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Abstract

We consider an integro-differential equation of hyperbolic type with a temporal non-local memory term. It will be shown by a new argument that the dissipation induced by the memory effect is strong enough to yield exponential decay of the energy.

Key words and phrases: Exponential decay, fading memory, viscoelastic damping.

AMS subject classifications: 35B40, 35L70, 45K05

1 Introduction

In this paper we investigate the following problem

$$\begin{cases} u_{tt} - k(0)\Delta u - \int\limits_0^\infty k'(s)\Delta u(t-s)ds + g(u) = f \text{ in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0,\ x \in \partial\Omega,\ t \in \mathbb{R}^+ \\ u(x,t) = u_0(x,t),\ x \in \Omega,\ t \leq 0 \end{cases} \tag{1}$$

with $k(0)$, $k(\infty) > 0$ and $k'(s) \leq 0$ for every $s \in \mathbb{R}^+$. Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, Δ is the Laplacian operator and $'$ denotes differentiation with respect to the time variable. The function

$u_0(x,t)$ is a given initial data. As for the non-linear term $g(u)$ and the time-varying source f, they will be specified later on.

This model arises, for instance, in viscoelasticity, in electromagnetism and in heat conduction theory, see [2,3,10]. It concerns materials with fading memory. The convolution term in the equation expresses the dependence on the entire past history of the solution.

This problem has been studied in [1,10,2,7,8,5,6,9] (see also references therein). In [5] Giorgi, Rivera and Pata showed the existence of absorbing sets and global attractors for solutions after proving in particular an exponential decay of the energy (in the linear homogeneous case). See [11] for a similar problem but in an abstract Cauchy form. Observe that there is no dissipation besides the weak viscoelastic damping provided by the long-range memory term. As is well known, the presence of an explicit dissipation (weak or strong) produces loss of energy and therefore it is considered as a favorable situation.

In this work we propose a new argument based on an appropriately chosen functional which satisfies a certain differential equation. The "standard" condition on the derivative of the kernel is relaxed.

2 Reformulation and assumptions

In this paper we shall try to keep the same notation as in [5]. In particular, the usual notation for Lebesgue and Sobolev spaces will be used together with their scalar products and norms. By $L^2_\mu(\mathbb{R}^+, H^1_0)$ we denote the weighted Hilbert spaces of H^1_0-valued functions on $\mathbb{R}^+$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_\mu(\mathbb{R}^+, H^1_0)} = \int_\Omega \left(\int_0^\infty \mu(s) \nabla\varphi \nabla\psi ds \right) dx.$$

We will denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} = H^1_0 \times L^2 \times L^2_\mu(\mathbb{R}^+, H^1_0)$ endowed with the usual inner product. The space $\mathcal{T}$ will stand for the space of all L^1_{loc}-translation bounded L^2-valued functions on $\mathbb{R}^+$, that is

$$\mathcal{T} = \left\{ f \in L^1_{loc}(\mathbb{R}^+, L^2) : \|f\|_{\mathcal{T}} = \sup_{\xi \geq 0} \int_\xi^{\xi+1} \left(\int_\Omega f(y) dx \right) dy \right\}.$$

(This space will be needed in case of the presence of f.)

Introducing the new variable

$$\eta^t(x,s) = u(x,t) - u(x,t-s)$$

and setting $\mu(s) = -k'(s)$, $k(\infty) = 1$ we may reformulate problem (1) as

$$\begin{cases} u_{tt} = a\Delta u + \int_0^\infty \mu(s)\Delta\eta^t(s)ds - g(u) + f \\ \eta_t = -\eta_s + u_t. \end{cases} \tag{2}$$

If we designate by

$$\begin{cases} u_0(x) = u_0(x,0), \\ v_0(x) = \partial_t u_0(x,t)\,|_{t=0}\,, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), \end{cases}$$

then the initial and boundary conditions are given by

$$\begin{cases} u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0 \\ \eta^t(x,s) = 0,\ (x,s) \in \partial\Omega \times \mathbb{R}^+,\ t \geq 0 \\ u_0(x,0) = u_0(x),\ x \in \Omega \\ u_t(x,0) = v_0(x),\ x \in \Omega \\ \eta^0(x,s) = \eta_0(x,s),\ (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{3}$$

We suppose that the new kernel μ satisfies

(h1) $\mu \in C^1(\mathbb{R}^+)$ and $e^{\alpha t}\mu \in L^1(\mathbb{R}^+)$ for some $\dot{\alpha} > 0$,

(h2) $\mu(s) \geq 0$ and $\mu'(s) \leq 0$, for all $s \in \mathbb{R}^+$,

(h3) $\int_0^\infty \mu(s)ds = k_0 > 0$.

For the nonlineariry we assume that $g \in C^1(\mathbb{R})$. Let us denote by

$$G(s) = \int_0^s g(y)dy \text{ and } \mathcal{G}(u) = \int_\Omega G(u(x))dx \text{ for } u \in H_0^1(\Omega).$$

We further require that g fulfills the following assumptions: there exist $C_0 > 0$ and $\Gamma > 0$ such that

(g1) $\liminf_{|y|\to\infty} \frac{G(y)}{y^2} \geq 0$, (g2) $\liminf_{|y|\to\infty} \frac{yg(y)-C_0G(y)}{y^2} \geq 0$,

(g3) $|g'(y)| \leq \Gamma$.

It has been proved in [4] that (g1)-(g2) imply

$$\mathcal{G}(u) + \frac{1}{4}\int_\Omega |\nabla u|^2\, dx \geq -C_1, \text{ for all } u \in H_0^1(\Omega),$$

$$\int_\Omega ug(u)dx - C_0\mathcal{G}(u) + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx \geq -C_2, \text{ for all } u \in H_0^1(\Omega),$$

for some positive constants C_1 and C_2. It turns out that these inequalities are of great help when examining the asymptotic behavior.

Finally, we make clear what we will mean by a solution to the initial boundary value problem (1.3)-(1.4).

Definition 1 *A function $z = (u, u_t, \eta) \in C(I, \mathcal{H})$ where $I = [0,T]$ is a solution to problem (2) in I, with initial data $z(0) = z_0 = (u_0, v_0, \eta_0) \in \mathcal{H}$ and $f \in L^1(I, L^2)$, if*

$$\langle u_{tt}, \tilde{v}\rangle = -\int_\Omega \nabla u \nabla \tilde{v} dx - \int_\Omega \left(\int_0^\infty \mu(s)\nabla\eta(s)ds\right)\nabla\tilde{v}dx - \int_\Omega g(u)\tilde{v}dx + \int_\Omega f\tilde{v}dx,$$

$$\int_\Omega \left(\int_0^\infty \mu(s)\left(\eta_t(s) + \eta_s(s)\right)\Delta\tilde{\eta}(s)ds\right)dx = \int_\Omega u_t\left(\int_0^\infty \mu(s)\Delta\tilde{\eta}(s)ds\right)dx,$$

for all $\tilde{v} \in H_0^1(\Omega)$ and $\tilde{\eta} \in L^2_\mu(\mathbb{R}^+, H^2 \cap H_0^1)$, and a.e. $t \in I$.

For simplicity, we shall treat here the case $f \equiv g \equiv 0$, focussing only on the main differences with the argument given in [5]. The case where f and g are not both zero can be handeled in exactly the same manner and under the same hypotheses as in [5] without any changes other than the ones we are exposing here. So our result holds also in this case.

3 Exponential decay

In this section we shall present and prove our result. First, let us introduce some functionals. The energy associated to problem (2) is defined by

$$\mathcal{E}(t) = \frac{1}{2}\left(\int\limits_{\Omega} a\left|\nabla u\right|^2 + \left|u_t\right|^2 + \int\limits_0^{\infty} \mu(s)\left|\nabla \eta^t(s)\right|^2 ds\right) dx. \tag{4}$$

The functionals to follow will help us built the Lyapunov functional $\mathcal{L}(t)$ which will satisfy a certain differential inequality. Then an application of Gronwall inequality (a generalized version of it) yields at once the desired estimates. We set,

$$\mathcal{F}(t) = -\int\limits_{\Omega} u_t \left(\int\limits_0^{\infty} \mu(s)\eta^t(s)ds\right) dx, \tag{5}$$

$$\mathcal{H}(t) = \int\limits_{\Omega} u_t u dx$$

and

$$\mathcal{K}(t) = \int\limits_{\Omega}\int\limits_{-\infty}^{t} P_\alpha(t-s)\left|\nabla u(s)\right|^2 dsdx, \tag{6}$$

where

$$P_\alpha(s) = e^{-\alpha s}\int\limits_s^{\infty} \mu(r)e^{\alpha r}dr.$$

Theorem 2 *Suppose that the hypotheses (h1)-(h3) hold. Then there exist positive constants C and ε such that the estimate*

$$\mathcal{E}(t) \leq Ce^{-\varepsilon t}$$

is true for every $t \geq 0$.

Proof. Differentiating (4) with respect to t and using (2), it appears that

$$\frac{d\mathcal{E}(t)}{dt} = \frac{1}{2}\int_{\Omega}\left(\int_0^{\infty}\mu'(s)\left|\nabla\eta^t(s)\right|^2 ds\right)dx. \tag{7}$$

Recall that $\mu'(s) \leq 0$. Therefore, the energy is uniformly bounded (by $\mathcal{E}(0)$)and decreasing.

A differentiation of $\mathcal{F}(t)$ (see (5)) with respect to t gives

$$\frac{d\mathcal{F}(t)}{dt} = -\int_{\Omega} u_{tt}\left(\int_0^{\infty}\mu(s)\eta^t(s)ds\right)dx - \int_{\Omega} u_t\left(\int_0^{\infty}\mu(s)\eta_t^t(s)ds\right)dx. \tag{8}$$

By the second equation in (2) and (h3) we may estimate the second term in the right hand side of (8) in the following manner

$$\begin{aligned}
-\int_{\Omega} u_t\left(\int_0^{\infty}\mu(s)\eta_t^t(s)ds\right)dx &= -\int_{\Omega} u_t\left(\int_0^{\infty}\mu(s)\left[u_t(t)-\eta_s^t(s)\right]ds\right)dx \\
&= -k_0\int_{\Omega}|u_t|^2 dx - \int_{\Omega} u_t\left(\int_0^{\infty}\mu'(s)\eta^t(s)ds\right)dx.
\end{aligned}$$

By the hölder inequality, Young inequality and Poincaré inequality (we shall use these inequalities repeatedly in the sequel), we have

$$\begin{aligned}
-\int_{\Omega} u_t\left(\int_0^{\infty}\mu(s)\eta_t^t(s)ds\right)dx &\leq -\left(k_0-\delta\mu(0)\right)\int_{\Omega}|u_t|^2 dx \\
&+\tfrac{1}{4\delta\lambda_0}\int_{\Omega}\left(\int_0^{\infty}-\mu'(s)\left|\nabla\eta^t(s)\right|^2 ds\right)dx,
\end{aligned} \tag{9}$$

where λ_0 is the Poincaré constant.

Using equation $(2)_1$, it is easily seen that

$$\begin{aligned}
-\int_{\Omega} u_{tt}\left(\int_0^{\infty}\mu(s)\eta^t(s)ds\right)dx &= a\int_{\Omega}\nabla u\left(\int_0^{\infty}\mu(s)\nabla\eta^t(s)ds\right)dx \\
&+\int_{\Omega}\left(\int_0^{\infty}\mu(s)\nabla\eta^t(s)ds\right)^2 dx.
\end{aligned}$$

Young's inequality, together with the Cauchy-Schwarz inequality, allow us to write

$$-\int_\Omega u_{tt}\left(\int_0^\infty \mu(s)\eta^t(s)ds\right)dx \le a\rho\int_\Omega |\nabla u|^2\,dx + \tfrac{a}{4\rho}\int_\Omega\left(\int_0^\infty \mu(s)\nabla\eta^t(s)ds\right)^2 dx$$
$$+\int_\Omega\left(\int_0^\infty \mu(s)\nabla\eta^t(s)ds\right)^2 dx,\ \rho>0$$

or

$$-\int_\Omega u_{tt}\left(\int_0^\infty \mu(s)\eta^t(s)ds\right)dx$$
$$\le a\rho\int_\Omega |\nabla u|^2\,dx + \left(\tfrac{a}{4\rho}+1\right)\int_\Omega\left(\int_0^\infty \mu(s)\nabla\eta^t(s)ds\right)^2 dx.$$

The assumption (h3) and Hölder's inequality imply

$$-\int_\Omega u_{tt}\left(\int_0^\infty \mu(s)\eta^t(s)ds\right)dx$$
$$\le a\rho\int_\Omega |\nabla u|^2\,dx + \left(\tfrac{a}{4\rho}+1\right)k_0\int_\Omega\int_0^\infty \mu(s)\left|\nabla\eta^t(s)\right|^2 dsdx.$$

Another use of Young's inequality leads to

$$-\int_\Omega u_{tt}\left(\int_0^\infty \mu(s)\eta^t(s)ds\right)dx \le a\rho\int_\Omega |\nabla u|^2\,dx + \left(\tfrac{a}{4\rho}+1\right)k_0$$
$$\times\left[(1+\sigma)k_0\int_\Omega |\nabla u|^2\,dx + \left(1+\tfrac{1}{\sigma}\right)\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx\right],\ \sigma>0$$

or

$$-\int_\Omega u_{tt}\left(\int_0^\infty \mu(s)\eta^t(s)ds\right)dx \le \left[a\rho+(1+\sigma)\left(\tfrac{a}{4\rho}+1\right)k_0^2\right]\int_\Omega |\nabla u|^2\,dx$$
$$+\left(1+\tfrac{1}{\sigma}\right)\left(\tfrac{a}{4\rho}+1\right)k_0\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx,\ \rho,\sigma>0. \tag{10}$$

Using the above inequalities (9) and (10) in (8), we get

$$\frac{d\mathcal{F}(t)}{dt} \le -(k_0-\delta\mu(0))\int_\Omega |u_t|^2\,dx + \frac{1}{4\delta\lambda_0}\int_\Omega\left(\int_0^\infty -\mu'(s)\left|\nabla\eta^t(s)\right|^2 ds\right)dx$$
$$+\left[a\rho+(1+\sigma)\left(\tfrac{a}{4\rho}+1\right)k_0^2\right]\int_\Omega |\nabla u|^2\,dx \tag{11}$$
$$+\left(1+\tfrac{1}{\sigma}\right)\left(\tfrac{a}{4\rho}+1\right)k_0\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx.$$

Next, we examine the derivative of $\mathcal{H}(t)$. By the equation (2), we find

$$\begin{aligned} \tfrac{d\mathcal{H}(t)}{dt} &= \tfrac{d}{dt}\int_\Omega u_t u dx = \int_\Omega |u_t|^2 dx + \int_\Omega u_{tt} u dx \\ &= \int_\Omega |u_t|^2 dx - a\int_\Omega |\nabla u|^2 dx - \int_\Omega \nabla u \left(\int_0^\infty \mu(s)\nabla\eta^t(s)ds\right) dx. \end{aligned} \tag{12}$$

The last term in the right hand side of (12) has already been estimated while examining the first term in the right hand side of (8) (see proof of (10)). Therefore

$$\begin{aligned} \tfrac{d\mathcal{H}(t)}{dt} &\le \int_\Omega |u_t|^2 dx - a\int_\Omega |\nabla u|^2 dx + \rho\int_\Omega |\nabla u|^2 dx \\ &+\tfrac{k_\Omega}{4\rho}\left[(1+\sigma)k_0\int_\Omega |\nabla u|^2 dx + \left(1+\tfrac{1}{\sigma}\right)\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx\right] \end{aligned}$$

or

$$\begin{aligned} \tfrac{d\mathcal{H}(t)}{dt} &\le \int_\Omega |u_t|^2 dx - \left[a-\rho-(1+\sigma)\tfrac{k_\Omega^2}{4\rho}\right]\int_\Omega |\nabla u|^2 dx \\ &+\tfrac{k_\Omega}{4\rho}\left(1+\tfrac{1}{\sigma}\right)\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx. \end{aligned} \tag{13}$$

Finally, we differentiate $\mathcal{K}(t)$ with respect to t,

$$\begin{aligned} \tfrac{d\mathcal{K}(t)}{dt} &= P_\alpha(0)\int_\Omega |\nabla u|^2 dx - \int_\Omega\int_{-\infty}^t \mu(t-s)\left|\nabla u(s)\right|^2 dsdx \\ &-\alpha\int_\Omega\int_{-\infty}^t P_\alpha(t-s)\left|\nabla u(s)\right|^2 dsdx. \end{aligned}$$

It is apparent that,

$$\frac{d\mathcal{K}(t)}{dt} = \left(\int_0^\infty \mu(r)e^{\alpha r}dr\right)\int_\Omega |\nabla u|^2 dx - \int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 dsdx - \alpha\mathcal{K}(t). \tag{14}$$

We are now ready to define the functional $\mathcal{L}(t)$,

$$\mathcal{L}(t) = N\mathcal{E}(t) + \mathcal{F}(t) + \nu\mathcal{H}(t) + \beta\mathcal{K}(t),\ N,\nu,\beta > 0.$$

Clearly, for small ν and large N, there exist positive constants $D_1 < 1$ and $D_2 > 1$ such that

$$D_1\left(\mathcal{E}(t) + \beta\mathcal{K}(t)\right) \le \mathcal{L}(t) \le D_2\left(\mathcal{E}(t) + \beta\mathcal{K}(t)\right). \tag{15}$$

Taking into account the relations (7), (11), (13) and (14) we infer that

$$\frac{d\mathcal{L}(t)}{dt} \leq \frac{N}{2}\int_\Omega\left(\int_0^\infty \mu'(s)\left|\nabla\eta^t(s)\right|^2 ds\right)dx - (k_0 - \delta\mu(0))\int_\Omega |u_t|^2\,dx$$
$$+\frac{1}{4\delta\lambda_0}\int_\Omega\left(\int_0^\infty -\mu'(s)\left|\nabla\eta^t(s)\right|^2 ds\right)dx + \left[a\rho + (1+\sigma)\left(\frac{a}{4\rho}+1\right)k_0^2\right]\int_\Omega |\nabla u|^2\,dx$$
$$+\left(1+\frac{1}{\sigma}\right)\left(\frac{a}{4\rho}+1\right)k_0\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 ds dx + \nu\int_\Omega |u_t|^2\,dx$$
$$-\nu\left[a-\rho-(1+\sigma)\frac{k_0^2}{4\rho}\right]\int_\Omega |\nabla u|^2\,dx + \nu\frac{k_0}{4\rho}\left(1+\frac{1}{\sigma}\right)\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 ds dx$$
$$+\beta\left(\int_0^\infty \mu(r)e^{\alpha r}dr\right)\int_\Omega |\nabla u|^2\,dx - \beta\alpha\mathcal{K}(t) - \beta\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 ds dx$$

or

$$\frac{d\mathcal{L}(t)}{dt} \leq \left(\frac{N}{2}-\frac{1}{4\delta\lambda_0}\right)\int_\Omega\int_0^\infty \mu'(s)\left|\nabla\eta^t(s)\right|^2 ds dx$$
$$-\left[\nu\left(a-\rho-(1+\sigma)\frac{k_0^2}{4\rho}\right) - a\rho - (1+\sigma)\left(\frac{a}{4\rho}+1\right)k_0^2 - \beta\left(\int_0^\infty \mu(r)e^{\alpha r}dr\right)\right]$$
$$\times\int_\Omega |\nabla u|^2\,dx - (k_0 - \nu - \delta\mu(0))\int_\Omega |u_t|^2\,dx - \beta\alpha\mathcal{K}(t)$$
$$-\left[\beta - \left(1+\frac{1}{\sigma}\right)\left(\frac{a}{4\rho}+1\right)k_0 - \nu\frac{k_0}{4\rho}\left(1+\frac{1}{\sigma}\right)\right]\int_\Omega\int_0^\infty \mu(s)\left|\nabla u(t-s)\right|^2 ds dx.$$

Choose $\nu < k_0$, then δ small such that $\delta\mu(0) < k_0 - \nu$. Choose N large enough so that $\frac{N}{2} > \frac{1}{4\delta\lambda_0}$ i.e. $N > \frac{1}{2\delta\lambda_0}$. For small values of the integral $\int_0^\infty \mu(r)e^{\alpha r}dr$ (and therefore for small values of k_0) we can choose N large enough and δ, ν and ρ small enough so that we can find a positive constant C for which the relation

$$\frac{d\mathcal{L}(t)}{dt} \leq -C\left(\mathcal{E}(t) + \mathcal{K}(t)\right)$$

holds for every $t \geq 0$. Making use of (15), we entail that

$$\frac{d\mathcal{L}(t)}{dt} \leq -\varepsilon\mathcal{L}(t),\ t \geq 0.$$

Therefore

$$\mathcal{L}(t) \leq e^{-\varepsilon t}\mathcal{L}(0),\ t \geq 0$$

and (15) again implies that

$$\mathcal{E}(t) \leq Ce^{-\varepsilon t}\left(\mathcal{E}(0)+\mathcal{K}(0)\right).$$

This completes the proof. ■

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