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Abstract

We consider a wave equation with an internal damping represented by a fractional derivative of lower order than one. An exponential growth result is proved in presence of a source of polynomial type. This result improves an earlier one where the initial data are supposed to be very large in some norm. A new argument based on a new functional is proposed.

Key words and phrases: Exponential growth, fractional derivative, internal damping

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1 Introduction

We are interested in the following fractional differential problem

$$\begin{cases} u_{tt} + \partial_t^{1+\alpha} u = \Delta u + |u|^{p-1} u, \ x \in \Omega, \ t > 0, \\ u(x,t) = 0, \ x \in \Gamma, \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1}$$

where $p > 1$, $-1 < \alpha < 1$, $u_0(x)$ and $u_1(x)$ are given functions. Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ. Here $\partial_t^{1+\alpha}$ is the Caputo's fractional derivative of order $1+\alpha$ (see [17], Chapter 2.4.1) defined by

$$\partial_t^{1+\alpha} w(t) := I^{-\alpha} \frac{d}{dt} w(t), \ -1 < \alpha < 0 \tag{2}$$

and

$$\partial_t^{1+\alpha} w(t) := I^{1-\alpha}\frac{d^2}{dt^2}w(t),\ 0<\alpha<1, \tag{3}$$

where I^β, $\beta>0$, is the fractional integral

$$I^\beta w(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}w(s)ds.$$

See also [16,17,19,4] for more on fractional derivatives and applications. In particular, in control theory, it is known that noise is amplified by the differentiation process. To attenuate this noise one is lead to use derivatives of lower order.

This problem was first studied for $\alpha = 1/2$ by Lokshin in [11] and Lokshin and Rok in [12]. Then, it has been considered (for $0<\alpha<2$) by Matignon et al. [13]. The authors have managed to replace the hereditary equation by a non-hereditary system for which the standard methods, such as the Galerkin method and LaSalle's invariance principle, apply. For the well posedness we refer the reader to this reference (see also [8] for more on existence results).

Let us mention here that the case $\alpha = 0$ corresponds to an internal damping. This damping competes with the polynomial source. As a result, it was proved (see [14,15,18,5]) that solutions exist globally in time when the initial data are in a *stable* set and blow up in a finite time when the initial data are in an *unstable* set.

The wave equation without damping corresponds to the case $\alpha = -1$. It has been extensively studied by many authors (see, for instance [1,7,10,2,3,6,20]). It has been proved, in particular, that solutions blow up in finite time for sufficiently large initial data (in some sens) and also for small initial data provided that the exponent p lies in some critical range.

In this paper we improve an earlier result by the present author with M. Kirane in [9]. There, for sufficiently large initial data (in some sense), it has been shown that the solution is unbounded provided that the initial data are very large in some norm. In fact, an exponential growth result was proved. Here we relax considerably this condition on the initial data. So the space of initial data is enlarged. To this end we present a different argument based on a new functional while the previous proof makes use of the Hardy-Littlewood-Sobolev inequality and some "convolution" inequalities.

2 Exponential growth

Let us define the classical energy by

$$E(t) := \int_\Omega \left\{ \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} \right\} dx$$

and the modified energy by

$$E_\varepsilon(t) := \int_\Omega \left\{ \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \varepsilon u u_t - \frac{1}{p+1}|u|^{p+1} \right\} dx, \qquad (4)$$

for some $0 < \varepsilon < 1$.

Theorem 1 *Let $u(x,t)$ be a regular solution of problem (1) with $-1 < \alpha < 0$. If the initial data u_0 and u_1 are such that $E_\varepsilon(0) < 0$, then the solution $u(x,t)$ grows up exponentially in the L_{p+1}-norm.*

Proof. Let us multiply $(1)_1$ by $(u_t - \varepsilon u)$ and integrate over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left\{ \tfrac{1}{2}u_t^2 + \tfrac{1}{2}|\nabla u|^2 - \varepsilon u u_t - \tfrac{1}{p+1}|u|^{p+1} \right\} dx \\ &+\tfrac{1}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx = \varepsilon \int_\Omega |\nabla u|^2 dx \\ &+\tfrac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx - \varepsilon\int_\Omega u_t^2 dx - \varepsilon \int_\Omega |u|^{p+1} dx. \end{aligned}$$

Recalling the definition (4) of $E_\varepsilon(t)$, we see that

$$\begin{aligned} &\tfrac{dE_\varepsilon(t)}{dt} + \tfrac{1}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx \\ &= \varepsilon\int_\Omega |\nabla u|^2 dx + \tfrac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx \\ &\quad -\varepsilon \int_\Omega u_t^2 dx - \varepsilon\int_\Omega |u|^{p+1} dx. \end{aligned} \qquad (5)$$

Next, we define the auxiliary functional

$$F_{\alpha,\beta,\sigma}(t) := \int_0^t \int_\Omega G_{\alpha,\beta}(t-s) e^{-\sigma\varepsilon s} |u_t|^2 \,dx\,ds \qquad (6)$$

with

$$G_{\alpha,\beta}(t) := e^{\beta t} \int_{t}^{+\infty} e^{-\beta s} s^{-(2\alpha+3)} ds. \tag{7}$$

where β and $\sigma\varepsilon$ are positive constants which will be precised later on. Since there is no risk of confusion in the notation, we will drop the subscripts of F and G for convenience.

A differentiation of $F(t)$ in (6) with respect to t gives

$$\begin{aligned} \frac{dF(t)}{dt} &= \int_\Omega G_\beta(0) e^{-\sigma\varepsilon t} |u_t|^2 \, dx \\ &+ \int_0^t \int_\Omega \left\{ -(t-s)^{-(2\alpha+3)} + \beta e^{\beta(t-s)} \int_{t-s}^{+\infty} e^{-\beta z} z^{-(2\alpha+3)} dz \right\} e^{-\sigma\varepsilon s} |u_t|^2 \, dx ds. \end{aligned} \tag{8}$$

Observe that

$$G(0) = \int_0^{+\infty} e^{-\beta s} s^{-(2\alpha+3)} ds = \beta^{2(\alpha+1)} \Gamma(2\alpha+4).$$

Then the relation (8) becomes

$$\begin{aligned} \frac{dF(t)}{dt} = {} & \beta^{2(\alpha+1)} \Gamma(2\alpha+4) e^{-\sigma\varepsilon t} \int_\Omega |u_t|^2 \, dx - \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s} |u_t|^2 \, dx ds \\ & + \beta F(t). \end{aligned} \tag{9}$$

Now, we consider the functional

$$H(t) = e^{-\sigma\varepsilon t} E_\varepsilon(t) + \mu F(t), \; t \geq 0 \tag{10}$$

for some $\mu > 0$ to be determined. Its derivative with respect to t is, according to (5) and (9), equal to

$$\begin{aligned} \frac{dH(t)}{dt} = {} & -\sigma\varepsilon e^{-\sigma\varepsilon t} E_\varepsilon(t) \\ & + e^{-\sigma\varepsilon t} \left\{ -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s) ds dx + \varepsilon \int_\Omega |\nabla u|^2 \, dx \right. \\ & \left. -\varepsilon \int_\Omega |u_t|^2 \, dx + \frac{\varepsilon}{\Gamma(-\alpha)} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_t(s) ds dx - \varepsilon \int_\Omega |u|^{p+1} \, dx \right\} \\ & + \mu \left\{ \beta^{2(\alpha+1)} \sigma\varepsilon (2\alpha+4) e^{-\sigma\varepsilon t} \int_\Omega |u_t|^2 \, dx \right. \\ & \left. - \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s} |u_t|^2 \, dx ds + \beta F(t) \right\}. \end{aligned}$$

Using the definition (4) of $E_\varepsilon(t)$, we may write

$$\begin{aligned}\frac{dH(t)}{dt} &= -\left(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right)e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2\,dx\\
&-\left(\frac{\sigma\varepsilon}{2}-\varepsilon\right)e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2\,dx+\sigma\varepsilon\varepsilon e^{-\sigma\varepsilon t}\int_\Omega u_t u\,dx-\left(\varepsilon-\frac{\sigma\varepsilon}{p+1}\right)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1}\,dx\\
&-\mu\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dxds+\frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx\\
&-\frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx+\mu\beta F(t).\end{aligned} \tag{11}$$

By the generalized Young inequality and the Poincaré inequality, we clearly have

$$\int_\Omega u_t u\,dx \le \frac{1}{4\varepsilon}\int_\Omega |u_t|^2\,dx+\varepsilon C_p\int_\Omega |\nabla u|^2\,dx, \tag{12}$$

where C_p is the Poincaré constant. The seventh term in the right hand side of (11) may be handled in the following manner. First note that

$$\begin{aligned}&e^{-\sigma\varepsilon t}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx\\
&=e^{-\frac{\sigma\varepsilon}{2}t}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}e^{-\frac{\sigma\varepsilon}{2}(t-s)}e^{-\frac{\sigma\varepsilon}{2}s}u_t(s)dsdx,\end{aligned}$$

then by the generalized Young inequality, we find

$$\begin{aligned}e^{-\sigma\varepsilon t}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx &\le \frac{\varepsilon\Gamma(-\alpha)}{2}e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2\,dx\\
&+\frac{1}{2\varepsilon\Gamma(-\alpha)}\int_\Omega\left(\int_0^t (t-s)^{-(\alpha+1)}e^{-\frac{\sigma\varepsilon}{2}(t-s)}e^{-\frac{\sigma\varepsilon}{2}s}u_t(s)ds\right)^2dx.\end{aligned}$$

Using the decomposition $\alpha+1=-\frac{1}{2}+\left(\alpha+\frac{3}{2}\right)$, we obtain by the hölder inequality

$$\begin{aligned}e^{-\sigma\varepsilon t}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx &\le \frac{\varepsilon\Gamma(-\alpha)}{2}e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2\,dx\\
&+\frac{1}{2\Gamma(-\alpha)\sigma\varepsilon^3}\int_\Omega\int_0^t (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dsdx.\end{aligned} \tag{13}$$

Similarly, we obtain for the sixth term in the right hand side of (11)

$$e^{-\sigma\varepsilon t}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx \le \delta C_p e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\delta\sigma\varepsilon^2}\int_\Omega\int_0^t (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dsdx,\ \delta>0. \tag{14}$$

Taking into account (12)-(14) in (11), we infer that

$$\begin{aligned}\frac{dH(t)}{dt} &\le -\left[\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)-\frac{1}{4}\sigma\varepsilon-\frac{\varepsilon}{2}\right]e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2\,dx\\ &-\left[\frac{\sigma\varepsilon}{2}-\varepsilon\left(1+\sigma\varepsilon^2C_p+\frac{C_p\delta}{\Gamma(-\alpha)}\right)\right]e^{-\sigma\varepsilon t}\int_\Omega|\nabla u|^2\,dx-\left(\varepsilon-\frac{\sigma\varepsilon}{p+1}\right)e^{-\sigma\varepsilon t}\int_\Omega|u|^{p+1}\,dx\\ &-\left[\mu-\frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\left(\frac{2}{\varepsilon\Gamma(-\alpha)}+\frac{\varepsilon}{\delta}\right)\right]\int_0^t\int_\Omega(t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dxds+\mu\beta F(t).\end{aligned}$$

This inequality may also be written as follows

$$\begin{aligned}\frac{dH(t)}{dt} &\le \sigma\varepsilon H(t)-\left[\sigma\varepsilon-\varepsilon\left(1+\sigma\varepsilon^2C_p+\frac{C_p\delta}{\Gamma(-\alpha)}\right)\right]e^{-\sigma\varepsilon t}\int_\Omega|\nabla u|^2\,dx\\ &-\left[\sigma\varepsilon+\varepsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)-\frac{1}{4}\sigma\varepsilon-\frac{\varepsilon}{2}\right]e^{-\sigma\varepsilon t}\int_\Omega|u_t|^2\,dx\\ &-\left[\mu-\frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\left(\frac{2}{\varepsilon\Gamma(-\alpha)}+\frac{\varepsilon}{\delta}\right)\right]\int_0^t\int_\Omega(t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dxds\\ &-\left(\varepsilon-\frac{2\sigma\varepsilon}{p+1}\right)e^{-\sigma\varepsilon t}\int_\Omega|u|^{p+1}\,dx+\sigma\varepsilon^2e^{-\sigma\varepsilon t}\int_\Omega u_tudx+\mu(\beta-\sigma\varepsilon)F(t).\end{aligned} \tag{15}$$

To get (15), we have added and sustracted $\sigma\varepsilon H(t)$ in the right hand side of the previous inequality.

Finally, we apply (12) to the term $\int_\Omega u_tudx$ in (15), to obtain

$$\begin{aligned}\frac{dH(t)}{dt} &\le \sigma\varepsilon H(t)-\varepsilon\left[\sigma-\left(1+2\sigma\varepsilon^2C_p+\frac{C_p\delta}{\Gamma(-\alpha)}\right)\right]e^{-\sigma\varepsilon t}\int_\Omega|\nabla u|^2\,dx\\ &-\frac{1}{2}\left[\sigma\varepsilon+\varepsilon-2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right]e^{-\sigma\varepsilon t}\int_\Omega|u_t|^2\,dx\\ &-\left[\mu-\frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\left(\frac{2}{\varepsilon\Gamma(-\alpha)}+\frac{\varepsilon}{\delta}\right)\right]\int_0^t\int_\Omega(t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2\,dxds\\ &-\varepsilon\left(1-\frac{2\sigma}{p+1}\right)e^{-\sigma\varepsilon t}\int_\Omega|u|^{p+1}\,dx+\mu(\beta-\sigma\varepsilon)F(t).\end{aligned} \tag{16}$$

Choosing $\delta = (p-1)\Gamma(-\alpha)/4C_p$, the inequality (16) reduces to

$$\begin{aligned}\frac{dH(t)}{dt} &\leq \sigma\varepsilon H(t) - \varepsilon\left[\sigma - \left(2\sigma\varepsilon^2 C_p + \tfrac{p+3}{4}\right)\right] e^{-\sigma\varepsilon t} \int_\Omega |\nabla u|^2\, dx \\ &-\tfrac{1}{2}\left[\sigma\varepsilon + \varepsilon - 2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right] e^{-\sigma\varepsilon t} \int_\Omega |u_t|^2\, dx \\ &- \left[\mu - \tfrac{1}{2\sigma^2\varepsilon^2\Gamma^2(-\alpha)}\left(\tfrac{1}{\varepsilon} + \tfrac{2\varepsilon C_p}{(p-1)}\right)\right] \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s} |u_t|^2\, dx ds \\ &-\varepsilon\left(1 - \tfrac{2\sigma}{p+1}\right) e^{-\sigma\varepsilon t} \int_\Omega |u|^{p+1}\, dx + \mu(\beta - \sigma\varepsilon) F(t).\end{aligned} \tag{17}$$

If we choose $\varepsilon < \min\left\{1, \frac{1}{C_p}, \left[\frac{p-1}{2(p+1)C_p}\right]^{\frac{1}{2}}\right\}$, then it is possible to select σ such that

$$\frac{p+3}{4(1-2C_p\varepsilon^2)} < \sigma < \frac{p+1}{2}.$$

This ensures the negativity of the coefficients of $\int_\Omega |\nabla u|^2\, dx$ and $\int_\Omega |u|^{p+1}\, dx$.

Next, assuming μ large enough, namely

$$\mu \geq \frac{1}{2\sigma^2\varepsilon^3\Gamma^2(-\alpha)}\left(1 + \frac{2\varepsilon^2 C_p}{p-1}\right)$$

and

$$\beta \leq \min\left\{\sigma\varepsilon, \left[\frac{\varepsilon}{2\mu\Gamma(2\alpha+4)}\right]^{\frac{1}{2(\alpha+1)}}\right\},$$

the remaining coefficients are also negative. Therefore (17) reduces to

$$\frac{dH(t)}{dt} \leq \sigma\varepsilon H(t),\ t \geq 0. \tag{18}$$

Observe that if

$$H(0) = E_\varepsilon(0) := \int_\Omega \left\{\frac{1}{2}u_1^2 + \frac{1}{2}|\nabla u_0|^2 - \varepsilon u_0 u_1 - \frac{1}{p+1}|u_0|^{p+1}\right\} dx$$

is negative, then defining

$$\Psi(t) = -H(t),\ t \geq 0,$$

we have $\Psi(0) > 0$. By Gronwall inequality it is easy to see from (18) that

$$\Psi(t) \geq \Psi(0)e^{\sigma\varepsilon t},\ t \geq 0. \tag{19}$$

On the other hand from the definition of $\Psi(t)$ and (12) (with $\varepsilon = 1/2$), we obtain

$$\Psi(t) \leq \tfrac{e^{-\sigma\varepsilon t}}{p+1} \int_\Omega |u|^{p+1}\,dx - \tfrac{e^{-\sigma\varepsilon t}}{2} \int_\Omega |u_t|^2\,dx - \tfrac{e^{-\sigma\varepsilon t}}{2} \int_\Omega |\nabla u|^2\,dx$$
$$+ \tfrac{\varepsilon e^{-\sigma\varepsilon t}}{2} \int_\Omega |u_t|^2\,dx + \tfrac{\varepsilon C_p e^{-\sigma\varepsilon t}}{2} \int_\Omega |\nabla u|^2\,dx,$$

or, for $t \geq 0$,

$$\Psi(t) \leq \frac{e^{-\sigma\varepsilon t}}{p+1} \int_\Omega |u|^{p+1}\,dx - \frac{(1-\varepsilon)e^{-\sigma\varepsilon t}}{2} \int_\Omega |u_t|^2\,dx - \frac{(1-\varepsilon C_p)e^{-\sigma\varepsilon t}}{2} \int_\Omega |\nabla u|^2\,dx.$$

From our choice of ε, it is clear,

$$\Psi(t) \leq \frac{e^{-\sigma\varepsilon t}}{p+1} \int_\Omega |u|^{p+1}\,dx. \tag{20}$$

The relations (19) and (20) imply that

$$\int_\Omega |u|^{p+1}\,dx \geq (p+1)\Psi(0)e^{(2\sigma\varepsilon)t},\ t \geq 0.$$

This completes the proof. ■

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