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Abstract

In this note, we prove through a short and direct argument, that for a left pure semisimple ring R, R is left (resp. right) serial if and only if it is left (resp. right) duo. As a corollary we obtain that if R is a counter-example to the *pure semisimple conjecture*, then it cannot be a duo ring.

1. Introduction. Throughout, R is a ring with identity and $J = J(R)$ is its Jacobson radical; and, unless stated otherwise, all modules are left unital. Consider the following conditions.

(i) R is artinian serial.

(ii) R has finite representation type.

(iii) R is left pure semisimple, that is, every R−module is pure-injective.

It is well-known that the implications (i)⇒(ii)⇒(iii) are always true, and that, if R is commutative, their converses hold as well (see for example [3, 12]). The equivalence of (iii) and (ii) for arbitrary rings, known as *the pure semisimple conjecture*, has so far eluded all attempts of proof, but has been established for several types of non-commutative rings (e.g. Artin algebras) mainly through the use of dualities and homological methods. We refer to [1, 4, 5, 6, 12] for more details.

In another direction, various authors have described classes of rings for which duoness implies that they are serial (see for example [7, 10]). The aim of this note is to discuss by using a direct, non-homological argument, the above conditions for left (or right) duo rings. As a corollary, we obtain that (iii)⇒(i) for duo rings, providing yet another evidence as to the truth of the pure semisimple conjecture. Let us first recall some definitions. Following Warfield [8], we say that a module is *serial* if its submodules are linearly ordered by inclusion. The ring R is *left (resp. right) serial* if it is a direct sum of serial modules, and is *serial* if it is both left and right serial. If R has only finitely many non-isomorphic indecomposable left modules, it is said to have *finite representation type.*(It is known that finite representation type is left-right symmetric.) We say that R is *left (resp. right) duo* if each left (resp. right) ideal is two-sided, and that it is *duo* if it is both left and right duo.

2. Results. The main result is the following

Theorem. *Let R be a left pure semisimple ring. Then R is left (resp. right) duo if and only if it is left (resp. right) serial.*

Proof. Since R is left pure semisimple, ${}_RR$ is $\sum-$ pure-injective and so, by [11], R is semiprimary. Suppose first that R is left (resp. right) duo, then by [2], R is a finite direct sum of left duo local rings R_n . Furthermore, R is easily seen to be left pure semisimple if, and only if, each R_n is left pure semisimple. We may thus assume that R itself is a left (resp. right) duo local ring with maximal ideal J. If R/J is finite, then R/J is clearly a field, and we infer from [6] that R is serial. Suppose therefore that R/J is infinite, and let H be an infinite subset of $R\backslash J$ whose elements are distinct modulo J. The proof of the 'only if' part is complete once we show that the two-sided ideals of R are linearly ordered with respect to inclusion. Let u, v be in J, we prove that either $u \in RvR$ or $v \in RuR$. For each $h \in H$, set

$$R_h = \begin{cases} Rr_h & \text{if } R \text{ is left duo} \\ r_hR & \text{if } R \text{ is right duo} \end{cases}$$

, where $r_h = u - vh$,and let $M_h = R/R_h$, $P = \prod_{h\in H} M_h$, $S = \bigoplus_{h\in H} M_h$. Denote by q_h $(h \in H)$ the canonical composition $P \xrightarrow{proj} M_h \xrightarrow{incl} S$, let $\mu \in P$ be given by $\mu(h) = 1 + R_h$ $(h \in H)$ and consider the following system **(1)** of equations

$$x + r_h y_h = q_h(v\mu) \ (h \in H)$$

with unknowns $x, (y_h)_{h\in H}$. If **(1)'** is the system obtained from **(1)** by restricting h to a finite subset $\{h_1, h_2, ..., h_n\}$ of H, and if w_{ij} $(1 \leq i, j \leq n,\ i \neq j)$ are elements of R with $(h_i - h_j)w_{ij} = 1$ (recall that when $i \neq j$, $h_i - h_j \notin J$ and so $h_i - h_j$ is a unit of R), then $x = \sum_{j=1}^{n} q_{h_j}(v\mu)$, $y_{h_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_{ij} q_{h_j}(\mu)$ $(1 \leq i \leq n)$ is easily seen to be a solution of **(1)'** in S. Since S is pure–injective, the system **(1)** is solvable by $a, (b_h)_{h\in H}$ in S, say. Now, a and $(b_h)_{h\in H}$ have finite support, and so there exist $h_0 \in H$ and $c, d \in R$ such that $v = r_{h_0}c + dr_{h_0}$. Next, as R is left (resp. right) duo, we obtain that $v \in R_h$, and, since R is local, this means $v \in RuR$ or $u \in RvR$.

For the 'if' part, observe first that the direct product of left duo rings is again left duo, and so R may be assumed to be a local left serial ring. Since R is perfect, it follows easily that it is left artinian, and has therefore a unique composition series of left ideals $R \supseteq J \supseteq J^2 \supseteq \dots \supseteq J^k = 0$, for some k. It is clear then that R is left duo. A symmetric argument shows that a right serial perfect ring is right duo.

The theorem immediately yields

Corollary 1. *For any ring R, the following statements are equivalent.*

(i) R is an artinian serial ring.

(ii) R is a left pure semisimple duo ring.

(iii) R is duo and has finite representation type.

Remark. Combining [9, Theorem 3] and [4, Corollary 5.3], we obtain that a left pure-semisimple duo ring R with $J^2 = 0$ has finite representation type. Corollary 1 shows that the condition $J^2 = 0$ is not necessary.

Corollary 2. *Let R be a local left pure semisimple one-sided duo ring. Then either R is serial or $J^2 = 0$.*

Proof. Use Theorem and [5, Theorem 2.2].

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