega, x))$. Obviously h is continuous and by [5, Lemma 6], h is measurable in ω (see Remark 4.4), so $h(\cdot, \cdot)$ is a Caratheodory function. Similarly $k(\cdot, \cdot)$ is also a Caratheodory function. By the continuity of functions involved, $H(\omega)$ is closed for each $\omega \in \Omega$.

Define $\phi(\omega, x) = h(\omega, x) - k(\omega, x)$. Clearly, $\phi(\cdot, \cdot)$ is jointly measurable. Observe that

$$Gr(H) = Gr(F) \cap \{(\omega, x) \in \Omega \times X : \phi(\omega, x) = 0\} \in \Sigma \times B(X).$$

Since Σ is a Souslin family, therefore by [17, Theorem 4.2], $H(\cdot)$ is weakly measurable. By the selection theorem in [8], $H(\cdot)$ has a measurable selector $\xi : \Omega \to X$. Consequently, $\xi(\omega) \in D(\omega)$ and

$$d(f(\omega, \xi(\omega)), g(\omega, \xi(\omega))) = d(f(\omega, \xi(\omega)), F(\omega))$$

for each $\omega \in \Omega$. ■

An immediate consequence of the above theorem when the underlying domain of the maps f and g is not varying stochastically is presented below; our result generalizes Corollary 2 in [2] to metrizable TVS.

COROLLARY 4.6. *Let (Ω, Σ) and X be as in Theorem 4.5 and C a nonempty convex subset of X. Assume that $f : \Omega \times C \to X$ is a random contraction and $g : \Omega \times C \to C$ is a continuous almost quasi-convex onto random operator. Let X_0 be a nonempty*

compact convex subset of C and K be a nonempty compact subset of C. If for each $y \in C \setminus K$, there exists $x \in X_0$ such that

$$d(g(\omega, x), f(\omega, y)) < d(g(\omega, y), f(\omega, y)),$$

then there exists a measurable mapping $\xi : \Omega \to K$ satisfying

$$d(g(\omega, \xi(\omega)), f(\omega, \xi(w)) = d(f(\omega, \xi(\omega)), C)$$

for each $\omega \in \Omega$.

REMARK 4.7. Theorem 4.5 extends Corollary 3.3 [1], Theorem 1 [2], Theorem 5 [3], Theorem 4 [10] and Theorem 5 [15] to the general framework of metrizable topological vector spaces.

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