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**Eigenvalues of Sturm-Liouville Problems with
Coupled Self-Adjoint Boundary Conditions**

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with Coupled Self-Adjoint Boundary Conditions

by

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Abstract

This paper is a sequel to our earlier paper on high order approximations of eigenvalues of regular Sturm-Liouville Problems with separable boundary conditions. We shall extend the approach based on sampling theory to the case of real coupled self-adjoint boundary conditions.

keywords Sturm-Liouville problems, Whittaker-Shannon-Kotel'nikov Sampling theorem, Eigenvalues, Coupled Boundary Conditions, Periodic and Semi-Periodic Sturm-Liouville problems.

Mathematics Subject Classification: 34L15

1. Introduction

Sampling theory [23] has been used in [6] to compute the Dirichlet eigenvalues of Sturm-Liouville problems. This approach has been extended in [9] to include general separable boundary conditions. In [10], high order approximations of the eigenvalues of regular SL problems were introduced together with very sharp error estimates of the eigenvalues. We shall extend this approach to include the case of real coupled self-adjoint boundary conditions for which periodic and semi-periodic BC constitute particular cases. Thus we are considering the Sturm-Liouville system

$$\begin{cases} -y'' + q(x)y = \lambda y & , \quad x \in [0, \pi] \\ \begin{pmatrix} y(\pi, \lambda) \\ y'(\pi, \lambda) \end{pmatrix} = K \begin{pmatrix} y(0, \lambda) \\ y'(0, \lambda) \end{pmatrix} \end{cases} \quad (1.1)$$

where q is a real valued function satisfying $q \in L^1_{loc}(0, \pi)$ and $K \in SL_2(R)$, that is K is a real 2 by 2 matrix whose determinant is 1. The special cases $K = I$ and $K = -I$ correspond to the periodic and semi-periodic problems respectively. It is well known [19] that the eigenvalues may be simple or double and can be ordered to satisfy $-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$.

We are interested in computing the positive eigenvalues; all the negative ones (a finite number of them) can be obtained by a simple shift in λ [10].

2. Main Results

Let $\lambda = \mu^2$ and $y_1(x, \mu^2)$ and $y_2(x, \mu^2)$ denote the solutions of the initial value problems

$$\begin{cases} -y_1'' + q(x)y_1 = \mu^2 y_1 \\ y_1(0, \mu^2) = 1 \\ y_1'(0, \mu^2) = 0 \end{cases} \quad (2.1)$$

and

$$\begin{cases} -y_2'' + q(x)y_2 = \mu^2 y_2 \\ y_2(0, \mu^2) = 0 \\ y_2'(0, \mu^2) = 1 \end{cases} \quad (2.2)$$

respectively. The general solution of

$$\begin{cases} -y'' + q(x)y = \mu^2 y \\ y(0, \mu^2) = \alpha \\ y'(0, \mu^2) = \beta \end{cases} \quad (2.3)$$

can be written as $y = \alpha y_1 + \beta y_2$ so that at $x = \pi$, we get

$$\begin{pmatrix} y(\pi, \mu^2) \\ y'(\pi, \mu^2) \end{pmatrix} = \begin{pmatrix} \alpha y_1(\pi, \mu^2) + \beta y_2(\pi, \mu^2) \\ \alpha y_1'(\pi, \mu^2) + \beta y_2'(\pi, \mu^2) \end{pmatrix} = K \begin{pmatrix} y(0, \mu^2) \\ y'(0, \mu^2) \end{pmatrix} = K \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (2.4)$$

that is,

$$(W(\pi, \mu^2) - K) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.5)$$

where $W(\pi, \mu^2)$ is the wronskian matrix of y_1, y_2 evaluated at $x = \pi$. It follows that

$$B(\pi, \mu) = 0 \quad (2.6)$$

where $B(x, \mu) := \det[W(x, \mu^2) - K]$. We shall call the function $B(\pi, \cdot)$ the boundary function.

The zeroes of $B(\pi, \cdot)$ are the square root of the sought eigenvalues. Using the fact $|K| = 1$

and the fact $|W(\pi, \mu^2)| = 1$, simple calculations show that the above expression for $B(\pi, \mu)$

simplifies to

$$B(\pi, \mu) := k_{12}y_1'(\pi, \mu^2) - k_{22}y_1(\pi, \mu^2) + k_{21}y_2(\pi, \mu^2) - k_{11}y_2'(\pi, \mu^2) + 2 \quad (2.7)$$

Generally $B(\pi, \cdot) \notin PW_\pi$, PW_x being the Paley-Wiener space

$$PW_x = \{f \text{ entire, } |f(\mu)| \leq C e^{x|\operatorname{Im}\mu|}, \int_{\mathbb{R}} |f(\mu)|^2 d\mu < \infty\}$$

However, an immediate consequence of Theorem 1 in [10] for the initial value problems (2.1), (2.2) gives,

Corollary 1 Let

$$v_{1,i}^{[0]}(x, \mu) = y_i(x, \mu) \quad , \quad i = 1, 2$$

$$v_{2,1}^{[0]}(x, \mu) = y_1'(x, \mu) + \mu \sin \mu x$$

$$v_{2,2}^{[0]}(x, \mu) = y_2'(x, \mu) - \cos \mu x$$

$$\varphi_{0,1}(x, \mu) = \cos \mu x$$

$$\varphi_{0,2}(x, \mu) = \frac{\sin \mu x}{\mu}$$

and

$$\varphi_{n,i}(x, \mu) = \int_0^x q(t) \frac{\sin \mu(x-t)}{\mu} \varphi_{n-1,i}(t, \mu) dt \quad , \quad i = 1, 2$$

$$v_{1,i}^{[n]}(x, \mu) = v_{1,i}^{[n-1]}(x, \mu) - \varphi_{n-1,i}(x, \mu) \quad , \quad i = 1, 2$$

$$v_{2,i}^{[n]}(x, \mu) = v_{2,i}^{[n-1]}(x, \mu) - \int_0^x q(t) \cos \mu(x-t) \varphi_{n-1,i}(t, \mu) dt \quad , \quad i = 1, 2$$

$$\tilde{B}^{[n]}(x, \mu) = k_{12}v_{2,1}^{[n]}(x, \mu) - k_{22}v_{1,1}^{[n]}(x, \mu) + k_{21}v_{1,2}^{[n]}(x, \mu) - k_{11}v_{2,2}^{[n]}(x, \mu)$$

for $n \geq 1$. Then

$$\begin{aligned} \varphi_{n,i}(x, \mu), v_{1,i}^{[n]}(x, \mu), v_{2,i}^{[n]}(x, \mu), \tilde{B}^{[n]}(x, \mu) &\in PW_x, \quad i = 1, 2 \\ \mu^{n-1}\varphi_{n,i}(\pi, \mu), \mu^{n-1}v_{1,i}^{[n]}(\pi, \mu), \mu^{n-1}v_{2,i}^{[n]}(\pi, \mu), \mu^{n-1}\tilde{B}^{[n]}(\pi, \mu) &\in L^2(-\infty, \infty), \quad i = 1, 2 \end{aligned}$$

for $n \geq 1$. Furthermore, we have the following estimates

$$\begin{aligned} |\varphi_{n,i}(x, \mu)| &\leq c_{4,i} \left(\frac{c_5}{1 + |\mu|\pi} \right)^n e^{x|\operatorname{Im}\mu|}, \quad i = 1, 2 \\ |v_{1,i}^{[n]}(x, \mu)| &\leq c_{1,i} \left(\frac{c_5}{1 + |\mu|\pi} \right)^n e^{x|\operatorname{Im}\mu|}, \quad i = 1, 2 \\ |v_{2,i}^{[n]}(x, \mu)| &\leq c_{2,i} \left(\frac{c_5}{1 + |\mu|\pi} \right)^n e^{x|\operatorname{Im}\mu|}, \quad i = 1, 2 \\ |\tilde{B}^{[n]}(x, \mu)| &\leq c_3 \left(\frac{c_5}{1 + |\mu|\pi} \right)^n e^{x|\operatorname{Im}\mu|} \end{aligned}$$

where

$$\begin{aligned} c_{4,1} &= 1, \quad c_{4,2} = c_0\pi, \quad c_5 = \pi c_0 \int_0^\pi |q(t)| dt, \quad c_{1,i} = c_{4,i} \exp c_5, \\ c_{2,i} &= c_{1,i} \int_0^\pi |q(t)| dt, \quad c_3 = |k_{12}|c_{2,1} + |k_{22}|c_{1,1} + |k_{21}|c_{1,2} + |k_{11}|c_{2,2} \end{aligned}$$

$i = 1, 2$.

Now recall the well-known

Theorem 1 (Whittaker-Shannon-Kotel'nikov) [23] *Let $f \in PW_\pi$ then*

$$f(\mu) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(\mu - k)}{\pi(\mu - k)}$$

where the series converges uniformly on compact set of \mathbb{R} and also in $L^2_{d\mu}$.

Since $v_{1,i}^{[n]}(\pi, \mu), v_{2,i}^{[n]}(\pi, \mu), \tilde{B}^{[n]}(\pi, \mu) \in PW_\pi$, the above theorem is applicable and we can reconstruct $v_{1,i}^{[n]}(\pi, \mu), v_{2,i}^{[n]}(\pi, \mu)$ for $i = 1, 2$ and $\tilde{B}^{[n]}(\pi, \mu)$ from their samples $v_{1,i}^{[n]}(\pi, k), v_{2,i}^{[n]}(\pi, k)$

for $i = 1, 2$ and $\tilde{B}^{[n]}(\pi, k)$, $k \in Z$ for any given n .

Once we reconstruct $\tilde{B}^{[n]}(\pi, \mu)$, we obtain

$$\begin{aligned}
B(\pi, \mu) &= \tilde{B}^{[n]}(\pi, \mu) + k_{12} \left\{ -\mu \sin \mu\pi + \int_0^\pi q(t) \cos \mu(\pi - t) \sum_{k=0}^{n-1} \varphi_{k,1}(t, \mu) dt \right\} \\
&\quad - k_{22} \left\{ \sum_{k=0}^{n-1} \varphi_{k,1}(\pi, \mu) \right\} + k_{21} \left\{ \sum_{k=0}^{n-1} \varphi_{k,2}(\pi, \mu) \right\} \\
&\quad - k_{11} \left\{ \cos \mu\pi + \int_0^\pi q(t) \cos \mu(\pi - t) \sum_{k=0}^{n-1} \varphi_{k,2}(t, \mu) dt \right\} + 2 \tag{2.8}
\end{aligned}$$

the zeroes of which are the square root of the sought eigenvalues.

For a given n , let $\tilde{B}_N^{[n]}(\pi, \mu)$ denote the truncation of $\tilde{B}^{[n]}(\pi, \mu)$

$$\tilde{B}_N^{[n]}(\pi, \mu) = \sum_{k=-N}^N \tilde{B}^{[n]}(\pi, k) \frac{\sin \pi(\mu - k)}{\pi(\mu - k)} \tag{2.9}$$

and $B_N^{[n]}(\pi, \mu)$ the corresponding approximation to $B^{[n]}(\pi, \mu)$. Since $\mu^{n-1} \tilde{B}^{[n]}(\pi, \mu) \in L^2(-\infty, \infty)$

Jagerman's result (see [23], Theorem 3.21, p.90) is applicable and yields the very sharp estimate

Lemma 1 Truncation error

$$|B^{[n]}(\pi, \mu) - B_N^{[n]}(\pi, \mu)| \leq \frac{|\sin \pi\mu| c_6}{\pi\sqrt{1-4^{-n+1}}} \left[\frac{1}{\sqrt{N-\mu}} + \frac{1}{\sqrt{N+\mu}} \right] \frac{1}{(N+1)^{n-1}}, \text{ for } |\mu| < N$$

where $c_6 = \|\mu^{n-1} \tilde{B}^{[n]}(\pi, \mu)\|_2$.

As in [10], if $\bar{\mu}^2$ is an exact eigenvalue and μ_N^2 is an approximation obtained as a square of a zero of $B_N^{[n]}$ then

$$\begin{aligned}
|B^{[n]}(\pi, \bar{\mu}) - B^{[n]}(\pi, \mu_N)| &= |B_N^{[n]}(\pi, \mu_N) - B^{[n]}(\pi, \mu_N)| \\
&\leq \frac{|\sin \pi\mu| c_6}{\pi\sqrt{1-4^{-n+1}}} \left[\frac{1}{\sqrt{N-\mu}} + \frac{1}{\sqrt{N+\mu}} \right] \frac{1}{(N+1)^{n-1}}
\end{aligned}$$

from which we get the following

Lemma 2 Error bounds

- Simple eigenvalue

$$|\bar{\mu} - \mu_N| \leq \frac{1}{\inf_{\bar{\mu} \leq \tilde{\mu} \leq \mu_N} |B^{[n]'}(\pi, \tilde{\mu})|} \frac{|\sin \pi \mu_N| c_6}{\pi \sqrt{1 - 4^{-n+1}}} \left[\frac{1}{\sqrt{N - \mu_N}} + \frac{1}{\sqrt{N + \mu_N}} \right] \frac{1}{(N + 1)^{n-1}}$$

- Double eigenvalue

$$|\bar{\mu} - \mu_N| \leq \left\{ \frac{2}{\inf_{\bar{\mu} \leq \tilde{\mu} \leq \mu_N} |B^{[n]''}(\pi, \tilde{\mu})|} \frac{|\sin \pi \mu_N| c_6}{\pi \sqrt{1 - 4^{-n+1}}} \left[\frac{1}{\sqrt{N - \mu_N}} + \frac{1}{\sqrt{N + \mu_N}} \right] \frac{1}{(N + 1)^{n-1}} \right\}^{\frac{1}{2}}$$

Remark 1 If the eigenvalue is simple $|B'_N(\mu_N)| \neq 0$, if it is a double eigenvalue $|B''_N(\mu_N)| \neq 0$, in both cases the *inf* is different from 0. The inequalities above result from the use of the mean value theorem.

3. Conclusion

Sampling theory has been used in [6] to compute the Dirichlet eigenvalues of regular second order Sturm-Liouville systems. We have shown in a recent paper [9] that the method is still valid in the non-Dirichlet case by showing that a transform \tilde{B} of the boundary function B was in the Paley-Wiener space PW_π and thus the Whittaker-Shannon-Kotel'nikov theorem was applicable. That is, we can recover \tilde{B} from its samples; thus B , whose zeroes are the square root of the sought eigenvalues of the problem. However the error bounds on the eigenvalues were not tight enough. We have introduced in [10] a transformation $\tilde{B}^{[n]}(\pi, \mu)$ of $B(\pi, \mu)$ satisfying $\tilde{B}^{[n]}(\pi, \mu) \in L^2(-\infty, \infty)$ and $\mu^{n-1} \tilde{B}^{[n]}(\pi, \mu) \in L^2(-\infty, \infty)$. This fact lead to high order approximations of the eigenvalues with very sharp error bounds. We have demonstrated in this paper that the method based on sampling theory which was successful in providing good estimates for the

eigenvalues of Sturm-Liouville problems with separable boundary conditions is still applicable when the boundary conditions are coupled by showing that a transform $\tilde{B}^{[n]}$ of the boundary function B is in the Paley-Wiener space PW_π and thus the Whittaker-Shannon-Kotel'nikov theorem is applicable. That is, we can recover \tilde{B} from its samples, thus B , whose zeroes are the square root of the sought eigenvalues of the problem. The computation of eigenvalues of Sturm-Liouville problems both regular and singular is well documented (see [1-7,9-22] and the references therein) and many packages/programs exist. We can quote SLEIGN [5], SLEIGN2 [4], SLEDGE [14], SL02F [20],...Extensive numerical computations based on our method will appear elsewhere and comparison will be made with output of the above packages.

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