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Nonexistence in the large of solutions to a nonlinear homogeneous wave equation

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Abstract
We consider a special type of quasilinear wave equation and show that classical solutions blow up in finite time even for small initial data.
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1. Introduction.

In this work, we are concerned with a one-dimensional first order quasilinear hyperbolic system of the form

$$\begin{cases} u_t(x,t) = a\left(\dfrac{u(x,t)}{v(x,t)}\right) v_x(x,t) \\ v_t(x,t) = u_x(x,t) \end{cases} \tag{1.1}$$

where a subscript denotes partial derivative with respect to the relevant variable; $x \in I = (0,1)$, and $t > 0$.

It is well known that, generally, classical solutions for such systems break down in finite time, even for smooth and small initial data. For instance, Lax [5] and MacCamy and Mizel [6] studied the system for a depending on v only. They showed that the solutions blow up in finite time even if the initial data are smooth and small. Note in this particular case, the system is reduced to the nonlinear wave equation. For systems with dissipation, similar results have been established by Slemrod [9], Kosinski [4] and Messaoudi [7] .

It is worth mentioning that global existence for the system considered in [9] has been established by Nishida [8]. Also, Aregba and Hanouzet [1] and Tartar [10] have considered a class of semilinear hyperbolic systems and proved some global existence and blow-up results.

In this paper, we study the system (1.1) together with initial and boundary conditions and prove a blow up result.

2. Local Existence.

We consider the following problem

$$u_t(x,t) = a\left(\frac{u(x,t)}{v(x,t)}\right) v_x(x,t) \tag{2.1}$$

$$v_t(x,t) = u_x(x,t), \qquad \forall\, x \in I = [0,\ 1),\ \ t \geq 0 \tag{2.2}$$

$$u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), \qquad \forall\, x \in I \tag{2.3}$$

$$u(0,t) = u(1,t) = 0,\ v_x(0,t) = v_x(1,t) = 0, \quad t \geq 0 \tag{2.4}$$

where a is a function satifying

$$a(\xi) \geq a_0 > 0, \quad \forall\, \xi \in \mathbb{R} \tag{2.5}$$

and the initial data satisfy

$$u_0 \in H^2(I) \cap H_0^1(I), \quad v_0 \in H^2(I),\ v_0(x) \neq 0,\ \forall\, x \in I. \tag{2.6}$$

Proposition. *Assume that a is a C^2 function satifying (2.5) and let u_0 and v_0 be given and satisfying (2.6). Then the problem (2.1) – (2.4) has a unique local solution (u, v), on a maximal time interval $[0, T)$, satisfying*

$$u, v \in C\left([0,T),\ H^2(I)\right) \cap C^1\left([0,T),\ H^1(I)\right). \tag{2.7}$$

This result can be proved by either using a classical energy argument [2] or the nonlinear semigroup theory [3].

Remark 2.1. u, v are in $C^1(I \times [0, T))$ by the Sobolev embedding theorem.

Remark 2.2. The local existence can be obtained even if (2.5) holds in a neighbourhood of zero. In this case, we have to be careful with the choice of the initial data.

3. Formation of singularities.

In this section, we state and prove our main result. We first begin with a lemma that gives a uniform bound on u/v in terms of the initial data.

Lemma. *Let a be as in the proposition. Then there exist initial data satisfying (2.6), for wich $|u(x,t)/v(x,t)|$ remains uniformly bounded on $I \times [0, T)$.*

Proof. We define the quantities

$$r(x,t) := \ln|v(x,t)| + \int_0^{\frac{u(x,t)}{v(x,t)}} \alpha(\xi)d\xi, \quad s(x,t) := \ln|v(x,t)| - \int_0^{\frac{u(x,t)}{v(x,t)}} \beta(\xi)d\xi, \tag{3.1}$$

and the differential operators :

$$\partial_t := \frac{\partial}{\partial t} - \rho\left(\frac{u(x,t)}{v(x,t)}\right)\frac{\partial}{\partial x}, \quad D_t := \frac{\partial}{\partial t} + \rho\left(\frac{u(x,t)}{v(x,t)}\right)\frac{\partial}{\partial x}. \tag{3.2}$$

We then choose α, β, and ρ in such a way that

$$\partial_t r = D_t s = 0. \tag{3.3}$$

Straightforward calculations then yield

$$\rho(\xi) = \sqrt{a(\xi)}, \quad \alpha(\xi) = \frac{1}{\sqrt{a(\xi)} + \xi}, \quad \beta(\xi) = \frac{1}{\sqrt{a(\xi)} - \xi}. \tag{3.4}$$

Therefore as long as a smooth solution continues to exist and

$$\left|\frac{u(x,t)}{v(x,t)}\right| < \sqrt{a(\frac{u(x,t)}{v(x,t)})}, \tag{3.5}$$

r and s remain constant along backward and forward characteristics, respectively; hence

$$\| r \|_\infty = \| r_0 \|_\infty, \quad \| s \|_\infty = \| s_0 \|_\infty. \tag{3.6}$$

To establish (3.5), we need to choose the initial data in a conviennient way. To this end we note, by (3.1), that

$$r - s = \phi\left(\frac{u(x,t)}{v(x,t)}\right), \tag{3.7}$$

where

$$\phi(\tau) = \int_0^\tau \frac{2\sqrt{a(\xi)}d\xi}{a(\xi) - \xi^2} \tag{3.8}$$

is continuous and admits a continuous inverse ψ, at least in a neighbouhood of zero. By noting that the function $g(\xi) = a(\xi) - \xi^2$ is continuous and $g(0) > a_0$,one can choose $\varepsilon > 0$ such that $g(\xi) > a_0/2$, for all $|\xi| < \varepsilon$. We then choose $\delta > 0$ so that $|\psi(\xi)| < \varepsilon$, for all $|\xi| < \delta$. Therefore as long as $(r - s) < \delta$, we have

$$\left|\frac{u}{v}\right| = |\psi(r-s)| < \varepsilon. \tag{3.9}$$

Consequently, by choosing u_0 and v_0 such that

$$||r_0||_\infty + ||s_0||_\infty < \delta, \tag{3.10}$$

the relation (3.5) holds and the proof of the lemma is completed.

Theorem. *Assume that, in addition to (2.5), a satisfies*

$$a'(0) > 0. \tag{3.11}$$

Then there exist initial data u_0, v_0 satifying (2.6), for which the solution of the problem (2.1) – (2.4) blows up in finite time.

Proof. We take an x-partial derivative of (3.3) to get

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x \rho_x = 0 \tag{3.12}$$

which, in turn, implies

$$\partial_t(r_x) = r_x \rho_x = \frac{a'}{2\sqrt{a}} \frac{vu_x - uv_x}{v^2} r_x. \tag{3.13}$$

By using

$$r_x = \frac{v_x}{v} + \alpha\left(\frac{u}{v}\right)\frac{vu_x - uv_x}{v^2}, \qquad s_x = \frac{v_x}{v} - \beta\left(\frac{u}{v}\right)\frac{vu_x - uv_x}{v^2} \tag{3.14}$$

and substituting in (3.13), we obtain

$$\partial_t r_x = \frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right) - \left(\frac{u}{v}\right)^2]\, r_x^2 - \frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right) - \left(\frac{u}{v}\right)^2] r_x s_x. \tag{3.15}$$

To handle the last term in (3.15), we set

$$W := \lambda\left(\frac{u}{v}\right).r_x \tag{3.16}$$

and substitute in (3.15), to get

$$\partial_t W = \lambda\left(\frac{u}{v}\right)\frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right)-\left(\frac{u}{v}\right)^2]\, r_x^2 - \lambda\left(\frac{u}{v}\right)\frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right)-\left(\frac{u}{v}\right)^2] r_x s_x$$

$$+ r_x\lambda'\left(\frac{u}{v}\right)\partial_t\left(\frac{u}{v}\right) \tag{3.17}$$

By using the equations, we estimate :

$$\partial_t\left(\frac{u}{v}\right) = \frac{v(u_t-\sqrt{a}u_x)-u(v_t-\sqrt{a}v_x)}{v^2}$$

$$= \frac{v(av_x-\sqrt{a}u_x)-u(u_x-\sqrt{a}v_x)}{v^2} = \frac{(\sqrt{a}v_x-u_x)(u+\sqrt{a}v)}{v^2}$$

At this point, we choose λ so that

$$-\lambda\left(\frac{u}{v}\right)\frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right)-\left(\frac{u}{v}\right)^2] r_x s_x + r_x\lambda'\left(\frac{u}{v}\right)\frac{(\sqrt{a}v_x-u_x)(u+\sqrt{a}v)}{v^2} = 0. \tag{3.18}$$

By using the fact that

$$s_x = \frac{\beta}{v}(\sqrt{a}v_x-u_x) = \frac{1}{v}\frac{1}{\sqrt{a\left(\frac{u}{v}\right)-\left(\frac{u}{v}\right)}}(\sqrt{a}v_x-u_x) \tag{3.19}$$

and substituting in (3.18) we arrive, by simple computations, at

$$\frac{\lambda'\left(\frac{u}{v}\right)}{\lambda\left(\frac{u}{v}\right)} = \frac{a'\left(\frac{u}{v}\right)}{4a\left(\frac{u}{v}\right)}, \tag{3.20}$$

which yields, by a direct integration,

$$\lambda(\xi) = a^{1/4}(\xi); \tag{3.21}$$

consequently (3.17) reduces to

$$\partial_t W = \frac{a'\left(\frac{u}{v}\right)}{4\lambda\left(\frac{u}{v}\right)a\left(\frac{u}{v}\right)}[a\left(\frac{u}{v}\right) - \left(\frac{u}{v}\right)^2]\, W^2. \tag{3.22}$$

Therefore W (hence r_x) blows up in a time, if we choose initial data satisfying (3.10) with derivatives satisfying

$$\frac{v_0'}{v_0} + \alpha\left(\frac{u_0}{v_0}\right)\frac{u_0'v_0 - u_0v_0'}{v^2} > 0 \tag{3.23}$$

Remark 3.1. Similar result can be established for $a'(0) < 0$. In this case consider the evolution of s_x on the forward characteristics.

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