King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 205

May 1996

An Equational Characterization of Quasi-injective Modules

A. Laradji

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

An Equational Characterization of Quasi-injective Modules

A. Laradji
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

Abstract

Injective modules are known to be precisely those over which every compatible system of linear equations is solvable. In this note, we introduce the concept of strong compatibility to give an analogous, but not generally known characterization of quasi-injectivity.

Throughout, R is an associative ring with 1, and M always denotes a left R-module. For any set I, M^I and $M^{(I)}$ denote respectively the direct product and the direct sum of $|I|$ copies of M, and their elements are represented by column vectors. By a system (S) of equations over an R-module M is meant a set of linear equations of the form $Ax = b$, where A is a row-finite $J \times K$ matrix over M and $b \in M^J$. (S) is said to be *compatible* if, whenever $p \in R^{(J)}$, $p^T A = 0$ implies $p^T b = 0$. It is well-known that a module is injective if and only if every compatible system of equations over it is solvable. Using Baer's criterion, we can even restrict to systems in one unknown. Our objective is to discuss an anologous equational characterization for quasi-injective modules, and which does not use the existence of injective modules. Recall that M is quasi-injective if every R-homomorphism $N \longrightarrow M$, where N is a submodule of M can be extended to an endomorphism of M. We first need the following

Definition. With the same notation above, we say that the system (S) is *strongly compatible* over M, if there exists $m \in M^K$ such that whenever $p \in R^{(J)}$, $p^T Am = 0$ implies $p^T b = 0$.

It is easy to see that for any system (S) we have the following implications: (S) is solvable$\Rightarrow$ (S) is strongly compatible$\Rightarrow$ (S) is compatible.

Proposition. For the R-module M, the following statements are equivalent.

(i) M is quasi-injective.

(ii) Every strongly compatible system of equations in one unknown over M is solvable.

(iii) Every strongly compatible system of equations over M is solvable.

Proof. (i)$\Rightarrow$(ii). Suppose that M is quasi-injective and let $Ax = b$, where $A = [r_j]$ is a vector in R^J and $b = [b_j]$ is a vector in M^J, be a strongly compatible system of equations in one unknown, so that there exists $m \in M$ for which $p^T Am = 0$ implies $p^T b = 0$, whenever $p \in R^{(J)}$. Let N be the submodule of M generated by $\{r_j m\}_{j \in J}$ and define a map $f : N \longrightarrow M$ by $f(r_j m) = b_j$. By strong compatibility f is a well-defined homomorphism, and so there exists a map $g : M \longrightarrow M$ extending f. It is clear that

$r_j g(m) = f(r_j m) = b_j$.

(ii)$\Rightarrow$(iii). Let (S) be the system $Ax = b$, where now $A = [r_{jk}]$ is a row-finite $J \times K$ matrix over R, and $b = [b_j]$ is a vector in M^J. Suppose that (S) is strongly compatible, i.e. that there exists $m = [m_k]$ in M^K such that $p^T Am = 0$ implies $p^T b = 0$, whenever $p \in R^{(J)}$. Denote by G the submodule of M^2 generated by $\{(-b_j, \sum_{k \in K} r_{jk} m_k) : j \in J\}$, let $T = M^2/G$ and let $f : M \longrightarrow T$ be the map given by $f(a) = \overline{(a, 0)}$. It is easy to check that f is a monomorphism and, by identifying each a in M with its image $\overline{(a, 0)}$, M becomes a submodule of T; furthermore, the system $Ax = b$ is solvable in T by $[\overline{(0, m_k)}]$. By adapting an argument in [1] almost verbatim for modules, in order to prove that (S) is solvable in M, it is enough to show that the inclusion f splits. Define a family $\mathcal{F}$ consisting of all pairs (Q, g), where Q is a submodule of T containing M, and $g : Q \longrightarrow M$ is a homomorphism extending f, and define a partial order $\leq$ on $\mathcal{F}$ by $(Q, g) \leq (Q', g')$ if and only if $Q \subseteq Q'$ and g' extends g. It is routine to check that Zorn's lemma applies and that $\mathcal{F}$ has a maximal member $\left(\overline{Q}, \overline{g}\right)$, say. Assume by way of contradiction that there exists $\overline{(\sigma, \tau)} \in T \backslash \overline{Q}$. Since $\overline{(\sigma, 0)} \in M \subseteq \overline{Q}$, it follows that $\overline{(0, \tau)} \in T \backslash \overline{Q}$. Denote by $\{r_i\}_{i \in I}$ the set of all elements $r \in R$ such that $r\ \overline{(0, \tau)} \in \overline{Q}$, and consider the system of equations (S') in one unknown over M given by

$$r_i x = \overline{g}(r_i \overline{(0, \tau)}) \qquad (i \in I)$$

We claim that (S') is strongly compatible. For, if $p = [p_i] \in R^{(I)}$ and $\sum p_i r_i \overline{(\tau, 0)} = 0$, then $\sum p_i r_i \tau = 0$ and so $\sum p_i r_i \overline{(0, \tau)}$ i.e. $\sum p_i \overline{g}(r_i \overline{(0, \tau)}) = 0$. By (ii), (S') has a solution $\overline{(\mu, 0)}$, say. If $h : \overline{Q} + R\overline{(0, \tau)} \longrightarrow M$ is the map given by $h(q + r(\overline{(0,\tau)})) = \overline{g}(q) + r\overline{(\mu,0)}$, then clearly h is a well-defined homomorphism that extends $\overline{g}$, contradicting the maximality of $\left(\overline{Q},\overline{g}\right)$. This proves (iii).

(iii)$\Rightarrow$(i). Let N be a submodule of M, let $f \in Hom_R(N,M)$ and let M/N be generated by $\{m_k + N\}_{k\in K}$. Consider the exact sequence

$$0 \longrightarrow B \longrightarrow R^{(K)} \xrightarrow{\pi} M/N \longrightarrow 0$$

where, for the standard basis $\{e_k\}_{k\in K}$ of $R^{(K)}$, $\pi(e_k) = m_k + N$ and $B = \ker\pi$ is generated by $\{b_j\}_{j\in J}$, say. Define an R–homomorphism $h : R^{(K)} \longrightarrow M$, by $h(e_k) = m_k$. Since, for some r_{jk} $(j \in J, k \in K)$ in R, $\sum\limits_{k\in K} r_{jk}e_k = b_j$ $(j \in J)$, it follows that the system $\sum\limits_{k\in K} r_{jk}x_k = f(h(b_j))$ $(j \in J)$ is strongly compatible in M, and so it is solvable by some $\{\mu_k\}_{k\in K}$ in M. For each $m \in M$, there exists $c = \sum\limits_{k\in K} c_k e_k$ in $R^{(K)}$ such that $\pi(c) = m + N$, and observing that $m - h(c) \in N$, define a map $g : M \longrightarrow M$ by $g(m) = f(m - h(c)) + \sum\limits_{k\in K} c_k\mu_k$. It is easy to check that g is a well-defined homomorphism that extends f. This proves that M is quasi-injective.

Remark. The equivalence of (i) and (iii) can be proved directly by similar arguments without recourse to the axiom of choice (which is used in the proof of (ii)$\Longrightarrow$(iii)). However, (ii) is easily seen to provide an equational, alternative proof of Fuchs' criterion in [2] or [3].

Acknowledgment. The author gratefully acknowledges the support provided by King Fahd University of Petroleum and Minerals.

REFERENCES

1. S. Balcerzyk, Remark on a paper of S. Gacsályi, Publ. Math. Debrecen 4 (1956), 357-358.

2. A. Cailleau and G. Renault, Etude des modules $\sum$-quasi-injectifs, C. R. Acad. Sc. Paris, t. 270 (1970), 1391-1394.

3. L. Fuchs, On quasi-injective modules, Scuola Norm. Sup. Pisa 23 (1968), 541-546.