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On Some Properties of Branch Operators

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ON SOME PROPERTIES OF BANACH OPERATORS

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Abstract

A mapping α from a normed space X into itself is called a Banach operator if there is a constant k such that $0 \leq k < 1$ and $\|\alpha^2(x) - \alpha(x)\| \leq k\|\alpha(x) - x\|$ for all $x \in X$. In this note we study some properties of Banach operators. Among other results we show that if α is a linear Banach operator on a normed space X, then $N(\alpha - 1) = N((\alpha - 1)^2)$, $N(\alpha - 1) \cap R((\alpha - 1)) = (0)$ and if X is finite-dimensional then $X = N(\alpha - 1) \oplus R(\alpha - 1)$, where $N(\alpha - 1)$ and $R(\alpha - 1)$ denote the null space and the range space of $(\alpha - 1)$, respectively and 1 is the identity mapping on X. We also obtain some commutativity results for a pair of bounded linear multiplicative Banach operators on normed algebras.

1. Introduction

Let X be a normed space and $\alpha : X \to X$ be a mapping. Following [5], α is said to be a Banach operator if there exists a constant k such that $0 \leq k < 1$ and $\|\alpha^2(x) - \alpha(x)\| \leq k\|\alpha(x) - x\|$ for all $x \in X$. Banach operators generalize contractions and play an important role in the fixed point theory; their consideration essentially goes back to Cheney and Goldstein [4] in the study of proximity maps on convex sets (see, e.g., [5] and references therein). The purpose of this note is to study some further properties of Banach operators. In section 2, we prove some decomposition results for Banach operators. For instance, we show that if α is a linear Banach operator on a normed space X, then $N(\alpha-1) = N((\alpha-1)^2), N(\alpha-1) \cap R(\alpha-1) = (0)$, and if X is finite-dimensional, then $X = N(\alpha-1) \oplus R(\alpha-1)$, where $N(\alpha-1)$ and $R(\alpha-1)$ denote the null space of $(\alpha-1)$ and the range space of $(\alpha-1)$, respectively and 1 is the identity mapping on X. This result is no longer true when X is infinite-dimensional. However, if α is a bounded linear operator on a Hilbert space H such that α and α^* have common

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fixed points, then it is shown that $N(\alpha-1)+R(\alpha-1)$ is dense in H. Though this result is not directly related to Banach operators, yet it provides information about decompositions; therefore, it is of independent interest and we include it here. The techniques in the proof of this decomposition theorem lead to an alternate proof of a well-known result on the residual spectrum of a bounded normal operator on H.

In section 3, we study the operator equation $\alpha+c\alpha^{-1}=\beta+c\beta^{-1}$ for a pair of invertible bounded linear Banach operators on a normed space X where c is an appropriate complex number. We briefly recall that the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ has been extensively studied for a pair of automorphisms α,β on von Neumann algebras, C^*-algebras and rings. Most of the results deal with decomposition of the underlying structures with an additional assumption of commutativity of α and β. Certain situations are identified when the equation itself implies the commutativity of α and β. Recently, this equation has been generalized as $a\alpha+b\alpha^{-1}=a\beta+b\beta^{-1}$ for linear combinations $a\alpha+b\alpha^{-1}$ for appropriate real or complex numbers a,b. If we put $c=\frac{b}{a}$, then the equation $a\alpha+b\alpha^{-1}=a\beta+b\beta^{-1}$ becomes $\alpha+c\alpha^{-1}=\beta+c\beta^{-1}$. We obtain here some Banach operator analogues of some results of [3]. For instance, we show that if α and β are invertible bounded linear multiplicative Banach operators on a normed algebra X with identity such that $\alpha(x)+c\alpha^{-1}(x)=\beta(x)+c\beta^{-1}(x)$ for all $x\in X$ and if β is inner (that is, there exists an invertible element $u\in X$ such that $\beta(x)=uxu^{-1}$ for all $x\in X$), then α,β commute. For more on these operator equations we refer to [1, 3, 8] where further references are given.

2. Decomposition Results

Let X be a normed space and α a linear operator on X. Denote by $N(\alpha)$, the null space of α and $R(\alpha)$, the range space of α. If X is finite-dimensional and $N(\alpha)=N(\alpha^2)$ then by the dimension theory of finite-dimensional spaces, $R(\alpha)=R(\alpha^2)$ and X admits the decomposition $X=N(\alpha)\oplus R(\alpha)$. This result may not be true when X is

infinite-dimensional. In fact, one cannot expect even a weaker result that $N(\alpha)+R(\alpha)$ is dense in X. For instance, one can find an injective operator whose range is not dense. However, if α is a bounded linear Banach operator on a normed space X, then $N(\alpha-1)=N((\alpha-1)^2), N(\alpha-1)\cap R(\alpha-1)=(0)$, where 1 denotes the identity operator on X and if X is finite-dimensional, then $X=N(\alpha-1)\oplus R(\alpha-1)$. These and some other results are proved in this section.

PROPOSITION 2.1. *Let α be a linear Banach operator on normed space X, then (i) $N(\alpha-1)=N((\alpha-1)^2)$, (ii) $N(\alpha-1)\cap R(\alpha-1)=(0)$.*

Proof. (i) $N(\alpha-1)\subseteq N((\alpha-1)^2)$ is obvious. Assume $x\in N((\alpha-1)^2)$. Then $(\alpha-1)^2(x)=\alpha^2x-2\alpha(x)+x=0$; so $\alpha(x)-x=\alpha^2(x)-\alpha(x)$ and since α is a Banach operator, we get $\|\alpha(x)-x\|=\|\alpha^2(x)-\alpha(x)\|\leq k\|\alpha(x)-x\|$. Since $0\leq k<1$, we get $\|\alpha(x)-x\|=0$; that is, $\alpha(x)=x$ or $x\in N(\alpha-1)$. Thus $N((\alpha-1)^2)\subseteq N(\alpha-1)$ and hence $N(\alpha-1)=N((\alpha-1)^2)$.

(ii) To prove $N(\alpha-1)\cap R(\alpha-1)=(0)$, assume that $y\in N(\alpha-1)\cap R(\alpha-1)$. Then $(\alpha-1)(y)=0$ and $y=(\alpha-1)(x)$ for some $x\in X$ and hence $(\alpha-1)(y)=(\alpha-1)^2(x)=0$ so that $x\in N((\alpha-1)^2)=N(\alpha-1)$, (by (i)). Thus $y=(\alpha-1)(x)=0$ and hence $N(\alpha-1)\cap R(\alpha-1)=(0)$.

It is obvious that $R((\alpha-1)^2)\subseteq R(\alpha-1)$. Thus $N((\alpha-1)^2)\cap R((\alpha-1)^2)\subseteq N((\alpha-1)^2)\cap R(\alpha-1)=N(\alpha-1)\cap R(\alpha-1)=(0)$, so that $N((\alpha-1)^2)\cap R((\alpha-1)^2)=(0)$.

If X is finite-dimensional then by the dimension theory of finite-dimensional spaces, we can conclude that $R((\alpha-1)^2)=R(\alpha-1)$ and hence by Taylor [7, pp. 271–273] $X=N(\alpha-1)\oplus R(\alpha-1)$. Thus we have the following proposition.

PROPOSITION 2.2. *Let α be a linear Banach operator on a finite-dimensional normed space X, then $X=N(\alpha-1)\oplus R(\alpha-1)$.*

For certain operators on Hilbert spaces, we are able to get somewhat general form of the above result though it would be interesting to obtain a similar result for Banach

operators. However, this result is of independent interest in operator theory, so we include it here.

PROPOSITION 2.3. *Let α be a bounded linear operator on a Hilbert space H such that α and α^* have common fixed points, then $N(\alpha-1)\cap R(\alpha-1) = (0)$ and $N(\alpha-1)+R(\alpha-1)$ is dense in H.*

Proof. If $x \in N(\alpha-1)\cap R(\alpha-1)$, then $\alpha(x) = x$. Thus (by assumption) $\alpha^*(x) = x$; that is, $x \in N(\alpha^*-1) = N((\alpha-1)^*) = \overline{R(\alpha-1)}^{\perp}$. This implies $\langle x, x\rangle = 0$ and hence $x = 0$. To prove the second part, assume that $N(\alpha-1)+R(\alpha-1)$ is not dense in H. Then there exists a nonzero continuous linear functional ϕ on H such that ϕ vanishes on $N(\alpha-1)$ and $R(\alpha-1)$. That ϕ vanishes on $R(\alpha-1)$ implies $\phi(\alpha(x)-x) = (\phi\circ\alpha)(x)-\phi(x) = 0$ or $(\phi\circ\alpha)(x) = \phi(x)$ for all $x \in H$. By the Riesz-representation theorem, there is a unique $y_0 \in H$ such that $\phi(x) = \langle x, y_0\rangle = (\phi\circ\alpha)(x) = \langle\alpha(x), y_0\rangle = \langle x, \alpha^*(y_0)\rangle$ for all $x \in H$ and hence $\alpha^*(y_0) = y_0$. By assumption, $\alpha(y_0) = y_0$; that is, $y_0 \in N(\alpha-1)$ and hence $\phi(y_0) = \langle y_0, y_0\rangle = \|y_0\|^2 = 0$; that is, $y_0 = 0$. It follows that $\phi(x) = 0$ for all $x \in H$, so $\phi = 0$, a contradiction. This completes the proof.

If α is a linear contraction on H then α and α^* have common fixed points (see, e.g., [2, p. 188]); therefore, we have the following corollary:

COROLLARY 2.4. *If α is a linear contraction on a Hilbert space H, then $N(\alpha-1)\cap R(\alpha-1) = (0)$ and $N(\alpha-1)+R(\alpha-1)$ is dense in H.*

If α is a bounded normal operator on H, then $\|(\alpha-1)(x)\| = \|(\alpha^*-1)x\|$ for all $x \in H$ and hence α and α^* have common fixed points; therefore, we have the following corollary:

COROLLARY 2.5. *If α is a bounded normal operator on a Hilbert space H, then $N(\alpha-1)\cap R(\alpha-1) = (0)$ and $N(\alpha-1)+R(\alpha-1)$ is dense in H.*

It is well known that the residual spectrum of a bounded normal operator on a Hilbert space H is empty (see, e.g., [6, p. 479]). The techniques in the proof of the

above proposition can be applied to obtain an alternate proof of this result.

PROPOSITION 2.6. *The residual spectrum $R_\sigma(\alpha)$ of a bounded normal operator α on a Hilbert space H is empty.*

Proof. Let $\lambda \in R_\sigma(\alpha)$. Then $N(\alpha - \lambda) = (0)$ and $\overline{R(\alpha - \lambda)} \neq H$. Let ϕ be a nonzero continuous linear functional on H such that ϕ vanishes on $\overline{R(\alpha - \lambda)}$. By the Riesz-representation theorem, there is a unique $y_0 \in H$ such that $\phi(x) = \langle x, y_0 \rangle$ for all $x \in H$. Now ϕ vanishes on $\overline{R(\alpha - \lambda)}$; therefore, $\phi(\alpha(x)) = (\phi \circ \alpha)(x) = \lambda\phi(x)$ for all $x \in H$ and hence $\phi(\alpha(x)) = \lambda\phi(x) = \lambda\langle x, y_0 \rangle = \langle x, \overline{\lambda} y_0 \rangle$. Also, $\phi(\alpha(x)) = \langle \alpha(x), y_0 \rangle = \langle x, \alpha^*(y_0) \rangle$, so that $\langle x, \overline{\lambda} y_0 \rangle = \langle x, \alpha^*(y_0) \rangle$ for all $x \in H$ and hence $\alpha^*(y_0) = \overline{\lambda} y_0$. Since α is normal, we get $\alpha(y_0) = \lambda y_0$ and hence $y_0 \in N(\alpha - \lambda)$ and consequently $y_0 = 0$. Thus $\phi = 0$, a contradiction and hence $\sigma_R(\alpha)$ is empty. This completes the proof.

3. Commutativity Results

In this section, we mainly obtain some commutativity results for a pair of invertible bounded linear Banach operators α, β satisfying the equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ on normed algebras. We begin with the following basic result:

PROPOSITION 3.1. *Let α be an invertible linear Banach operator on a normed space X and $x_0 \in X$ such that $\alpha(x_0) + c\alpha^{-1}(x_0) = (1 + c)x_0$ where c is a real or complex number with $|c| \geq 1$, then x_0 is a fixed point of α.*

Proof. The equation implies that $\alpha^2(x_0) + cx_0 = (1 + c)\alpha(x_0)$ and hence $c(\alpha(x_0) - x_0) = \alpha^2(x_0) - \alpha(x_0)$. Since α is a Banach operator, therefore there exists a constant $0 \leq k < 1$ such that $|c| \, \|\alpha(x_0) - x_0\| = \|\alpha^2(x_0) - \alpha(x_0)\| \leq k\|\alpha(x_0) - x_0\|$ and hence $\|\alpha(x_0) - x_0\| \leq \frac{k}{|c|}\|\alpha(x_0) - x_0\| \leq k\|\alpha(x_0) - x_0\|$ and since $0 \leq k < 1$, we obtain $\alpha(x_0) - x_0 = 0$ or $\alpha(x_0) = x_0$.

The following proposition shows that under certain situations the equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ implies the commutativity of α, β.

PROPOSITION 3.2. *Let α, β be invertible linear multiplicative Banach operators on a normed algebra X with identity such that $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X$ where c is a real or complex number with $|c| \geq 1$. If β is inner, then α, β commute.*

Proof. Assume that $\beta(x) = uxu^{-1}$ for all $x \in X$, $u \in X$. Putting $x = u$ in the equation, we get $\alpha(u)+c\alpha^{-1}(u) = (1+c)u$ and by Proposition 3.1, $\alpha(u) = u$, and hence $(\beta\alpha)(x) = \beta(\alpha(x)) = u\alpha(x)u^{-1} = \alpha(u)\alpha(x)\alpha(u^{-1}) = \alpha(uxu^{-1}) = \alpha(\beta(x)) = (\alpha\beta)(x)$ for all $x \in X$. This proves that α, β commute.

PROPOSITION 3.3. *Let α, β be invertible bounded linear multiplicative Banach operators on a normed algebra X with identity such that $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X, \|\alpha - 1\| < 1, \|\beta - 1\| < 1$, where c is a real or complex number with $|c| > 4$ and β is inner. Then $\alpha = \beta$.*

Proof. By Proposition 3.2, α, β commute. The equation $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ together with the commutativity of α, β implies $(\alpha\beta - c)(\beta^{-1} - \alpha^{-1})(x) = 0$. Put $(\beta^{-1} - \alpha^{-1})(x) = y$. Then $(\alpha\beta - c)(y) = 0$ or $\alpha\beta(y) = cy$, so that $cy - \beta y = \alpha\beta y - \beta y = (\alpha - 1)\beta(y)$ and hence $\|cy - \beta(y)\| = \|(\alpha - 1)\beta(y)\| \leq \|\alpha - 1\| \, \|\beta(y)\| < \|\beta(y)\|$. That is, $\Big|\|cy\| - \|\beta(y)\|\Big| < \|\beta(y)\|$. This implies that $|c| \, \|y\| < 2\|\beta(y)\| \leq 2\|\beta\| \, \|y\|$. Since $\|\beta - 1\| < 1$, we get $\Big|\|\beta\| - 1\Big| \leq \|\beta - 1\| < 1$ and hence $\|\beta\| < 2$ so that $|c| \, \|y\| < 4\|y\|$. If $y \neq 0$, then we get $|c| < 4$, a contradiction; therefore $y = 0$ and hence $\beta^{-1}(\alpha) - \alpha^{-1}(x) = 0$ for all $x \in X$. Since α, β commute and $\alpha\beta$ is linear, we get $\alpha(x) - \beta(x) = 0$; that is, $\alpha(x) = \beta(x)$ for all $x \in X$ and hence $\alpha = \beta$. This completes the proof.

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