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On a Property of Large Inverse Systems

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Given a direct system $\{M_i\}_{i\in I}$ of modules, it is well-known that $\varinjlim M_i$ is a pure quotient of the direct sum $\oplus_{i\in I} M_i$. In contrast, the dual statement that inverse limits are pure submodules of corresponding direct products is not always true:
For each prime number p, we can construct a descending chain $\{A_n\}_{n\in \mathbb{N}}$ of divisible abelian groups whose intersection A is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (see [2, Exercise 6 p. 101]). Since divisibility is inherited by pure subgroups and direct products and since A is not divisible, it follows that the inverse limit A of the divisible groups A_n is not pure in $\prod_n A_n$. However, as we shall show in this note, when certain set-theoretic conditions are imposed on an inverse system of modules, the inverse limit is a *direct summand* of the corresponding direct product. This is motivated by the following observation:
Let p be a prime number and let J_p be the p-adic group $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. As each $\mathbb{Z}/p^n\mathbb{Z}$ is finite, J_p is linearly, and hence, algebraically compact. (See [1] and [2].) Since, as can easily be proved, J_p is pure in $\prod_n \mathbb{Z}/p^n\mathbb{Z}$, it follows that the canonical monomorphism $0 \longrightarrow \varprojlim \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \prod_n \mathbb{Z}/p^n\mathbb{Z}$ splits.

The purpose of this note is to generalize this result in both set-theoretic and universal algebraic directions. We refer to [4] and [3] for the various notions used here from the theory of large cardinals and universal algebra respectively. Let us call a subalgebra B of an algebra A a *retract* of A if there exists a homomorphism $g : A \longrightarrow B$ whose restriction to B is the identity on B; such a g is called a *retraction.* A directed set

$\{I; \leq\}$ is λ-*directed* for some infinite cardinal λ, if every subset of I of size less than λ has an upper bound in I.

First, we have

Lemma 1. *A subalgebra B of an algebra A is a retract of A if and only if every system Σ of equations over B and with a solution in A has a solution in B.*

Proof. (Cf. [2, Proposition 22.3].) Suppose B is a retract of A, with retraction g, and let Σ be a system of equations over B with set of unknowns $\{x_s\}_{s\in S}$. If $\{a_s\}_{s\in S}$ is a solution in A of Σ, then, clearly, $\{g(a_s)\}_{s\in S}$ is a solution of Σ in B. Conversely, let Σ be the system over B

$$x_{f((a_i)_{i\in r(f)})} = f((x_{a_i})_{i\in r(f)})$$

$$x_b = b$$

for any $a_i \in A$, $b \in B$ and any operation f on A (with arity $r(f)$), and where the unknowns are indexed by A. This system is solvable in A by $x_a = a$ $(a \in A)$. Thus, if $x_a = g(a)$ $(a \in A)$ is a solution of Σ in B, then the mapping $g : A \longrightarrow B$ is a retraction.□

Proposition 2. *Let α be a limit ordinal, κ be an infinite cardinal and $\{A_i; \sigma_i^j\}_{i\leq j<\alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa < cf(\alpha)$. Then the inverse limit $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$.*

Proof. We first show that $\varprojlim A_i$ is a subalgebra of $\prod_{i<\alpha} A_i$, i.e. that $\varprojlim A_i$ is not empty. For each $i < \alpha$, choose p_i in A_i, and let $T_i = \{\sigma_i^j(p_j) : i \leq j < \alpha\}$. Partial-order $T = \bigcup_{i<\alpha} T_i$ by setting $x < y$ when $x \in T_i$, $y \in T_j$ and $\sigma_i^j(y) = x$ for some $i < j < \alpha$. It is easy to see that $(T, <)$ is a tree of height α. For any $x = \sigma_i^j(p_j)$ in T_i, we have $x = p_i$ or $x = \sigma_i^{i+1}\sigma_{i+1}^j(p_j) \in \sigma_i^{i+1}(A_{i+1})$, so that $T_i \subseteq \{p_i\} \cup \sigma_i^{i+1}(A_{i+1})$, and therefore $|T_i| < \kappa$. If α is a limit ordinal, then T has a branch $b = \{x_i\}_{i<\alpha}$ of length α, by [5, Proposition 2.32 p. 304]. Clearly $\sigma_i^j(x_j) = x_i$ whenever $i < j < \alpha$, so that $(x_i)_{i<\alpha} \in \varprojlim A_i$, and hence $\varprojlim A_i \neq \emptyset$. Next, let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s\in S}$ and constants $\{c\}_{c\in C}$, and suppose it is solvable in $\prod_{i<\alpha} A_i$ by $\{a_s\}_{s\in S}$, say. For each $i < \alpha$, let Σ^i be the system obtained from Σ by replacing each c in C by its i-th coordinate in A_i. Fix s in S and denote by R the set consisting of all initial segments of the sequences $(\sigma_i^j(a_s(j)))_{i\leq j}$ $(j < \alpha)$ (where $a_s(j)$ is the j-th coordinate of a_s). It is easy to see that R is a tree of height α (ordered by inclusion). By an argument similar to the one used for T above, we infer that R has a branch $(\mu_s(i))_{i<\alpha}$. Since $\sigma_i^j(\mu_s(j)) = \mu_s(i)$ for all $i \leq j < \alpha$, we obtain that $(\mu_s(i))_{i<\alpha} \in \varprojlim A_i$. Now we have $c(i) = \sigma_i^j(c(j))$ for all $j \geq i$ (since $C \subseteq \varprojlim A_i$), so that for all $j \geq i$, $\{\sigma_i^j(a_s(j))\}_{s\in S}$ is a solution of Σ^i. By definition of R, for each $i < \alpha$, $\mu_s(i) = \sigma_i^j(a_s(j))$ for some $j \geq i$, i.e. $\{\mu_s(i)\}_{s\in S}$ is a solution of Σ^i. Since $(\mu_s(i))_{i<\alpha} \in \varprojlim A_i$ for all s in S, the proof is complete by Lemma 1.□

We next turn our attention to cardinals with the tree property. Recall that $\aleph_0$ and weakly compact (e.g. measurable) cardinals have the tree property, whereas $\aleph_1$ and singular cardinals do not.

Proposition 3. *Let α be a limit ordinal, κ be an infinite cardinal with the tree property, and $\{A_i; \sigma_i^j\}_{i\leq j<\alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa \leq cf(\alpha)$. Then $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$.*

Proof. If $\kappa < cf(\alpha)$, use Proposition 2. Suppose that $\kappa = cf(\alpha)$ with $\alpha = \Sigma_{t<\kappa}\alpha_t$, where $\alpha_t < \kappa$. Then, using the tree property of κ and an argument similar to that of Proposition 2, we obtain that $\varprojlim A_i$ is a subalgebra of $\prod_{i<\alpha} A_i$, and that $\varprojlim A_{\alpha_t}$, the inverse limit of the inverse family $\{A_{\alpha_t}; \sigma_{\alpha_t}^{\alpha_s}\}_{t\leq s<\kappa}$, is a retract of $\prod_{t<\kappa} A_{\alpha_t}$. Let $\varphi : \prod_{i<\alpha} A_i \longrightarrow \prod_{t<\kappa} A_{\alpha_t}$ be the canonical projection. Then (see for example the proof of [3, Lemma 7 p.133]), the restriction ψ of φ to $\varprojlim A_i$ is an isomorphism $\varprojlim A_i \longrightarrow \varprojlim A_{\alpha_t}$ and we have $\varphi f = g\psi$, where $f : \varprojlim A_i \longrightarrow \prod_{i<\alpha} A_i$ and $g : \varprojlim A_{\alpha_t} \longrightarrow \prod_{t<\kappa} A_{\alpha_t}$ are the inclusion mappings. If $\pi : \prod_{t<\kappa} A_{\alpha_t} \longrightarrow \varprojlim A_{\alpha_t}$ is such that πg is the identity, then $\psi^{-1}\pi\varphi f = \psi^{-1}\pi g\psi$ is the identity mapping on $\varprojlim A_i$, and so $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$.□

The conclusion of Proposition 3 can be arrived at for a wider class of inverse systems, provided κ is a compact cardinal. (An infinite cardinal λ is *compact* if, for any set S, every λ-complete proper filter on S can be extended to a λ-complete ultrafilter.) To prove that, the following lemma is needed.

Lemma 4. *Let $\{A_i; \sigma_i^j\}_{i\in I}$ be an inverse system of nonempty sets and let κ be a compact cardinal such that $\{I; \leq\}$ is κ-directed and $\left|\bigcup_{j>i} \sigma_i^j(A_j)\right| < \kappa$, for every $i \in I$. Then $\varprojlim A_i$ is nonempty.*

Proof. For each $i \in I$, let $p_i \in A_i$, $\pi_i : \prod_{j\in I} A_j \longrightarrow A_i$ be the i-th canonical projection, and let $T_i = \{\sigma_i^j(p_j) : i, j \in I, i \leq j\}$. For every $J \in [I]^{<\kappa} = \{S \subseteq I : |S| < \kappa\}$, let $X_J = \{x \in \prod_{i\in I} T_i : \sigma_i^j(p_j) = p_i$, for all $i, j \in J$ and $i \leq j\}$. Since I is κ-directed and κ is regular (compact cardinals are regular), $\varnothing \subset X_{\bigcup_{\tau<\lambda} J_\tau} \subseteq \bigcap_{\tau<\lambda} X_{J_\tau}$, whenever $J_\tau \in [I]^{<\kappa}$ and λ is a cardinal less than κ. It follows that the set $\{X_J\}_{J\in[I]^{<\kappa}}$ generates on $\prod_{i\in I} T_i$ a κ-complete proper filter, which, as κ is compact, can be extended to a κ-complete ultrafilter U. For each $Y \in U$, let $Y_i = \{x_i \in T_i : x = (x_i)_{i\in I} \in Y\}$ and let $U_i = \{Y_i : Y \in U\}$. As in the proof of [3, Theorem 1, p.132], we obtain that U_i is a κ-complete ultrafilter on T_i. By hypothesis $|T_i| < \kappa$, so that U_i is principal generated by a singleton $\{y_i\}$,say. Now, for all $i, j \in I$, $\pi_i^{-1}(\{y_i\})$, $\pi_j^{-1}(\{y_j\})$ and $X_{\{i,j\}}$ are in U, so that $\pi_i^{-1}(\{y_i\}) \cap \pi_j^{-1}(\{y_j\}) \cap X_{\{i,j\}} \in U$. Therefore, if $i \leq j$, there exists $x = (x_i)_{i\in I} \in X_{\{i,j\}}$ such that $\sigma_i^j(y_j) = \sigma_i^j(x_j) = x_i = y_i$. This proves that $\varprojlim A_i$ is nonempty.□

Remark. The foregoing proof is a straightforward adaptation of an argument of Grätzer [3, Theorem 1, p.132], where he used ultrafilters to prove the classical theorem that inverse limits of finite nonempty sets are nonempty. Indeed, since $\aleph_0$

is compact, the following proposition generalizes both [3, Theorem 1, p.132] (and hence König's Graph Lemma) and the observation on J_p mentioned above.

Proposition 5. *Let* $\{A_i; \sigma_i^j\}_{i\in I}$ *be an inverse system of algebras and let* κ *be a compact cardinal such that* $\{I; \leq\}$ *is* κ*-directed and* $\left|\bigcup_{j>i} \sigma_i^j(A_j)\right| < \kappa$*, for every* $i \in I$*. Then* $\varprojlim A_i$ *is a retract of* $\prod_{i\in I} A_i$*.*

Proof. That $\varprojlim A_i$ is a subalgebra of $\prod_{i\in I} A_i$ follows from Lemma 4. As in the proof of Proposition 2, let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s\in S}$ and constants $\{c\}_{c\in C}$, and suppose it is solvable in $\prod_{i\in I} A_i$ by $\{a_s\}_{s\in S}$. For each $i \in I$, let Σ^i be the system obtained from Σ by replacing each c in C by its i-th coordinate $c(i)$ in A_i. Fix s in S, and set $B_i^s = \{\sigma_i^j(a_s(j)) : j \in I, i \leq j\}$. It is easy to see that $\{B_i^s\}_{i\in I}$ can be regarded as an inverse system of nonempty sets with bonding maps σ_i^j $(i \leq j)$. By Lemma 4 again, $\varprojlim B_i^s$ is nonempty. Clearly, if $\mu_s \in \varprojlim B_i^s$, then $\{\mu_s\}_{s\in S}$ is a solution of Σ in $\varprojlim A_i$. Now use Lemma 1.□

Corollary 6. *Let* κ *be a compact cardinal and let* $\{A_i\}_{i\in I}$ *be an inverse system of algebras such that* I *is* κ*-directed and* $|A_i| < \kappa$ for all $i \in I$. *Then* $\varprojlim A_i$ *is a retract of* $\prod_{i\in I} A_i$.□

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