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# Alpha - Limits of Algebras


A. Laradji

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

# $\alpha$-Limits of Algebras*

A. Laradji

**Abstract**

In this note we prove some universal algebraic properties of $\alpha$-limits, and apply them to characterize a certain class of $\Sigma$-pure injective rings. This will provide an answer to a generalized version of a problem of Jensen and Lenzing. Our results also yield a module-theoretic characterization of singular cardinals.

Let $A$ be a general algebra and let $\alpha$ be an ordinal. Following [12], a retraction $f : A^{\omega_\alpha} \longrightarrow A$ such that $f(\{x_\tau\}_{\tau<\omega_\alpha}) = f(\{y_\tau\}_{\tau<\omega_\alpha})$ whenever there exists an ordinal $\tau_0 < \omega_\alpha$ with $x_\tau = y_\tau$ for all $\tau \geq \tau_0$, is called an $\alpha$-limit over $A$. It is easy to see that if $F_\alpha$ is the filter $\{\{\tau : \tau_0 \leq \tau < \omega_\alpha\} : \tau_0 < \omega_\alpha\}$ on $\omega_\alpha$ , then $A$ has $\alpha$-limits if and only if it is a retract of the reduced power $A^{\omega_\alpha}/F_\alpha$. $\alpha$-limits were first used by Łoś [8] to characterize algebraically compact abelian groups (see also [12] and [2]), and in [11] Wenzel proved that an algebra $A$ of cardinality $\omega_\alpha$ is equationally compact (in the sense of Mycielski [10]) if and only if it has $\beta$-limits for all ordinals $\beta < \alpha$.

In [4], Jensen and Lenzing posed the following problem: Is a module $M$ necessarily equationally compact if the diagonal map $M \longrightarrow M^{\mathbb{N}}/M^{(\mathbb{N})}$ splits? Since algebras are equationally compact

*2000 Mathematics Subject Classification. Primary 08A45, 13C11 if and only if they have $\alpha$-limits for all ordinals $\alpha$ (see [12]), this question can be rephrased as: Does the existence of a 0-limit over a module $M$ force it to have $\alpha$-limits for all ordinals $\alpha$? The question has a negative answer as was shown in [7] by using reduced powers of certain complete local noetherian rings, but the construction there cannot be extended to uncountable cardinals. It is worthwhile therefore to pose a generalized version of Jensen and Lenzing's question, namely:

*If a module $M$ has $\beta$-limits for all ordinals $\beta < \alpha$, then does $M$ necessarily have $\alpha$-limits?*

In our attempt to answer this question, we not only establish a cardinal-oriented characterization of commutative $\Sigma$-pure injective rings that are not principal ideal rings, but we also obtain, as a by-product, the following module-theoretic characterization of infinite regular cardinals:

*An infinite cardinal $\omega_\alpha$ is regular if and only if there exists a module without $\alpha$-limits but which has $\beta$-limits for all $\beta < \alpha$.*

Throughout, a theory of ordinals is assumed where an ordinal is the set of all smaller ordinals, and where cardinals are initial ordinals, $R$ is an associative ring with 1 and all modules are unitary left $R$-modules. Given a family $\{A_i\}_{i\in I}$ of similar algebras and a filter $F$ on $I$, an *$F$-restricted product* of the $A_i$'s is a subalgebra $B$ of $\prod_{i\in I} A_i$ such that (i) $\{i \in I : a_i = b_i\} \in F$ whenever $a = (a_i), b = (b_i) \in B$ and (ii) if $a = (a_i) \in \prod_{i\in I} A_i$ , $b = (b_i) \in B$, and $\{i \in I : a_i = b_i\} \in F$ then $a \in B$. (See [3].)

We first start with some preliminary results on $\alpha$-limits.

**Proposition 1.** *Let $F$ be an $\omega_\alpha$-complete filter on a set $I$ and let $\beta < \alpha$. If $\{A_i\}_{i\in I}$ is a family of similar algebras that have $\beta$-limits then so too does $B = \prod_{i\in I} A_i/F$.*

*Proof.* For each $i \in I$ let $f_i : A_i^{\omega_\beta} \longrightarrow A_i$ be a $\beta$-limit, and define $f : B^{\omega_\beta} \longrightarrow B$ by

$$f(\{\overline{(a_{\tau i})_{i\in I}}\}_{\tau<\omega_\beta}) = \overline{\{f_i(\{a_{\tau i}\}_{\tau<\omega_\beta})\}_{i\in I}}.$$

Then $f$ is a well-defined mapping. For if $\{\overline{(a_{\tau i})_{i\in I}}\}_{\tau<\omega_\beta} = \{\overline{(b_{\tau i})_{i\in I}}\}_{\tau<\omega_\beta}$ then the set $X_\tau = \{i \in I : a_{\tau i} = b_{\tau i}\} \in F$, for all $\tau < \omega_\beta$, so that, since $F$ is $\omega_\alpha$-complete, $X = \{i \in I : a_{\tau i} = b_{\tau i}$ for all

$\tau < \omega_\beta\}$ is also in $F$, and therefore

$$\{i \in I : f_i(\{a_{\tau i}\}_{\tau<\omega_\beta}) = f_i(\{b_{\tau i}\}_{\tau<\omega_\beta})\} \in F.$$

It is routine to check that $f$ is a homomorphism and that if $\partial : B \longrightarrow B^{\omega_\beta}$ is the diagonal mapping then $f\partial$ is the identity on $B$, that is, $f$ is a retraction. Moreover, if for some $\tau_0 < \omega_\beta$ $\{\overline{(a_{\tau i})_{i\in I}}\}_{\tau_0\leq\tau<\omega_\beta} = \{\overline{(b_{\tau i})_{i\in I}}\}_{\tau_0\leq\tau<\omega_\beta}$, then the same argument we used for the well-definedness of $f$, can be applied to show that $f(\{\overline{(a_{\tau i})_{i\in I}}\}_{\tau_0\leq\tau<\omega_\beta}) = f(\{\overline{(b_{\tau i})_{i\in I}}\}_{\tau_0\leq\tau<\omega_\beta})$.□

**Remark.** It follows from Proposition 1 that direct products of algebras with $\alpha$-limits also have $\alpha$-limits.

**Proposition 2.** *Let $\{A_i\}_{i\in I}$ be a family of similar algebras with $\beta$-limits, and let $F$ be an $\omega_{\beta+1}$-complete filter on $I$. Then every $F$-restricted product of the $A_i$'s has $\beta$-limits.*

*Proof.* Let $B$ be an $F$-restricted product of the $A_i$'s, and let $f_i : A_i^{\omega_\beta} \longrightarrow A_i$ be a $\beta$-limit ($i \in I$). Define $f : B^{\omega_\beta} \longrightarrow B$ by $f(\{(a_{\tau i})_{i\in I}\}_{\tau<\omega_\beta}) = (f_i(\{a_{\tau i}\}_{\tau<\omega_\beta}))_{i\in I}$. For each pair $(\sigma, \tau)$ of ordinals less than $\omega_\beta$, denote by $X_{\sigma\tau}$ the set $\{i \in I : a_{\sigma i} = a_{\tau i}\}$. Clearly the set $X = \bigcap_{\sigma,\tau<\omega_\beta} X_{\sigma\tau}$ is a member of $F$, since $F$ is $\omega_{\beta+1}$-complete and $B$ is an $F$-restricted product. For each $i \in X$, set $a_i = a_{\tau i}$ ($\tau < \omega_\beta$). We then have $f_i(\{a_{\tau i}\}_{\tau<\omega_\beta}) = a_i$ and therefore $(f_i(\{a_{\tau i}\}_{\tau<\omega_\beta}))_{i\in I} \in B$. This shows that $f$ is a well-defined mapping. $f$ is also a retraction, as can easily be verified, and if there exists $\tau_0 < \omega_\beta$ such that $(a_{\tau i})_{i\in I} = (b_{\tau i})_{i\in I} \in B$ for each $\tau \geq \tau_0$, then $f_i(\{a_{\tau i}\}_{\tau<\omega_\beta}) = f_i(\{b_{\tau i}\}_{\tau<\omega_\beta})$ for each $i \in I$.□

**Proposition 3**. *Let $\omega_\beta = cf(\omega_\alpha)$. Then an algebra $A$ has $\alpha$-limits if and only if it has $\beta$-limits.*

*Proof.* Let $\omega_\alpha = \bigcup_{\tau<\omega_\beta} \upsilon_\tau$ where $\upsilon_\tau < \omega_\alpha$, so that $\omega_\alpha = \bigcup_{\tau<\omega_\beta} \left((\upsilon_\tau \setminus \bigcup_{\sigma<\tau} \upsilon_\sigma)\right)$. Assume first that $f : A^{\omega_\alpha} \longrightarrow A$ is an $\alpha$-limit, and define a mapping $g : A^{\omega_\beta} \longrightarrow A$ by $g((a_\tau)_{\tau<\omega_\beta}) = f((a'_i)_{i<\omega_\alpha})$, where $a'_i = a_\tau$ if $i \in \upsilon_\tau \setminus \bigcup_{\sigma<\tau} \upsilon_\sigma$. $g$ is clearly a retraction, and if $(a_\tau)_{\tau_0\leq\tau<\omega_\beta} = (b_\tau)_{\tau_0\leq\tau<\omega_\beta}$ for some $\tau < \omega_\beta$, then $a'_i = b'_i$ for all $i \geq \upsilon_{\tau_0}$. Thus $f((a'_i)_{i<\omega_\alpha}) = f((b'_i)_{i<\omega_\alpha})$ and $g$ is a $\beta$-limit. Conversely, if $g : A^{\omega_\beta} \longrightarrow A$ is a $\beta$-limit, define $f : A^{\omega_\alpha} \longrightarrow A$ by $f((a_i)_{i<\omega_\alpha}) = g((a_{\upsilon_\tau})_{\tau<\omega_\beta})$.

It is easily checked that $f$ is a retraction and that if $(a_i)_{i_0 \leq i < \omega_\alpha} = (b_i)_{i_0 \leq i < \omega_\alpha}$ for some $i_0 < \omega_\alpha$, then $f((a_i)_{i<\omega_\alpha}) = f((b_i)_{i<\omega_\alpha})$.□

Given an algebra $A$ and infinite cardinals $\kappa, \lambda$ we shall say that $A$ is $(\kappa, \lambda)$-*compact* if every system of $\lambda$ equations over $A$ all of whose subsystems of size less than $\kappa$ are solvable, is itself solvable. Thus, in particular, $\lambda$-(equationally) compact algebras are precisely the $(\omega, \lambda)$-compact ones. The following result is a universal algebraic counterpart of [8, Corollary 5.2 and Proposition 5.3] .

**Proposition 4.** (Cf. [11, Theorem 2].) *Let $A$ be an algebra.*

*(a) If $A$ has $\alpha$-limits, then it is $(\omega_\alpha, \omega_\alpha)$-compact.*

*(b) If $A$ is $(\omega_\alpha, \lambda)$-compact for all cardinals $\lambda$, then it has $\sigma$-limits for all regular cardinals $\omega_\sigma \geq \omega_\alpha$.*

*Proof.* (a) Let $\{R_j\}_{j<\omega_\alpha}$ be a system of equations over $A$ with a set of unknowns $\{x_s\}_{s\in S}$, and for each ordinal $\tau < \omega_\alpha$, let $\{a_s^\tau\}_{s\in S}$ be a solution to the subsystem $\{R_j\}_{j\leq\tau}$. If $f : A^{\omega_\alpha} \longrightarrow A$ is an $\alpha$-limit, then $\{f((a_s^\tau)_{\tau<\omega_\alpha})\}_{s\in S}$ is a solution of the system $\{R_j\}_{j<\omega_\alpha}$.

(b) It is clear that if $A$ is $(\omega_\alpha, \lambda)$-compact for all cardinals $\lambda$, then it is $(\omega_\sigma, \lambda)$-compact for all $\sigma \geq \alpha$. Hence, assuming without loss of generality that $\omega_\alpha$ is regular, we need only show that $A$ has $\alpha$-limits. By identifying each element of $A$ with its image in $A^{\omega_\alpha}/F_\alpha$ via the diagonal mapping $h : A \longrightarrow A^{\omega_\alpha}/F_\alpha$, we may assume that $A$ is a subalgebra of $A^{\omega_\alpha}/F_\alpha$. Let $\{R_j\}_{j\in J}$ be a system of equations over $A$ with a set of unknowns $\{x_s\}_{s\in S}$ and constants $\{a_c\}_{c\in C}$. Suppose that this system is solvable in $A^{\omega_\alpha}/F_\alpha$ by $\{\overline{(b_{s\tau})_{\tau<\omega_\alpha}}\}_{s\in S}$ and let $J_o$ be a subset of $J$ of size less than $\omega_\alpha$. We claim that $\{R_j\}_{j\in J_0}$ is solvable in $A$. For if $R_{j\tau}$ is the equation obtained from $R_j$ by replacing each constant $a_c$ by its $\tau$ th coordinate in $A$, then the set $X_j = \{\tau < \omega_\alpha : \{b_{s\tau}\}_{s\in S}$ solves $R_{j\tau}\}$ is in $F_\alpha$. As $\omega_\alpha$ is regular, $\bigcap_{j\in J_0} X_j \in F_\alpha$, and hence, choosing $\tau_0$ in $\bigcap_{j\in J_0} X_j$, we obtain that $\{b_{s\tau_0}\}_{s\in S}$ solves $\{R_{j\tau_0}\}_{j\in J_0}$. It is clear now that $\{h(b_{s\tau_0})\}_{s\in S}$ solves $\{R_j\}_{j\in J_0}$. Consider the system over $A$

$$x_{f((b_i)_{i\in r(f)})} = f((x_{b_i})_{i\in r(f)}),\ x_a = a \tag{1}$$

for any $b_i \in A^{\omega_\alpha}/F_\alpha$, $a \in A$, and any operation $f$ on $A^{\omega_\alpha}/F_\alpha$ (with arity $r(f)$), and where the unknowns are indexed by $A^{\omega_\alpha}/F_\alpha$. This system is solvable in $A^{\omega_\alpha}/F_\alpha$ by $x_b = b$ ($b \in A^{\omega_\alpha}/F_\alpha$) and hence, by the claim above, each of its subsystems of cardinality less than $\omega_\alpha$ is solvable in $A$. Since $A$ is $(\omega_\alpha, \lambda)$-compact for all cardinals $\lambda$, it follows that (1) is solvable in $A$ by $x_b = \pi(b)$ ($b \in A^{\omega_\alpha}/F_\alpha$). It is now easy to see that $\pi$ is a homomorphism of $A^{\omega_\alpha}/F_\alpha$ into $A$ and that $\pi h(a) = a$ for all $a$ in $A$. This shows that $A$ is a retract of $A^{\omega_\alpha}/F_\alpha$, and therefore has $\alpha$-limits.□

**Theorem 5**. (Cf. [5] and [6].) *Let $R$ be a $\Sigma$-pure injective commutative ring with $K_i$ ($1 \leq i \leq n$) the residue fields of its local ring factors, and let $\omega_\alpha$ be an infinite regular cardinal $\leq \min\{|K_i| : 1 \leq i \leq n\}$. Then the following statements are equivalent.*

*(i) $R$ is not a serial ring.*

*(ii) There exists an $R$-module $M$ with $\beta$-limits for all $\beta < \alpha$ but without $\alpha$-limits.*

*Proof.* If $R$ is a serial ring then it is an artinian principal ideal ring (since it is perfect) and so every $R$-module is equationally compact, i.e. has $\beta$-limits for all ordinals $\beta$. We therefore have (ii)⇒(i). Suppose now that (i) holds. Without loss of generality, we may assume that $R$ is local with maximal ideal $J$, and that there are elements $u$, $v$ in $R$ with $u \notin (v)$ and $v \notin (u)$. Since $|R/J| \geq \omega_\alpha$, there exists a subset $H$ of $R\backslash J$ with $|H| = \omega_\alpha$, whose elements are distinct modulo $J$. For each $h \in H$, let $M_h = R/(u - hv)$, and let $M$ be the filter sum

$$\sum_{F_\alpha} M_h = \{m \in \prod_{h\in H} M_h : \{h \in H : m(h) = 0\} \in F_\alpha\}.$$

Since $\omega_\alpha$ is regular, $F_\alpha$ is $\omega_\alpha$-complete and $M$ is an $F_\alpha$-restricted product of the $M_h$'s. By hypothesis, $R$ is equationally compact, and hence so is each cyclic module $M_h$. Proposition 2 now implies that $M$ has $\beta$-limits for all $\beta < \alpha$. But $F_\alpha$ is not a principal filter, and so $M$ cannot be $\omega_\alpha$-compact by [5, Lemma 1], hence by Proposition 4, $M$ does not have $\alpha$-limits.□

**Remark.** As in the case of [5, Lemma 1], one can easily modify the proof of Theorem 5 and show that it holds for the more general $\Sigma$-pure injective duo rings. (A ring is duo if all its one-sided ideals are two-sided.)

Combining Proposition 3 and Theorem 5 we obtain

**Corollary 6.** *A cardinal $\omega_\alpha$ is regular if and only if there exists a module with $\beta$-limits for all $\beta < \alpha$ but without $\alpha$-limits.*□

In [11] Wenzel conjectured that there exist algebras of size $\omega_\alpha$ and with $\alpha$-limits but without $\beta$-limits for some $\beta < \alpha$, and in [1] Bulman-Fleming and Taylor provided examples of unary algebras satisfying Wenzel's conjecture. Along this direction, we apply our results to construct an infinite cardinal $\omega_\alpha$ and a *module* of size $\omega_\alpha$ with $\alpha$-limits but without $n$-limits for any positive integer $n$.

**Construction.** Let $\sigma$ be an infinite ordinal and let $R$ be a commutative artinian local ring that is not a principal ideal ring of size $|R| \geq \omega_\sigma$. (Such a ring is for example the $K$-algebra $R$ generated by $\{1, a, b\}$ where $K$ is a field of size $\omega_\sigma$ and $a^2 = b^2 = ab = ba = 0$; see [13].) By Theorem 5, for each positive integer $n$ there exists an $R$-module $M_n$ with 0-limits but without $n$-limits. Let $M = \prod_{n\in\mathbb{N}} M_n$ and let $\alpha$ be the ordinal sum $|M| + \omega_\sigma + \omega$. By Proposition 3, and since $cf(\omega_\alpha) = \omega$, $M$ has $\alpha$-limits. The free module $R^{(\omega_\alpha)}$ is equationally compact (since $R$ is artinian), and this implies that the module $N = M \times R^{(\omega_\alpha)}$, of cardinality $\omega_\alpha$, has $\alpha$-limits but no $n$-limits for any positive integer $n$.

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