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Blow up in the solutions of an equation describing a transverse motion of a nonhomogeneous string

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Abstract

We prove a blow up result for the equation $w_{tt} = a(w_t)\varphi(w_x)w_{xx}$, which can be regarded as a model for a transverse motion of a string with a density depending on the velocity.

Keywords : Wave equation, Smooth solutions, Singularities, Global solutions, Nonlinear response, Blow up.

1 Introduction.

The aim of this paper is to study the existence and nonexistence of classical solutions to the one-dimensional nonlinear equation of the form

$$w_{tt}(x,t) = a(w_t(x,t))b(w_x(x,t))w_{xx}(x,t), \tag{1.1}$$

where $x \in I$ (bounded or unbounded interval), $t \geq 0$. This equation can be regarded as a model for a transverse motion of a nonhomogeneous vibrating string, where the density is depending of the velocity w_t. By assuming that

$$a(\xi) > 0, \qquad b(\xi) > 0, \quad \forall\, \xi \in \mathbb{R}, \tag{1.2}$$

the equation (1.1) is strictly hyperbolic.

Generally, classical solutions of problems associated with (1.1) develop singularities in finite time, if the elastic response function a and b satisfy some 'genuine' nonlinearity conditions. For $a \equiv 1$, Lax [5] and MacCamy and Mizel [9] studied the problem and showed that classical solutions break down in finite time even for smooth and small initial data. In his work, Lax assumed that b' does not change sign, whereas MacCamy and Mizel allowed b' to change sign. They also showed, under appropriate conditions on b, that

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intervals of x can exist, in which the solution must exist for all time t even though it breaks down for values x outside these intervals. For a depending on x, Messaoudi [11] discussed the Cauchy problem and established a blow up result for smooth initial data.

In the dissipative case, the situation is different. For initial data small and smooth enough, the effect of the damping term dominates the nonlinear elastic response and global solutions can be obtained (see [12]). However, for large initial data the nonlinearity in the elastic response takes over and classical solutions may develop singularities in finite time. These results have been established by several authors (see [4], [10], [14]).

It is interesting to mention that nonlinear hyperbolic systems, of which equation (1.1) with $a \equiv 1$ is a special case, have attracted the attention of many authors and several results concerning global existence and blow up have been established (see [6], [7], [8], [13]).

This work will be divided into two parts. In the first part we state, without proof, a local existence theorem. In the second part, we state and prove our main blow up result.

2 Local Existence.

In this section we state a local existence theorem. The proof is omitted. It can be easily established by either using a classical energy argument [1], or applying the nonlinear semigroup theory presented in [2]. We set

$$u(x,t) := w_t(x,t), \qquad v(x,t) := w_x(x,t)$$

and substitute in (1.1) to get the system

$$\begin{aligned} u_t(x,t) &= a(u(x,t))\, b(v(x,t)) v_x(x,t) \\ v_t(x,t) &= u_x(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0. \end{aligned} \tag{2.1}$$

We consider (2.1) together with the initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

In order to state the local existence result, we make the following hypotheses

(H1) a and $b \in C^2(\mathbb{R})$

(H2) $a(\xi) \geq \alpha, \quad b(\xi) \geq \alpha, \quad \xi \in \mathbb{R}, \quad \alpha > 0.$

Proposition. *Assume that (H1), (H2) hold and let u_0, v_0 in $H^2(\mathbb{R})$ be given. Then the initial value problem (2.1) , (2.2) has a unique local solution (u, v) defined on a maximal time interval $[0,T)$ such that*

$$u, v \in C([0,T); H^2(\mathbb{R})) \cap C^1([0,T); H^1(\mathbb{R})). \tag{2.3}$$

Remark 2.1. The Sobolev embedding theorem implies that u, v are C^1 functions on $\mathbb{R} \times [0,T)$. Hence (u,v) is a classical solution.
Remark 2.2. If φ is a C^{k+1} function and $u_0, v_0 \in H^k(\mathbb{R})$, then $u(\cdot,t), v(\cdot,t) \in H^k(\mathbb{R}),\ k \geq 1$.
Remark 2.3. A similar result can be obtained, if (H1) and (H2) holds only in a neighbourhood of zero. In this case, we have to be carefull with the choice of the initial data.

3 Formation of singularities.

In this section, we state and prove our main result. We first start with establishing uniform bounds on the solution (u,v) in terms of the initial data.
Lemma. *Assume (H1), (H2) hold. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that given any u_0, v_0 in $H^2(\mathbb{R})$ satisfying*

$$|u_0(x)| < \delta, \qquad |v_0(x)| < \delta, \quad \forall x \in \mathbb{R}, \tag{3.1}$$

the solution (2.3) obeys

$$|u(x,t)| < \varepsilon, \qquad |v(x,t)| < \varepsilon, \quad \forall x \in \mathbb{R}, \quad t \in [0,T). \tag{3.2}$$

Proof. We define the quantities

$$\begin{aligned} r(x,t) :=& \quad A(u(x,t)) + B(v(x,t)) \\ s(x,t) :=& \quad A(u(x,t)) - B(v(x,t)) \end{aligned} \tag{3.3}$$

where

$$A(z) = \int_0^z \sqrt{\frac{1}{a(\xi)}} d\xi, \quad B(z) = \int_0^z \sqrt{b(\xi)} d\xi, \tag{3.4}$$

and the differential operators

$$\begin{aligned} \partial_t^- :=& \quad \frac{\partial}{\partial t} - \rho(x,t)\frac{\partial}{\partial x} \\ \partial_t^+ :=& \quad \frac{\partial}{\partial t} + \rho(x,t)\frac{\partial}{\partial x}, \end{aligned} \tag{3.5}$$

where

$$\rho(x,t) = \sqrt{a(u(x,t))b(v(x,t))}. \tag{3.6}$$

Straightforward computations yield

$$\partial_t^- r = \partial_t^+ s = 0. \tag{3.7}$$

Therefore as long as a smooth solution continues to exist, r and s remain constant along backward and forward characteristics respectively; hence

$$\| r \|_\infty = \| r_0 \|_\infty, \quad \| s \|_\infty = \| s_0 \|_\infty. \tag{3.8}$$

By using (3.3), (H1), and (H2), the lemma is established.
Theorem. *Assume that, in addition to (H1 and (H2), a and b satisfy*

$$\frac{a'(0)}{\sqrt{a(0)}} + \frac{b'(0)}{b(0)\sqrt{b(0)}} > 0. \tag{3.9}$$

Then there exist initial data u_0, v_0 satisfying (2.3), for which the solution of the problem (2.1), (2.2) blows up in finite time.
Proof. We take an x-partial derivative of (3.6) to get

$$(\partial_t^- r)_x = r_{xt} - \rho r_{xx} - r_x \rho_x = 0 \tag{3.10}$$

which implies

$$\partial_t^- (r_x) = r_x \rho_x = \frac{[a(u)b(v)]_x}{2\sqrt{a(u)b(v)}}\, r_x = \frac{a'(u)b(v)u_x + a(u)b'(v)v_x}{2\sqrt{a(u)b(v)}}\, r_x. \tag{3.11}$$

By using

$$r_x = \frac{u_x}{\sqrt{a(u)}} + \sqrt{b(v)}\; v_x, \quad s_x = \frac{u_x}{\sqrt{a(u)}} - \sqrt{b(v)}\; v_x \tag{3.12}$$

and substituting in (3.11) we obtain, by direct calculations,

$$\partial_t^- r_x = \frac{1}{4}\sqrt{ab}\left(\frac{a'}{\sqrt{a}} + \frac{b'}{b\sqrt{b}}\right)\, r_x^2 + \frac{1}{4}\sqrt{ab}\left(\frac{a'}{\sqrt{a}} - \frac{b'}{b\sqrt{b}}\right) r_x s_x. \tag{3.13}$$

To handle the last term in (3.13), we set

$$W := \alpha\,(u)\,\beta(v) r_x, \tag{3.14}$$

for α and β to be chosen suitably; thus we have

$$\partial_t^- W = \frac{1}{4}\alpha\,(u)\,\beta(v)\sqrt{ab}\left(\frac{a'}{\sqrt{a}} + \frac{b'}{b\sqrt{b}}\right)\, r_x^2 + \frac{1}{4}\alpha\,(u)\,\beta(v)\sqrt{ab}\left(\frac{a'}{\sqrt{a}} - \frac{b'}{b\sqrt{b}}\right) r_x s_x$$

$$+ r_x\left(\alpha'\beta u_t + \alpha\beta' v_t - \sqrt{ab}\alpha'\beta u_x - \sqrt{ab}\alpha\beta' v_x\right). \tag{3.15}$$

At this point, we choose α and β so that

$$\frac{1}{4}\alpha\beta\sqrt{ab}\left(\frac{a'}{\sqrt{a}}-\frac{b'}{b\sqrt{b}}\right)r_x s_x + r_x\left(\alpha'\beta u_t+\alpha\beta' v_t\right)$$

$$-r_x\left(\sqrt{ab}\alpha'\beta u_x-\sqrt{ab}\alpha\beta' v_x\right)=0. \tag{3.16}$$

By using (3.12) to substitute for s_x and (2.1), (2.2) to substitute for u_t and v_t we arrive, by simple computations, at

$$\frac{1}{4}\alpha\beta\sqrt{ab}\left(\frac{a'}{\sqrt{a}}-\frac{b'}{b\sqrt{b}}\right)\left(\frac{u_x}{\sqrt{a}}-\sqrt{b}\,v_x\right)r_x+\Big((\alpha'\beta ab-\alpha\beta'\sqrt{ab})v_x$$

$$+(\alpha\beta'-\alpha'\beta\sqrt{ab})u_x\Big)r_x=0, \tag{3.17}$$

which yields

$$\frac{1}{4}\alpha\beta\sqrt{ab}\left(\frac{a'}{\sqrt{a}}-\frac{b'}{b\sqrt{b}}\right)\left(\frac{u_x}{\sqrt{a}}-\sqrt{b}\,v_x\right) \tag{3.18}$$

$$+\sqrt{a}\left((\alpha\beta'-\alpha'\beta\sqrt{ab})\right)\left(\frac{u_x}{\sqrt{a}}-\sqrt{b}\,v_x\right)=0$$

hence

$$\frac{1}{4}\alpha\beta\sqrt{ab}\left(\frac{a'}{\sqrt{a}}-\frac{b'}{b\sqrt{b}}\right)+\sqrt{a}\left((\alpha\beta'-\alpha'\beta\sqrt{ab})\right)=0. \tag{3.19}$$

By using the fact that a and α depend on u only and b and β depend on v only, we get

$$\frac{1}{4}\frac{a'}{a}=\frac{\alpha'}{\alpha},\quad \frac{1}{4}\frac{b'}{b}=\frac{\beta'}{\beta}. \tag{3.20}$$

Direct integration then yields

$$\alpha(u)=a^{1/4}(u),\qquad \beta(v)=b^{1/4}(v)$$

Consequently (3.13) is reduced to

$$\partial_t^- W=\frac{1}{4}a^{1/4}b^{1/4}\left(\frac{a'}{\sqrt{a}}+\frac{b'}{b\sqrt{b}}\right)W^2. \tag{3.21}$$

Therefore W (hence r_x) blows up in a time, if we choose initial data small enough in L^∞ norm with derivatives satisfying

$$\frac{u_0'}{\sqrt{a(u_0)}}+\sqrt{b(v_0)}v_0'>0 \tag{3.22}$$

Remark 3.1. Similar result can be established for $\frac{a'}{\sqrt{a}}+\frac{b'}{b\sqrt{b}}<0$. In this case

consider the evolution of s_x on the forward characteristics.
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