King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 286

January 2003

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Blow up for the Wave Equation with a Nonlinear Dissipation of Cubic Convolution type in $\mathbb{R}^N$

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Abstract

It is shown that, the solution of the wave equation in $\mathbb{R}^N$ with a nonlinear source of polynomial type and a nonlinear dissipation of nonlocal nature, blows up in finite time. Precisely, the dissipation is of cubic convolution type involving a singular kernel.

AMS Subject Classification: 35B40, 35L15, 35L70.

Key words: Blow up, cubic convolution, singular kernel, nonlocal dissipation.

1 Introduction

We consider the following equation

$$u_{tt} + \lambda u + u_t \left(V_\gamma * u_t^2\right) = \Delta u + a\,|u|^{p-1}\,u, \text{ in } \mathbb{R}^N \times (0,\infty) \tag{1}$$

with initial data

$$u(x,0) = u_1(x) \text{ and } u_t(x,0) = u_2(x), \;\; x \in \mathbb{R}^N \tag{2}$$

where

$$\left(V_\gamma * u_t^2\right)(x,t) = \int_{\mathbb{R}^N} V_\gamma(x-y)u_t^2(y,t)dy$$

and

$$V_\gamma(x) = |x|^{-\gamma},\ 0<\gamma<N,\ \lambda\geq 0,\ a>0,\ p>1.$$

We shall make use of the usual L^p, $1\leq p\leq\infty$ spaces and Sobolev spaces H^k, $k=1,2,\ldots$(see [1] for instance)

If $a=0$ and $\{u_1,u_2\}\in H^2\times(H^1\cap L^q)$, $q=\frac{6N}{3N-2\gamma}$, then the problem (1)-(2) admits a global solution (see [5,10,11]) satisfying

(i) $u(t)\in C([0,\infty);E)$ and is such that

$$\|u(t)\|_E^2 + \int_0^t\int_{\mathbb{R}^N} u_t(x,s)^2\int_{\mathbb{R}^N}|x-y|^{-\gamma}u_t(y,s)^2dydxds = \|u(0)\|_E^2$$

for any $t>0$, where

$$E=\left\{w=(w_1,w_2):\ \|w\|_E=\frac{1}{2}\left(\int_{\mathbb{R}^N}\left[\lambda w_1^2+|\nabla w_1|^2+w_2^2\right]dx\right)^{1/2}<\infty\right\}.$$

(ii) $u(t)\in C([0,T];L^2)$ for any $T>0$.

(iii) $u_{tt}(t)$, $\nabla u_t(t)$, $\Delta u(t)$, $u_t(t)\int_{\mathbb{R}^N}|x-y|^{-\gamma}u_t(y,t)^2dy\ \in L^\infty([0,T];L^2)$ for any $T>0$.

In [9], Mochizuki and Motai proved some decay and non-decay results depending on the initial data and the values of γ. With the help of weighted energy norms they obtained logarithmic and polynomial decay results for a dense class of initial data in $H^2\times(H^1\cap L^q)$.

The case of a source of the form $h(t)u\left(V_\gamma * u^2\right)(x,t)$ has been investigated by the present author in [12]. We showed, among other results, that the solution grows up polynomially as $t\to+\infty$ for a weakly decaying potential $h(t)$. A strongly decaying potential forces the energy to remain uniformly bounded. Moreover, we obtain an asymptotic stability result.

The difficulties we encounter here arise mainly from the singularity of the kernel in the convolution term in addition to the unboundedness of the region and the nonlocal nature of the dissipation. To overcome these difficulties we appeal to the Hardy-Littlewood-Sobolev inequality (see [3] or [8]) (see also Lemma 1 below) and a convolution property satisfied by the kernel in question.

Here, for the case $a \neq 0$, $a > 0$ *i.e.* in presence of a source of power type, we shall prove a blow up result in finite time. We will use a, by now well known, argument due to Georgiev and Todorova [2] which has been proved efficient for nonlinear dissipations (see [2,4,6,7,13]). Roughly, it consists in verifying an ordinary differential inequality for an appropriately chosen functional. The functional we propose in our proof has an advantage on those usually used in the litterature in that it allows for a larger class of initial data. Indeed, in contrast with the previous works, the initial energy we consider may take positive values.

It is easy to see, using some Sobolev embeddings and (3) below, that if the initial data $u_1(x)$ and $u_2(x)$ are of compact support, (say *supp* $u_1(x) \cup$ *supp* $u_2(x) \subset \{|x| < R\}$, for some $R > 0$), then the solution $u(x,t)$ is also of compact support (*supp* $u(t,.) \subset \{|x| < R + t\}$, for any $t < T_m$, where T_m is the maximal time of existence).

2 Blow up in finite time

Lemma 1 *(Hardy-Littlewood-Sobolev inequality, see [3] or [8])*
Let $u \in L^p(\mathbb{R}^N)$ $(p > 1)$, $0 < \gamma < N$ and $\frac{\gamma}{N} > 1 - \frac{1}{p}$, then $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^N)$ with $\frac{1}{q} = \frac{\gamma}{N} + \frac{1}{p} - 1$. Also the mapping from $u \in L^p(\mathbb{R}^N)$ into $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^N)$ is continuous.

Theorem 2 *Let $p > 3$ and assume the above hypotheses. For any $T > 0$ we can find initial data $u_1(x)$ and $u_2(x)$ (of compact support) for which the corresponding solution $u(x,t)$ blows up at a finite time $T^* \leq T$.*

Proof. The multiplication of equation (1) by u_t and the integration over $\mathbb{R}^N$ yield

$$\frac{dE(t)}{dt} = -\int_{\mathbb{R}^N} u_t^2 \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2(y,t)\,dy\,dx, \tag{3}$$

with

$$E(t) = \int_{\mathbb{R}^N} \left[\frac{1}{2}\lambda u^2 + \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{a}{p+1}|u|^{p+1}\right] dx.$$

Observe that $\frac{dE(t)}{dt} \leq 0$ and then

$$E(t) \leq E(0), \text{ for all } t \geq 0. \tag{4}$$

Let us introduce the functional

$$H(t) = \int_0^t \int_{\mathbb{R}^N} \left\{ \frac{a}{p+1} |u|^{p+1} - \frac{1}{2}\lambda u^2 - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2 \right\} dxds + (dt+l) \int_{\mathbb{R}^N} u_1^2 dx.$$

The positive constants d and l are to be chosen later on. A differentiation of this functional (with the above observation (4)) implies that

$$H'(t) = -E(t) + d\int_{\mathbb{R}^N} u_1^2 dx \geq d\int_{\mathbb{R}^N} u_1^2 dx - E(0). \tag{5}$$

We readily choose d so that

$$d\int_{\mathbb{R}^N} u_1^2 dx - E(0) = H'(0) > 0. \tag{6}$$

It appears then from (5) and (6) that

$$H'(t) \geq H'(0) > 0 \text{ for all } t \geq 0.$$

Moreover the identity (3) yields

$$H'(0) - H'(t) = -\int_0^t \int_{\mathbb{R}^N} u_t^2 \left(V_\gamma * u_t^2\right) dxds \leq 0. \tag{7}$$

Now we choose a second auxilliary functional

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{2}\left(\int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} u_1^2 dx\right)$$

with $\varepsilon > 0$ and $0 < \sigma = \frac{p-3}{6(p+1)} < 1$. Our goal is to show that $L(t)$ satisfies a differential inequality of the form

$$L'(t) \geq CL^q(t), \; q > 1.$$

This will yield blow up in finite time.

It is clear that

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_{\mathbb{R}^N} u_1 u_2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dx ds + \varepsilon \int_0^t \int_{\mathbb{R}^N} u u_{tt} dx ds. \tag{8}$$

The last term in (8) may be evaluated by multiplying equation (1) by u and integrating over $\mathbb{R}^N \times (0,t)$. Indeed,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^N} u u_{tt} dx ds = -\lambda \int_0^t \int_{\mathbb{R}^N} u^2 dx ds - \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 dx ds + a \int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dx ds \\ - \int_0^t \int_{\mathbb{R}^N} u u_t \left(V_\gamma * u_t^2\right) dx ds. \end{aligned} \tag{9}$$

We have, by Parseval equality and a convolution property enjoyed by the kernel $V_\gamma(x)$ (see [3, chapter 7.1 and 3.4])

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^N} u u_t \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2 dy dx ds \\ \leq & \int_0^t \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * (u_t u)\right)^2 dx\right]^{\frac{1}{2}} ds. \end{aligned}$$

In fact,

$$\begin{aligned} \int_{\mathbb{R}^N} u u_t \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2 dy dx &= \int_{\mathbb{R}^N} \widehat{u u_t}\, \overline{\widehat{V_\gamma * u_t^2}} dx \\ &= \int_{\mathbb{R}^N} \widehat{u u_t} \left|\widehat{V_{\frac{N+\gamma}{2}}}\right|^2 \overline{\widehat{u_t^2}} dx \\ \leq \left[\int_{\mathbb{R}^N} \left(\widehat{V_{\frac{N+\gamma}{2}} * u_t^2}\right)^2 dx\right]^{\frac{1}{2}} & \left[\int_{\mathbb{R}^N} \left(\widehat{V_{\frac{N+\gamma}{2}} * (u_t u)}\right)^2 dx\right]^{\frac{1}{2}}. \end{aligned} \tag{10}$$

The hat $\widehat{\ }$ stands for the Fourier transform. Also, by Cauchy-Schwarz inequality

$$\begin{aligned} V_{\frac{N+\gamma}{2}} * (u_t u) &= \int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y) u_t u(y) dy \\ \leq \left(\int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y) u_t^2(y) dy\right)^{\frac{1}{2}} & \left(\int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y) u^2(y) dy\right)^{\frac{1}{2}} \end{aligned}$$

That is

$$V_{\frac{N+\gamma}{2}} * (u_t u) \le \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^{\frac{1}{2}} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^N} u u_t \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2 dy dx ds \\ &\le \int_0^t \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)\left(V_{\frac{N+\gamma}{2}} * u^2\right) dx\right]^{\frac{1}{2}} ds \\ &\le \int_0^t \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{\frac{3}{4}} \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx\right]^{\frac{1}{4}} ds.\end{aligned}$$

By the Young inequality, we get

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^N} u u_t \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2 dy dx ds \\ &\le \delta \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx ds + \frac{1}{4\delta^3} \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx ds, \ \delta > 0.\end{aligned} \tag{11}$$

Taking into account (9)-(11) in (8), we find

$$\begin{aligned}L'(t) &\ge (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_{\mathbb{R}^N} u_1 u_2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dx ds \\ &-\lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dx ds - \varepsilon \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 dx ds + a\varepsilon \int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dx ds \\ &-\varepsilon\delta \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx ds - \frac{\varepsilon}{4\delta^3} \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx ds.\end{aligned} \tag{12}$$

By a similar argument to that in (10) we obtain

$$\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx = \int_{\mathbb{R}^N} u_t^2 \left(V_\gamma * u_t^2\right) dx.$$

It follows then from (7) that

$$\int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx ds = H'(t) - H'(0). \tag{13}$$

The last term in the right hand side of (12) may be handeled in the following fashion, by Hölder's inequality we see that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dxds = \int_0^t \int_{\mathbb{R}^N} u^2 \left(V_\gamma * u^2\right) dxds \\
&\leq \int_0^t \left(\int_{\mathbb{R}^N} |u|^{p+1}\, dx\right)^{\frac{2}{p+1}} \left(\int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{\frac{p+1}{p-1}} dx\right)^{\frac{p-1}{p+1}} ds \qquad (14)\\
&\leq \left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\, dxds\right)^{\frac{2}{p+1}} \left(\int_0^t \int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{\frac{p+1}{p-1}} dxds\right)^{\frac{p-1}{p+1}}.
\end{aligned}$$

Using the Hardy-Littlewood-Sobolev inequality (Lemma 1), we get

$$\int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{\frac{p+1}{p-1}} dx \leq A \left(\int_{\mathbb{R}^N} u^{2r} dx\right)^{\frac{1}{r}\frac{p+1}{p-1}}$$

with $A > 0$ and $r = \frac{N(p+1)}{2pN - \gamma(p+1)}$.

Observe that as $p > 3$ we have $2r \leq p+1$ (in fact $2r < p+1$). Indeed, $p > 3$ implies $\gamma < N \leq \frac{2N(p-1)}{p+1}$ then $2r < p+1$. Hölder's inequality implies

$$\int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{\frac{p+1}{p-1}} dx \leq C(R+T)^{\mu_1} \left(\int_{\mathbb{R}^N} |u|^{p+1}\, dx\right)^{\frac{2}{p-1}}$$

for some positive constant C and $\mu_1 = \frac{N}{r(p-1)}(p+1-2r)$. Therefore, as $p > 3$

$$\begin{aligned}
\left(\int_0^t \int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{\frac{p+1}{p-1}} dxds\right)^{\frac{p-1}{p+1}} &\leq \hat{C}(R+T)^{\mu_2} \left(\int_0^t \left(\int_{\mathbb{R}^N} |u|^{p+1}\, dx\right)^{\frac{2}{p-1}} ds\right)^{\frac{p-1}{p+1}} \\
&\leq \hat{C}(R+T)^{\mu_2} \left(\int_0^t 1^{\frac{p-1}{p-3}} ds\right)^{\frac{p-3}{p+1}} \left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\, dxds\right)^{\frac{2}{p+1}}
\end{aligned}$$

where $\hat{C} = C^{\frac{p-1}{p+1}}$ and $\mu_2 = \frac{N}{r}\left(1 - \frac{2r}{p+1}\right)$. From now on C will denote a generic positive constant which may change from line to line. Hence, from

(14)

$$\int_0^t\int_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}}*u^2\right)^2dxds\leq C(R+T)^{\mu_2}T^{\frac{p-3}{p+1}}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{\frac{4}{p+1}}. \quad (15)$$

By (13) and (15) we obtain from (12) that

$$\begin{aligned}L'(t)\geq{}&(1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\delta H'(t)+\varepsilon\delta H'(0)+\varepsilon\int_{\mathbb{R}^N}u_1u_2dx\\&+\varepsilon\int_0^t\int_{\mathbb{R}^N}u_t^2dxds-\lambda\varepsilon\int_0^t\int_{\mathbb{R}^N}u^2dxds-\varepsilon\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dxds\\&+a\varepsilon\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds-\tfrac{\varepsilon}{4\delta^3}C(R+T)^{\mu_2}T^{\frac{p-3}{p+1}}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{\frac{4}{p+1}}.\end{aligned} \quad (16)$$

Selecting $\delta=MH^{-\sigma}(t)$, the inequality (16) becomes

$$\begin{aligned}L'(t)\geq{}&((1-\sigma)-\varepsilon M)H^{-\sigma}(t)H'(t)+\varepsilon MH^{-\sigma}(t)H'(0)+\varepsilon\int_{\mathbb{R}^N}u_1u_2dx\\&+\varepsilon\int_0^t\int_{\mathbb{R}^N}u_t^2dxds-\lambda\varepsilon\int_0^t\int_{\mathbb{R}^N}u^2dxds-\varepsilon\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dxds+a\varepsilon\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\\&-\tfrac{\varepsilon}{4M^3}C(R+T)^{\mu_2}T^{\frac{p-3}{p+1}}H^{3\sigma}(t)\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{\frac{4}{p+1}}.\end{aligned} \quad (17)$$

Now we want to estimate the last term in the right hand side of (17), from the definition of $H(t)$

$$H^{3\sigma}(t)\leq2^{3\sigma-1}\left[\left(\frac{a}{p+1}\right)^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{3\sigma}+(dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N}u_1^2dx\right)^{3\sigma}\right].$$

Therefore,

$$\begin{aligned}H^{3\sigma}(t)\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{\frac{4}{p+1}}\leq{}&2^{3\sigma-1}\left(\tfrac{a}{p+1}\right)^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{3\sigma+\frac{4}{p+1}}\\&+2^{3\sigma-1}(dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N}u_1^2dx\right)^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{\frac{4}{p+1}}.\end{aligned}$$

As $\sigma = \frac{p-3}{6(p+1)}$ and $p > 3$ we have $3\sigma + \frac{4}{p+1} \leq 1$. In this case

$$H^{3\sigma}(t)\left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds\right)^{\frac{4}{p+1}}$$
$$\leq 2^{3\sigma-1}\left[\left(\tfrac{a}{p+1}\right)^{3\sigma} + (dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N} u_1^2 dx\right)^{3\sigma}\right]\left(1 + \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds\right).$$

Inserting this estimate in (17) and choosing $\varepsilon \leq \frac{1-\varrho}{M}$, we obtain

$$L'(t) \geq \varepsilon \int_{\mathbb{R}^N} u_1 u_2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds - \lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dxds - \varepsilon \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2\,dxds$$
$$+a\varepsilon \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds - \tfrac{\varepsilon}{M} B(T) \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds - \tfrac{\varepsilon}{M} B(T)$$

where

$$B(T) = 2^{3(\sigma-1)} C(R+T)^{\mu_2} T^{\frac{p-3}{p+1}}\left[\left(\frac{a}{p+1}\right)^{3\sigma} + (dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N} u_1^2 dx\right)^{3\sigma}\right].$$

For a positive constant K to be determined we may also write

$$L'(t) \geq KH(t) - \tfrac{aK}{p+1} \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds + \lambda\tfrac{K}{2} \int_0^t \int_{\mathbb{R}^N} u^2 dxds + \tfrac{K}{2} \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds$$
$$+\tfrac{K}{2} \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2\,dxds - K(dT+l) \int_{\mathbb{R}^N} u_1^2 dx + \varepsilon \int_{\mathbb{R}^N} u_1 u_2 dx - \tfrac{\varepsilon}{M} B(T)$$
$$+\varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds - \lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dxds - \varepsilon \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2\,dxds$$
$$+a\varepsilon \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds - \tfrac{\varepsilon}{M} B(T) \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds.$$

That is,

$$L'(t) \geq KH(t) + \left[\varepsilon\left(a - \tfrac{B(T)}{M}\right) - \tfrac{aK}{p+1}\right] \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\,dxds$$
$$+\lambda\left(\tfrac{K}{2} - \varepsilon\right) \int_0^t \int_{\mathbb{R}^N} u^2 dxds + \left(\tfrac{K}{2} - \varepsilon\right) \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2\,dxds + \left(\tfrac{K}{2} + \varepsilon\right) \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds$$
$$+\varepsilon \int_{\mathbb{R}^N} u_1 u_2 dx - K(dT+l) \int_{\mathbb{R}^N} u_1^2 dx - \tfrac{\varepsilon}{M} B(T).$$

Putting $K = 2\varepsilon$, we infer that

$$L'(t) \geq 2\varepsilon H(t) + \varepsilon \left[a\tfrac{p-1}{p+1} - \tfrac{B(T)}{M} \right] \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\, dxds + 2\varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds$$
$$+\varepsilon \left\{ \int_{\mathbb{R}^N} u_1 u_2 dx - 2(dT + l) \int_{\mathbb{R}^N} u_1^2 dx - \tfrac{B(T)}{M} \right\}.$$

Choose u_1 and u_2 such that

$$\int_{\mathbb{R}^N} u_1 u_2 dx - 2(dT + l) \int_{\mathbb{R}^N} u_1^2 dx > 0 \tag{18}$$

(this is possible, see Proposition 3 below) and then pick M large enough so that

$$\int_{\mathbb{R}^N} u_1 u_2 dx - 2(dT + l) \int_{\mathbb{R}^N} u_1^2 dx \geq \frac{B(T)}{M} > 0.$$

The constant M must also be sufficiently large so that

$$a\frac{p-1}{p+1} > \frac{B(T)}{M}.$$

Once this is satisfied we select b such that

$$a\frac{p-1}{p+1} - \frac{B(T)}{M} \geq b > 0.$$

It follows that

$$L'(t) \geq 2\varepsilon H(t) + \varepsilon b \int_0^t \int_{\mathbb{R}^N} |u|^{p+1}\, dxds + 2\varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds. \tag{19}$$

Next, it is clear that

$$L^{\frac{1}{1-\sigma}}(t) \leq 2^{\frac{1}{1-\sigma}} \left\{ H(t) + \varepsilon^{\frac{1}{1-\sigma}} \left(\int_0^t \int_{\mathbb{R}^N} u_t u dxds \right)^{\frac{1}{1-\sigma}} \right\}. \tag{20}$$

By the Cauchy-Schwarz inequality and Hölders inequality we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^N} u_t u\,dxds &\le \int_0^t\left(\int_{\mathbb{R}^N} u^2dx\right)^{\frac12}\left(\int_{\mathbb{R}^N} u_t^2dx\right)^{\frac12}ds\\
&\le C(R+T)^{\mu_3}\int_0^t\left(\int_{\mathbb{R}^N} u^{p+1}dx\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N} u_t^2dx\right)^{\frac12}ds\\
&\le C(R+T)^{\mu_3}\left\{\int_0^t\left(\int_{\mathbb{R}^N} u^{p+1}dx\right)^{\frac{2}{p+1}}ds\right\}^{\frac12}\left\{\int_0^t\int_{\mathbb{R}^N} u_t^2dxds\right\}^{\frac12}\end{aligned}$$

where $\mu_3 = \frac{N}{2}\left(\frac{p-1}{p+1}\right)$. Therefore,

$$\begin{aligned}&\left(\int_0^t\int_{\mathbb{R}^N} u_t u\,dxds\right)^{\frac{1}{1-\sigma}}\\
&\le C(R+T)^{\mu_4}T^{\alpha}\left\{\int_0^t\int_{\mathbb{R}^N} u^{p+1}dxds\right\}^{\frac{1}{(p+1)(1-\sigma)}}\left\{\int_0^t\int_{\mathbb{R}^N} u_t^2dxds\right\}^{\frac{1}{2(1-\sigma)}} \qquad (21)\\
&\le C(R+T)^{\mu_4}T^{\alpha}\left\{\int_0^t\int_{\mathbb{R}^N} u_t^2dxds+\left(\int_0^t\int_{\mathbb{R}^N} u^{p+1}dxds\right)^{\frac{2}{(p+1)(1-2\sigma)}}\right\},\end{aligned}$$

where $\mu_4 = \frac{\mu_3}{1-\sigma} = \frac{N}{2(1-\sigma)}\left(\frac{p-1}{p+1}\right) = \frac{N(p-1)}{p+3}$ and $\alpha = \frac{p-1}{p+3}$. We have used Young s inequality with $2(1-\sigma)$ and $\frac{2(1-\sigma)}{(1-2\sigma)}$ in the last inequality.

Finally, it is easy to see, from (19)-(21), that we can find a sufficiently large constant $\tilde{C} > 0$ such that

$$L^{\frac{1}{1-\sigma}}(t) \le \tilde{C}L'(t).$$

An integration over $(0, t)$ yields

$$L^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{\sigma t}{\tilde{C}(1-\sigma)}}.$$

So $L(t)$ blows up at a finite time

$$T^* \le \frac{(1-\sigma)\tilde{C}L(0)^{-\frac{\sigma}{1-\sigma}}}{\sigma}.$$

As $L(0)=H^{1-\sigma}(0)=\left(l\int\limits_{\mathbb{R}^N}u_1^2dx\right)^{1-\sigma}$, choosing l such that

$$l\geq\left(\frac{(1-\sigma)\tilde{C}}{\sigma T}\right)^{\frac{1}{\sigma}}\left(\int\limits_{\mathbb{R}^N}u_1^2dx\right)^{-1}$$

we see that $T^*\leq T$. ∎

Proposition 3 *The set of initial data satisfying (6) and (18) is not empty.*

Proof. Precisely, we would like to prove that we can always select initial data u_1 and u_2 satisfying both conditions

$$d\int\limits_{\mathbb{R}^N}u_1^2dx-E(0)>0$$

and

$$\int\limits_{\mathbb{R}^N}u_1u_2dx-2(dT+l)\int\limits_{\mathbb{R}^N}u_1^2dx>0.$$

Let us , first, prove that we can find u_1 such that

$$4(dT+l+\frac{\lambda}{8})\int\limits_{\mathbb{R}^N}u_1^2dx+\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla u_1|^2dx<\frac{a}{p+1}\int\limits_{\mathbb{R}^N}|u_1|^{p+1}dx+d\int\limits_{\mathbb{R}^N}u_1^2dx. \quad (22)$$

Suppose for contradiction that this is not true, that is we always have

$$\frac{a}{p+1}\int\limits_{\mathbb{R}^N}|u_1|^{p+1}dx+d\int\limits_{\mathbb{R}^N}u_1^2dx\leq 4(dT+l+\frac{\lambda}{8})\int\limits_{\mathbb{R}^N}u_1^2dx+\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla u_1|^2dx.$$

Let $u_1=\delta v_1$, then

$$\begin{aligned}\tfrac{a\delta^{p-1}}{p+1}\int\limits_{\mathbb{R}^N}|v_1|^{p+1}dx &\leq \tfrac{a\delta^{p-1}}{p+1}\int\limits_{\mathbb{R}^N}|v_1|^{p+1}dx+d\int\limits_{\mathbb{R}^N}v_1^2dx\\ &\leq 4(dT+l+\tfrac{\lambda}{8})\int\limits_{\mathbb{R}^N}v_1^2dx+\tfrac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla v_1|^2dx.\end{aligned}$$

This is impossible and therefore the inequality (22) holds. Next, we select $u_2 > 2\sqrt{2}(dT+l)u_1$ (in case $dT+l<1$ we choose $u_2 > 2\sqrt{2(dT+l)}u_1$) and satisfying

$$4(dT+l)\int\limits_{\mathbb{R}^N} u_1^2dx < \frac{1}{2}\int\limits_{\mathbb{R}^N} u_2^2dx$$
$$< \frac{a}{p+1}\int\limits_{\mathbb{R}^N} |u_1|^{p+1}\,dx + \left(d-\frac{\lambda}{2}\right)\int\limits_{\mathbb{R}^N} u_1^2dx - \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla v_1|^2\,dx.$$

In this way, it appears that

$$\int\limits_{\mathbb{R}^N} u_1u_2dx > 2\sqrt{2}(dT+l)\int\limits_{\mathbb{R}^N} u_1^2dx > 2(dT+l)\int\limits_{\mathbb{R}^N} u_1^2dx.$$

■

Acknowledgment: The author would like to thank King Fahd University of Petroleum and Minerals, Mathematical Sciences Department for its support.

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