King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 284

January 2003

Gradient catastrophe in heat Propogation with second sound

Salim A. Messaoudi and Aicha S. Al-Shehri

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Gradient catastrophe in heat propagation with second sound.

Salim A. Messaoudi [1] and Aicha S. Al Shehri[2]
1)Mathematical Sciences Department,
KFUPM, Dhahran 31261,
Saudi Arabia.
E-mail : messaoud@kfupm.edu.sa
2) Math. Department, School of Sciences
Girl's College, Dammam
Saudi Arabia

Abstract

In this work we consider a hyperbolic nonlinear system describing heat propagation with second sound in an inhomogeneous material. We establish a blow up result for classical solution with large-gradient initial data.
Keywords heat, second sound, nonlinear, hyperbolic, blow up.
AMS Subject Classification 35L45 - 35K05 - 35K65.

1. Introduction

In the absence of deformation and external sources, the equation of balance of energy in the one-dimendional heat propagation is

$$\mathcal{E}(\theta)_t + q_x = 0, \tag{1.1}$$

where $\theta > 0$ is the difference temperature, q is the heat flux, and $\mathcal{E}$ is a positive strictly increasing function. In the classical theory, the flux q is given by Fourier' s law

$$q + \kappa(\theta)\theta_x = 0$$

where κ is a strictly positive function characterizing the material in consideration. In the case where $c = \mathcal{E}'$ and κ are independent of θ, we get the familiar linear heat equation

$$\theta_t = k\theta_{xx}, \quad k = \frac{\kappa}{c}.$$

This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation; that is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. This is not always the case. In fact, experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances which are almost entirely thermal propagate in a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this contradictory paradox, many theories have merged. One of which suggests that we should account for memory effects (See [7], [13]. [14]). For this purpose, an internal parameter p has been introduced as

$$q = -\alpha(\theta)p \tag{1.2}$$

If the memory effect is considered as a functional of a history of temperature gradient then

$$p(x,t) = \int_{-\infty}^{t} e^{-b(t-s)}\theta_x(x,s)ds, \qquad b > 0. \tag{1.3}$$

A differentiation of (1.3) with respect to time gives

$$p_t = -bp + \theta_x \tag{1.4}$$

If $\alpha(\theta)$ is constant then (1.2) and (1.4) yield

$$q_t + bq = -a\theta_x.$$

This is a linear equation and does not fully describe the heat propagation in solids (See [7], [13]). In fact this is a special case of Cattaneo' s law [1], which has the form

$$\tau(\theta)q_t + q = -\kappa(\theta)\theta_x.$$

Here τ and κ are strictly positive functions depending on the absolute temperature and characterizing the material on consideration. In this case the system governing the evolution of θ and q become

$$\begin{aligned} c(\theta)\theta_t + q_x &= 0 \\ \tau(\theta)q_t + q + \kappa(\theta)\theta_x &= 0. \end{aligned} \tag{1.5}$$

Global existence and decay of classical solutions, for smooth and small initial data, to the Cauchy problem, as well as to some initial boundary value problems, have been established by Coleman, Hrusa, and Owen [2]. In their work, the authors considered a system, which satisfies the requirements imposed by the second law of thermodynamics discussed in [3], and showed that (θ, q) tends to the equilibrium state, however no rate of decay has been discussed. Messaoudi [11] showed that if the initial data are small enough then the solution decays exponentially to the rest state.

Concerning formation of singularities, Messaoudi [9], [10] showed, under the same restrictions on τ, c and κ, that classical solutions to the Cauchy problem of (1.5) break down in finite time if the initial data are chosen small in the L^∞ norm with large enough derivatives.

In this work, we consider the situation when a in (1.2) is a function of x only. This may be regarded as inhomogeneity in the material in consideration. Therefore the system we study takes the form

$$\begin{aligned} c(\theta(x,t))\theta_t(x,t) + q_x(x,t) = 0 \\ q_t(x,t) + bq(x,t) = -a(x)\theta_x(x,t), \quad x \in I = (0,1), \qquad t \geq 0 \end{aligned} \tag{1.6}$$

This is a hyperbolic system for (θ, q) and it will take care of the paradox of infinite speed propagation known in the classical theory of heat propagation. We associate with (1.6), the initial and the boundary conditions

$$\begin{aligned} \theta(x,0) = \theta_0(x), \ \ q(x,0) = q_0(x), \quad x \in I = [0,1] \\ \theta(0,t) = \theta(1,t) = 0, \quad t \geq 0 \end{aligned} \tag{1.7}$$

and prove a finite time blow up result similar to one in [9]. We should note here that hyperbolic systems similar to (1.6) have been discussed by many mathematicians [6], [8], [14] and various results concerning global existence and blow up have been established

In order to make this paper self contained we state, without proof, a local existence result. The proof can be established by either a classical energy argument [4] or by using the nonlinear semigroup theory [5]. We first start with the hypotheses on the functions a, c, and the initial data.

(H1) $a \in C^2([0,1])$ such that $a \geq a_0 > 0$

(H2) $c \in C^2(\mathbb{R})$ such that $c \geq c_0 > 0$

(H3) $\theta_0 \in H^2(I) \cap H_0^1(I)$ *and* $q_0 \in H^2(I)$

Proposition. *Assume that (H1), (H2), and (H3) hold. Then problem (1.6) – (1.7) has a unique local solution (θ, q), on a maximal time interval $[0,T)$, satisfying*

$$\begin{aligned} \theta \in C([0,T), \quad H^2(I) \cap H_0^1(I)) \cap C^1([0,T), \quad H_0^1(I)) \\ q \in C([0,T), \quad H^2(I)) \cap C^1([0,T), \quad H^1(I)). \end{aligned} \tag{1.8}$$

Remark 2.1. θ, q are in $C^1([0,1] \times [0,T))$ by the Sobolev embedding theorem.

2. Formation of singularities.

In this section, we state and prove our main result. We first begin with a result, which gives uniform bounds on the solution in terms of the initial data.

Theorem 1. *Assume that (H1), (H2), and (H3) hold. Then the solution (1.8) satisfies*

$$\max_{(x,t)\in[0,1]\times[0,T)} \{|\theta(x,t)| + |q(x,t)|\} \leq \Gamma \max_{x\in[0,1]} \{|\theta_0(x)| + |q_0(x)|\} \tag{2.1}$$

where Γ *is a constant independent of* θ, q, *and* t.
Proof. We introduce the quantities

$$r(x,t) := \frac{q(x,t)}{\sqrt{a(x)}} - A(\theta(x,t)), \qquad s(x,t) := \frac{q(x,t)}{\sqrt{a(x)}} + A(\theta(x,t)) \tag{2.2}$$

and the differential operators

$$\partial_t^- := \frac{1}{\rho}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}, \quad \partial_t^+ := \frac{1}{\rho}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$$

where

$$\rho(x,t) = \sqrt{a(x)/c(\theta(x,t))}, \qquad A(\theta) = \int_0^\theta \sqrt{c(\xi)}d\xi.$$

We then compute

$$\begin{aligned}
\partial_t^- r &= \frac{1}{\rho}r_t - r_x \\
&= \sqrt{\frac{c(\theta)}{a}}(\frac{q_t}{\sqrt{a}} - \sqrt{c(\theta)}\theta_t) - (\frac{q_x}{\sqrt{a}} - \sqrt{c(\theta)}\theta_x) + \frac{1}{2}a'a^{-3/2}q \\
&= -\frac{1}{\sqrt{a}}(c\theta_t + q_x) + \frac{\sqrt{c}}{a}(q_t + a\theta_x) + \frac{1}{2}a'a^{-3/2}q \\
&= (-b\frac{\sqrt{c}}{a} + \frac{1}{2}a'a^{-3/2})q(x,t) = (-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r+s}{2},
\end{aligned} \tag{2.3}$$

by virtue of the system (1.6). Simillar computations also yield

$$\partial_t^+ s = (-b\sqrt{\frac{c}{a}} - \frac{a'}{2a})\frac{r+s}{2}. \tag{2.4}$$

We then define the nonnegative Lipschitz functions

$$R(t) := \max_{x\in[0,1]} |r(x,t)|, \quad S(x,t) := \max_{x\in[0,1]} |s(x,t)|. \tag{2.5}$$

For each fixed $t > 0$, we pick x_1 and x_2 in $[0,1]$ so that

$$R(t) = |r(x_1,t)|, \qquad S(t) = |s(x_2,t)|; \tag{2.6}$$

therefore for so small $h \in (0,t)$, we have

$$\begin{aligned}
R(t-h) &\geq |r(x_1 + h\rho(x_1,t), t-h)| \\
S(t-h) &= |s(x_1 - h\rho(x_2,t), t-h)|.
\end{aligned} \tag{2.7}$$

By substracting (2.7) from (2.6), dividing by h, and then letting h go to zero we get

$$\begin{aligned}
R'(t) &\leq \rho(x_1,t)|\partial_t^- r(x_1,t)| \leq \sqrt{\frac{a}{c}}|-b\sqrt{\frac{c}{a}} + \frac{a'}{2a}|\frac{r+s}{2} \\
&\leq |-b + \frac{a'}{2\sqrt{ac}}|\frac{r+s}{2} \leq \frac{\gamma}{2}(R(t) + S(t))
\end{aligned}$$

and

$$S'(t) \leq \frac{\gamma}{2}(R(t) + S(t))$$

where $\gamma = b + (\max a')/a_0 c_0$. Therefore we have

$$\frac{d}{dt}[R(t) + S(t)] \leq \gamma[R(t) + S(t)].$$

A simple integration leads to

$$[R(t) + S(t)] \leq [R(0) + S(0)]e^{\gamma T} \tag{2.8}$$

We using (2.2) and (2.5), the assertion of theorem 1 is established.

Theorem 2. *Assume that* $(H1)$, $(H2)$, *and* $(H3)$ *hold. Assume further that* $c'(0) > 0$. *Then there exist initial data* θ_0 *and* q_0, *for which the solution (1.8) blows up in finite time.*

Proof. We take an t-partial derivative of (2.3) to have

$$(\partial_t^- r)_t = [(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r+s}{2}]_t \tag{2.9}$$

which, in turn, implies

$$\begin{aligned} \partial_t^- r_t &= \frac{\rho_t}{\rho^2} r_t + [(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r+s}{2}]_t \\ &= \frac{-c'}{2\sqrt{ac}}\theta_t r_t + (-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r_t + s_t}{2} - \frac{bc'}{4\sqrt{ac}}(r+s)\theta_t. \end{aligned} \tag{2.10}$$

By using (2.2), it is easy to see that

$$\theta_t = \frac{s_t - r_t}{2c(\theta)};$$

thus substituting in (2.10), we obtain

$$\begin{aligned} \partial_t^- r_t &= \frac{-c'}{2\sqrt{ac}}\frac{s_t - r_t}{2c} r_t + (-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r_t + s_t}{2} - \frac{bc'}{4\sqrt{ac}}(r+s)\frac{s_t - r_t}{2c} \\ &= \frac{c'}{4c\sqrt{ac}} r_t^2 - \frac{c'}{4c\sqrt{ac}} r_t s_t + (-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r_t + s_t}{2} - \frac{bc'}{4\sqrt{ac}}(r+s)\frac{s_t - r_t}{2c}. \end{aligned} \tag{2.11}$$

In order to eliminate the second term in the RHS of (2.11), we set

$$W := c^{1/4} r_t,$$

consequently we get

$$\begin{aligned} \partial_t^- W &= c^{1/4}\partial_t^- r_x + \frac{1}{4}c^{-3/4} r_t \partial_t^- c \\ &= \frac{c^{-7/4}c'}{4\sqrt{a}} W^2 - \frac{c^{-3/4}c'}{4\sqrt{a}} r_t s_t + c^{1/4}[(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r_t + s_t}{2} \\ &\quad - \frac{bc'}{4\sqrt{ac}}(r+s)\frac{s_t - r_t}{2c}] + \frac{1}{4}c^{-3/4} r_t c'(\sqrt{\frac{c}{a}}\theta_t - \theta_x) \end{aligned} \tag{2.12}$$

At this point we should note that, by (1.6) and (2.2), we have

$$-\theta_x = \frac{1}{a}(q_t + bq) = \frac{s_t}{\sqrt{a}} - \frac{\sqrt{c}}{\sqrt{a}}\theta_t + \frac{b}{a}q.$$

Therefore (2.12) becomes

$$\begin{aligned}\partial_t^- W &= \frac{c^{-7/4}c'}{4\sqrt{a}}W^2 - \frac{c^{-3/4}c'}{4\sqrt{a}}r_t s_t + c^{1/4}[(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r_t + s_t}{2} \\ &\quad -\frac{bc'}{4\sqrt{ac}}(r+s)\frac{s_t - r_t}{2c}] + \frac{1}{4}c^{-3/4}r_t c'(\frac{s_t}{\sqrt{a}} + \frac{b}{a}q) \\ &= \frac{c^{-7/4}c'}{4\sqrt{a}}W^2 + \frac{1}{2}[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} + \frac{bc'}{8c\sqrt{ac}}(r+s) + \frac{c'}{4c}(r+s)]W \\ &\quad +\frac{1}{2}[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} - \frac{bc'}{8c\sqrt{ac}}(r+s)]s_t\end{aligned} \tag{2.13}$$

Direct computation, using (1.6) and (2.2) again, gives

$$s_t = \sqrt{a}\partial_t^-\theta - \frac{b}{2\sqrt{a}}(s+r), \qquad r + s = 2r + 2A(\theta).$$

So substituting in (2.13) yields

$$\begin{aligned}\partial_t^- W &= \frac{c^{-7/4}c'}{4\sqrt{a}}W^2 + \frac{1}{2}[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} + \frac{bc'}{8c\sqrt{ac}}(r+s) + \frac{c'}{4c}(r+s)]W \\ &\quad +[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} - \frac{bc'}{8c\sqrt{ac}}(r+A)]\sqrt{a}\partial_t^-\theta \\ &\quad -\frac{b}{4\sqrt{a}}(s+r)[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} - \frac{bc'}{8c\sqrt{ac}}(r+s)]\end{aligned} \tag{2.14}$$

We now estimate the third term of (2.14) as follows

$$\begin{aligned}\frac{a'}{2\sqrt{a}}\partial_t^-\theta &= \partial_t^-(\frac{a'}{2\sqrt{a}}\theta) + \theta(\frac{a'}{2\sqrt{a}})' \\ -b\sqrt{c(\theta)}\partial_t^-\theta &= -\partial_t^- bA \\ -\frac{bc'}{8c\sqrt{c}}r\partial_t^-\theta &= -\frac{b}{8}rc^{-3/2}(\theta)c'(\theta)\partial_t^-\theta \\ &= \frac{b}{4}r\partial_t^- c^{-1/2}(\theta) = \frac{b}{4}\partial_t^-[rc^{-1/2}(\theta)] - \frac{b}{4}c^{-1/2}(\theta)\partial_t^- r \\ &= \frac{b}{4}\partial_t^-[rc^{-1/2}(\theta)] - \frac{b}{4}c^{-1/2}(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r+s}{2} \\ -\frac{bc'}{8c\sqrt{c}}A\partial_t^-\theta &= -\frac{b}{8}c^{-3/2}(\theta)c'(\theta)A(\theta)\partial_t^-\theta = -\frac{b}{8}\partial_t^-(\int_0^\theta c^{-3/2}c'A(\xi)d\xi)\end{aligned} \tag{2.15}$$

By setting

$$f(x,t) = (\frac{a'}{2\sqrt{a}}\theta) - bA + \frac{b}{4}rc^{-1/2}(\theta) - \frac{b}{8}(\int_0^{\theta} c^{-3/2}c' A(\xi)d\xi)$$

the estimate (2.14) takes the form

$$\partial_t^- W = \frac{c^{-7/4}c'}{4\sqrt{a}}W^2 + \partial_t^- f + MW + N$$

where

$$\begin{aligned} M &= \frac{1}{2}[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} + \frac{bc'}{8c\sqrt{ac}}(r+s) + \frac{c'}{4c}(r+s)] \\ N &= \theta(\frac{a'}{2\sqrt{a}})' - \frac{b}{4}c^{-1/2}(-b\sqrt{\frac{c}{a}} + \frac{a'}{2a})\frac{r+s}{2} \\ &\quad - \frac{b}{4\sqrt{a}}(s+r)[\frac{a'}{2a} - b\sqrt{\frac{c}{a}} - \frac{bc'}{8c\sqrt{ac}}(r+s)] \end{aligned}$$

When set $F = W - f$ to obatain, from (2.16),

$$\partial_t^- F = \frac{c^{-7/4}c'}{4\sqrt{a}}F^2 + BF + C \tag{2.16}$$

where $B = (M - 2\frac{c^{-7/4}c'}{4\sqrt{a}}f)$ and $C = N + \frac{c^{-7/4}c'}{4\sqrt{a}}f^2 - Mf$ are functions depending on a, b, θ, and q only. Therefore by choosing the initial data small enough in L^∞ norm, with sufficiently large derivatives (hence F is large enough), it is standard to deduce that F blows up in finite time.

Remark 2.1. The same result holds for $c'(0) < 0$. In this case consider the evolution of s_t on the forward characteristics.

Remark 2.2. Similar results can be obtained for the Cauchy problem, as well as other types of boundary conditions.

Acknowledgment The authors would like to thank KFUPM for its continuous support.

3. References

1. Cattaneo, C., Sulla conduzione del calore, *Atti Sem. Math. Fis Univ. Modena* **3** (1948), 83–101.

2. Coleman, B.D., W. J. Hrusa, and D. R. Owen, Stability of equilibrium for a nonlinear hyperbolic system describing heat propagation by second sound in solids, *Arch. Rational Mech. Anal.* **94** (1986), 267–289.

3. Coleman, B. D., M. Fabrizio, and D. R. Owen, On the thermodynamics of second sound in dielectric crystals, *Arch. Rational Mech. Analysis* **80** (1982), 135–158.

4. Dafermos, C.M. and W.J. Hrusa, Energy methods for quasilinear hyperbolic initial-boundary value problems. Applications to Elastodynamics, *Arch. Rational Mech. Anal.* **87** (1985), 267–292.

5. Hughes, T.J.R. and Kato, T., and Marsden, J.E., Well-posed quasilinear second order hyperbolic systems with applications to nonlinear elastodynamic and general relativity, *Arch. Rational Mech. Anal.* **63** (1977), 273–294.

6. Lax, P.D., Development of singularities in solutions of nonlinear hyperbolic partial differential equations, *J. Math. Physics* **5** (1964), 611–613.

7. Li H. and K. Saxton, Asymptotic behavior of solutions to quazilinear hyperbolic equations with nonlinear damping, *Quarterly of Applied Math.* (to appear)

8. MacCamy, R.C. and Mizel, V. J., Existence and nonexistence in the large solutions of quasilinear wave equations, *Arch. Rational Mech. Anal.* **25** (1967), 299–320.

9. Messaoudi, S. A., Formation of singularities in heat propagation guided by second sound, *J. Diff. Eqns.* **130** (1996), 92–99.

10. Messaoudi S. A., On the existence and nonexistence of solutions of a nonlinear hyperbolic system describing heat propagation by second sound, *Applicable Analysis* **73** (1999) 485 - 496

11. Messaoudi S. A., Decay of solutions of a nonlinear hyperbolic system describing heat propagation by second sound. *Appl. Anal.* **81** no. **2** (2002), 201–209.

12. K. Saxton, R. Saxton, and W. Kosinsky, On second sound at the critical temperature, *Quarterly of Applied Math.* **57 # 4** (1999), 723 - 740

13. K. Saxton, R. Saxton, and W. Kosinsky, Nonlinearity and Memory Effects in low temperature heat propagation, *Arch. Rational Mech.* **52** (2000), 127 - 142

14. Slemrod, M., Instability of steady shearing flows in nonlinear viscoelastic fluid, *Arch. Rational Mech. Anal.* **3** (1978), 211–225.