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Nonlinear Systems: Functional Expansion, Realization and Systems Inversion

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NONLINEAR SYSTEMS:

FUNCTIONAL EXPANSION, REALIZATION

and

SYSTEMS INVERSION

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Abstract

In this paper, we shall introduce a new functional series expansion of the response map of a general nonlinear system, it is a power series in powers of the input components. We shall illustrate the usefulness of this series by considering two important problems namely those of realization and systems inversion.

Keywords: Functional Expansion, Lie series, realization, system inversion

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1. Introduction

Functional expansions of nonlinear response maps have been the subject of extensive research ever since the work of Volterra [1], Wiener [2], Brillant [3], and George [4]. The Volterra series, introduced by Wiener in the 1940's has played an important role in the analysis of nonlinear systems (Brockett [5], Lesiak and Krener [6], Sussman [7], Boyd, Chua and Desoer [8], Isidori [9], Sandberg [10], Banks [11]). In 1981, Fliess [12], using the work of Chen [13] on iterated integrals, introduced an algebraic approach to nonlinear functional expansion: it is a formal power series in noncommuting variables (see also Fliess, Lamnabhi and Lamnabhi-Laguarigue [14]).

Recently a new series expansion of the input output map of general nonlinear systems in powers of the input components was introduced, Chanane and Banks [15], assuming that the input is a polynomial satisfying $u(0) = 0$, $u'(0) \neq 0$. The approach used was based on the notion of delta operators and their basic polynomial sequences developed by Rota et al [16].

In this paper, we shall derive the same expansion using the Lie series and relax the condition on the input. We shall assume the input to satisfy any autonomous nonlinear differential equation with $u(0) = 0$, $u'(0) \neq 0$.

The outline of the paper is as follows, in section 2, we introduce the new series solution to nonlinear differential equations. This result is used to obtain, in section 3, the series expansion for the i/o map of single-input single-output systems in powers of the input. In section 4, we extend this result to multi-input multi-output systems and in section 5, we illustrate the usefulness of the expansion by tackling the problems of realization and systems inversion and draw some conclusions in the final section.

2. Series solution to nonlinear differential equations

Consider the differential equation

$$\frac{dx}{dt} = f(x) \qquad , \qquad x(0) = x_0 \tag{2.1}$$

where f is an analytic function from $\mathbf{R}^n$ to $\mathbf{R}^n$. We assume this Cauchy problem to have a unique solution. We shall present in this section a series expansion of the solution of the above problem in powers of a given function $u : \mathbf{R} \rightarrow \mathbf{R}$ satisfying the following,

Hypothesis (H1):

(i) $u(0) = 0$, $u'(0) \neq 0$;

(ii) $\exists h : \mathbf{R}^m \rightarrow \mathbf{R}$ analytic such that $\frac{d^m u}{dt^m} = h(u, u', ..., u^{(m-1)})$

Let $y_1 = u$, $y_2 = u', ..., y_m = u^{(m-1)}$, $y^T = (y_1, ..., y_m)$, we have

$$\frac{dy}{dt} = (y_2, ..., y_m, h(y_1, ..., y_m))^T \tag{2.2}$$

Thus,

$$\frac{d}{dt}(x^T, y^T)^T = (f_1(x), ..., f_n(x), y_2, ..., y_m, h(y_1, ..., y_m))^T \tag{2.3}$$

but $\frac{du}{dt} = y_2$ therefore,

$$\frac{\partial}{\partial u}(x^T, y^T)^T = (f_1(x)/y_2, ..., f_n(x)/y_2, 1, y_3/y_2, ..., y_m/y_2, h(y_1, ..., y_m)/y_2)^T \tag{2.4}$$

whose solution can be obtained using the Lie series approach as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \exp\{uL_{f,h}\} \begin{pmatrix} x \\ y \end{pmatrix}|_{\binom{x}{y}=\binom{x_0}{y_0}} \tag{2.5}$$

where

$$L_{f,h} = \frac{1}{y_2}[\sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m-1} y_{j+1}\frac{\partial}{\partial y_j} + h(y)\frac{\partial}{\partial y_m}] \tag{2.6}$$

since $u'(0) \neq 0$ in the neighbourhood of 0 , $y_2 \neq 0$. Define

$$x^{[0]} = x \quad , \quad x^{[k]} = L_{f,h} x^{[k-1]} \tag{2.7}$$

so that according to (2.5)

$$x = \sum_{k \geq 0} \frac{u^k}{k!} x^{[k]}_{|t=0} \tag{2.8}$$

We also take the following hypothesis

Hypothesis (H2):

$\exists \alpha, \rho > 0$ such that $||x^{[k]}_{|t=0}|| < \rho \alpha^k k!$, $k \geq 0$ and $||u|| < \frac{1}{\alpha}$

to conclude

Theorem 2.1. *Let $u \in C^m(\mathbf{R})$ satisfy hypothesis (H1). If hypothesis (H2) is satisfied then the solution to the Cauchy problem (2.1) can be expanded in a uniformly and absolutely convergent series (2.8) and $||x|| \leq \frac{\rho}{1-\alpha||u||}$*

Example 2.2. *If u satisfies a second order differential equation, we have,*

$$x^{[0]}_{|t=0} = x_0$$

$$x^{[1]}_{|t=0} = \{L_{f,h} x^{[0]}\}_{|t=0} = \frac{1}{y_2} f(x)_{|t=0} = \frac{1}{u'(0)} f(x_0)$$

$$x^{[2]}_{|t=0} = \{L_{f,h}[\frac{1}{y_2} f(x)]\}_{|t=0} = \{\frac{1}{y_2}[\frac{1}{y_2} f(x) \frac{\partial f}{\partial x} - \frac{1}{y_2^2} f(x) h(y_1, y_2)]\}_{|t=0}$$

$$= \frac{1}{[u'(0)]^2} f(x_0) f'(x_0) - \frac{1}{[u'(0)]^3} f(x_0) h(0, u'(0))$$

etc...

so that,

$$x = x^{[0]}_{|t=0} + ux^{[1]}_{|t=0} + \frac{u^2}{2!}x^{[2]}_{|t=0} + \ldots \tag{2.9}$$

Remark 1. *It is clear from the fact that the approach needs only differentiations and functions evaluations that we can generate as many terms of the series as desired using any symbolic mathematics package (maple, mathematica,...)*

Remark 2. *This expansion reduces to the Taylor series expansion of x around $t = 0$ if $u(t) = t$ since h in this case is defined by $h(u) = 1$.*

3. Input-output map for single-input systems

Consider the nonlinear system described by

$$\begin{cases} \frac{dx}{dt} = f(x,u) \quad , \quad x(0) = x_0 \\ z = g(x) \end{cases} \tag{3.1}$$

where $x(t) \in \mathbf{R}^n$ for each $t \geq 0$ and suppose that f and g are analytic functions from $\mathbf{R}^n \times \mathbf{R}$ to $\mathbf{R}^n$ and $\mathbf{R}^n$ to $\mathbf{R}$ respectively. Suppose that u , the input, satisfies Hypothesis (H1) of the previous section. Assume that g satisfies the inequality $||g(x)|| \leq G(||x||)$ for some analytic increasing function G.

Turning u into a state as usual, by letting $y_1 = u$, and the y 's are defined as in section 2, we obtain,

$$\frac{d}{dt}(x^T, y^T)^T = (f_1(x, y_1), ..., f_n(x, y_1), y_2, ..., y_m, h(y_1, ..., y_m))^T \tag{3.2}$$

but $\frac{du}{dt} = y_2$, so

$$\frac{\partial}{\partial u}(x^T, y^T)^T = (f_1(x, y_1)/y_2, ..., f_n(x, y_1)/y_2, 1, y_3/y_2, ..., y_m/y_2, h(y_1, ..., y_m)/y_2)^T \tag{3.3}$$

whose solution can be obtained as in the previous section, using the Lie series, as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \exp\{uL_{f,h}\}\begin{pmatrix} x \\ y \end{pmatrix}|_{\binom{x}{y}=\binom{x_0}{y_0}} \tag{3.4}$$

where this time,

$$L_{f,h} = \frac{1}{y_2}[\sum_{i=1}^{n} f_i(x, y_1)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m-1} y_{j+1}\frac{\partial}{\partial y_j} + h(y)\frac{\partial}{\partial y_m}] \tag{3.5}$$

so that (2.8) still holds, where the $x^{[k]}$ are defined as before and satisfy hypothesis ($\dot{\text{H}}$2).

Returning to the output equation in (3.1), we may expand g in a Taylor's series around x_0 and get,

$$g(x) = g(x_0) + \sum_{p\geq 1} g_p(x_0)(x - x_0, ..., x - x_0) \tag{3.6}$$

where $g_p(x_0)(., ..., .)$ are p-linear maps. Replacing x by its expansion in powers of u yields, corresponding to (2.8),

$$z = \sum_{m\geq 0} \frac{u^m}{m!} z^{[m]}_{|t=0} \tag{3.7}$$

where

$$z^{[m]} = \begin{cases} \sum_{p=1}^{m} \sum_{\substack{k_1+...+k_p=m \\ k_1,...,k_p\geq 1}} \binom{m}{k_1 \ldots k_p} g_p(x_0)(x^{[k_1]}, ..., x^{[k_p]}) \text{ if } m \geq 1 \\ g(x^{[0]}) \text{ if } m = 0 \end{cases} \tag{3.8}$$

Theorem 3.1. *Let $u \in C^m(\mathbf{R})$ satisfy hypothesis (H1). If hypothesis (H2) is satisfied then the input-output map associated with the system (3.1) can be expanded in a uniformly and absolutely convergent series (3.7) and $||z|| \leq G(\frac{\rho}{1-\alpha||u||})$*

4. Input-output map for multi-input systems

Consider the nonlinear system described by

$$\begin{cases} \frac{dx}{dt} = f(x,u) \quad , \quad x(0) = x_0 \\ z = g(x) \end{cases} \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$ for each $t \geq 0$ and suppose that f and g are analytic functions from $\mathbf{R}^n \times \mathbf{R}^r$ to $\mathbf{R}^n$ and $\mathbf{R}^n$ to $\mathbf{R}^q$ respectively. Assume that g satisfies the inequality $||g(x)|| \leq G(||x||)$ for some analytic increasing function G. Suppose that $u : \mathbf{R} \to \mathbf{R}^r$, the input, satisfies the following

Hypothesis (H1'):

(i) $u(0) = 0$, $u'(0) \neq 0$

(ii) $\exists\, h : \mathbf{R}^{m \times r} \to \mathbf{R}^r$ analytic such that $\frac{d^m u}{dt^m} = h(u, u', ..., u^{(m-1)})$

Let $y_1 = u$, $y_2 = u', ..., y_m = u^{(m-1)}$, $y^T = (y_1^T, ..., y_m^T)$, we have

$$\frac{dy}{dt} = (y_2^T, ..., y_m^T, h^T(y_1, ..., y_m))^T \tag{4.2}$$

Thus,

$$\frac{d}{dt}(x^T, y^T)^T = (f_1(x, y_1), ..., f_n(x, y_1), y_2^T, ..., y_m^T, h^T(y_1, ..., y_m))^T \tag{4.3}$$

but $\frac{du}{dt} = y_2$, therefore,

$$\frac{\partial}{\partial u_j}(x^T, y^T)^T = (f_1(x, y_1), ..., f_n(x, y_1), y_2^T, ..., y_m^T, h(y_1, ..., y_m))^T / y_{2j} \tag{4.4}$$

for $j = 1, ..., r$, whose solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \exp\{\sum_{j=1}^{r} u_j L_{f,h,j}\} \begin{pmatrix} x \\ y \end{pmatrix}_{|\binom{x}{y} = \binom{x_0}{y_0}} \tag{4.5}$$

where

$$L_{f,h,j} = \frac{1}{y_{2j}} [\sum_{i=1}^{n} f_i(x, y_1) \frac{\partial}{\partial x_i} + \sum_{k=1}^{m-1} y_{k+1} \frac{\partial}{\partial y_k} + h(y) \frac{\partial}{\partial y_m}] \tag{4.6}$$

so

$$x = \sum_{k \geq 0} \frac{1}{k!} \{\sum_{j=1}^{r} u_j L_{f,h,j}\}^k x_{|\binom{x}{y} = \binom{x_0}{y_0}} \tag{4.7}$$

and if the vector fields are commuting this expansion can be rewritten as

$$x = \sum_{K \geq 0} \frac{1}{K!} a_K u^K \tag{4.8}$$

where $K = (k_1, ..., k_r)$ is a multi-index, $u^K = u_1^{k_1} ... u_r^{k_r}$, $K! = k_1! ... k_r!$, $u = (u_1, ... u_r)^T$, $a_K \in \mathbf{R}^n$ for each K. A similar expansion holds for z,

$$z = \sum_{K \geq 0} \frac{1}{K!} c_K u^K \tag{4.9}$$

where $c_K \in \mathbf{R}^q$ for each r-index K

Theorem 4.1. *Let $u \in C^m(\mathbf{R})$ satisfy hypothesis (H1'). If hypothesis (H2) is satisfied then the input-output map associated with the system (3.1) can be expanded in a uniformly and absolutely convergent series (4.9) and* $||z|| \leq G(\frac{\rho}{1-\alpha||u||})$

5. Realization and Systems inversion

From the very nature of the expansion introduced, two important problems can be tackled, namely, the realization problem and the problem of the inversion of a nonlinear system. Alternative approaches can be found for example in Jakubczyk [17] , Sontag and Wang [18] and Hirschorn [19] respectively.

5.1. Realization

In the following we shall solve the problem for single-input single-output system. Suppose that a pair of input and output (u, z) is available. We shall assume that these functions are differentiable.

Now, recall Teixeira's extended form of Burmann's theorem , Whitaker and

Watson [20],

Theorem 5.1. *Let f be a function of z , analytic in a ring-shaped region A, bounded by an outer curve Γ and an inner curve γ . Let θ be a function of z analytic on and inside Γ and have only one simple zero a within this contour. Then for $x \in A$,* we have

$$f(x) = \sum_{n\geq 0} A_n\{\theta(x)\}^n + \sum_{n\geq 1} \frac{B_n}{\{\theta(x)\}^n} \quad , \tag{5.1}$$

where

$$\begin{aligned} A_0 &= f(a) \quad , \\ A_n &= \tfrac{1}{2\pi i n} \textstyle\int_\Gamma \frac{f'(z)}{\{\theta(z)\}^n} dz, n \geq 1 \quad , \\ B_n &= \tfrac{-1}{2\pi i n} \textstyle\int_\gamma \{\theta(z)\}^n f'(z) dz, n \geq 1 \quad . \end{aligned} \tag{5.2}$$

Therefore, we have the expansion,

$$z = \sum_{k\geq 0} \frac{\alpha_k}{k!} \cdot u^k \tag{5.3}$$

where

$$\alpha_k = \begin{cases} z_0 & , \quad k = 0 \quad , \\ \frac{d^{k-1}}{dt^{k-1}}[z'(t)\{\frac{t}{u(t)}\}^k]_{|t=0} & , \quad k \geq 1 \quad , \end{cases} \tag{5.4}$$

From the uniqueness of the power series representation, we obtain the following realization result,

Corollary 5.2. *The coefficient of the same powers of u in the two series (5.3) and (3.7) must be equal, that is*

$$z^{[k]}_{|t=0} = \alpha_k \quad , \qquad k \geq 0 \tag{5.5}$$

Now, to get a realization in the form of (3.1), we proceed as follows. We assume first that the output map is given by

$$z = cx \quad , \tag{5.6}$$

where for simplicity we take $c = (1\ 1\ ...\ 1)$. Then, solve

$$cx^{[k]}_{|t-0} = \alpha_k \quad , \qquad k \geq 0 \tag{5.7}$$

for x_0 and f ,where $x^{[k]}$ are given by (2.7) and (3.5).

Example 5.3. *If u satisfies a second order differential equation, we have,*

$$x^{[0]}_{|t=0} = x_0$$

$$x^{[1]}_{|t=0} = \{L_{f,h}x^{[0]}\}_{|t=0} = \frac{1}{y_2}f(x, y_1)_{|t=0} = \frac{1}{u'(0)}f(x_0, 0)$$

$$x^{[2]}_{|t=0} = \{L_{f,h}[\frac{1}{y_2}f(x)]\}_{|t=0} = \{\frac{1}{y_2}[\frac{1}{y_2}f(x, y_1)\frac{\partial f}{\partial x} - \frac{1}{y_2^2}f(x, y_1)h(y_1, y_2)]\}_{|t=0}$$

$$= \frac{1}{[u'(0)]^2}f(x_0, 0)f'(x_0, 0) - \frac{1}{[u'(0)]^3}f(x_0, 0)h(0, u'(0))$$

etc...where we have made use of the condition $u(0) = 0$.

Thus,

$$cx_0 = \alpha_0$$

$$c\frac{1}{u'(0)}f(x_0, 0) = \alpha_1$$

$$c[\frac{1}{[u'(0)]^2}f(x_0, 0)f'(x_0, 0) - \frac{1}{[u'(0)]^3}f(x_0, 0)h(0, u'(0))] = \alpha_2$$

etc...from which we get sufficient information to determine $x_0, f(x_0, 0), f'(x_0, 0)$, etc... to within $n - 1$ parameters each. Hence, x_0 and $f(x, u)$ (see [21] for an alternative approach).

5.2. Inversion of nonlinear systems

If we assume the system at hand is single-input single-output, we can immediately obtain a result concerning the inversion (compositionwise) of the system by just considering the inversion of a power series expansion of the input-output map of general nonlinear systems in powers of the input components. We have illustrated the usefulness of this approach by considering two important problems, those of realization and systems inversion. In a future paper, we shall develop further these ideas and present some symbolic algebra tools to overcome the computational burden involved when dealing with nonlinear systems.

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7. References

1. V. Volterra, 1959, **Theory of Functionals and Integral and Integro-Differential Equations**, Dover, New-York,

2. N. Wiener, 1958, **Nonlinear Problems in Random Theory**, MIT Press, Cambridge, MA.

3. M. B. Brillant, 1958, 'Theory of the Analysis of Nonlinear Systems, Report RLE-345, MIT.

4. D. A. George, 1959, "Continuous Nonlinear Systems' MIT, Research Lab. of Electronics, Cambridge,

5. R. W. Brockett, 1976, 'Volterra Series and Geometric Control Theory',

Automatica, **12**, pp. 167-176, (Addendum with E. Gilbert Vol. 12),

6. C. M. Lesiak, A. J. Krener, 1978, ' The existence and uniqueness of Volterra series for nonlinear systems', IEEE Trans. Auto. Control, AC-23, pp 1090-1095,

7. H. T. Sussman, 1984, 'A Lie-Volterra Expansion for Nonlinear Systems' in Mathematical Theory of Networks and Systems, (P. Fuhrman Ed.) Lect. Notes Control Inf. Sci. **58**, pp822-828, Springer, Berlin,

8. S. Boyd, L. Chua, C. Desoer, 1985, 'Analytic Foundations of Volterra Series' I.M.A. J. of Math. Contr. & Info., Vol. I, No.1,

9. A. Isidori, 1985, **Nonlinear Control Systems: An introduction**, Springer, Berlin,

10. I. W. Sandberg, 1985, 'Criteria for the global existence of functional expansions for input-output maps', AT&T Tech. J. (USA), Vol.64, No7 (Sept) PP1639-1658,

11. S. P. Banks, 1988, **Mathematical Theories of Nonlinear Systems**, Prentice-Hall, London,

12. M. Fliess, 1981, 'Fonctionelles causalles nonlineares et indeterminees non commutatives', Bull. Soc. Math. France, **109**, pp 3-40,

13. K. T. Chen, 1958, 'Integration of paths- a faithful representation of paths by noncommutative formal power series', Trans. Amer. Math. Soc. Vol.89, pp395-407,

14. M. Fliess, M. Lamnabhi, F. Lamnabhi-Lagarrigue, 1983, 'An algebraic approach to nonlinear functional expansions', IEEE CAC-30, No8, pp554-570.

15. B. Chanane, S. P. Banks, 1994, 'A new series solution for nonlinear differential equation'.

16. G. C. Rota, D. Kahaner and A. Odlyzko, 1973, 'On the foundations of combinatorial theory, VIII, finite operator calculus', J. Math. Anal. App., 42, pp684-760.

17. B. Jakubczyk, 1985, 'Realization theory for nonlinear systems: Three approaches', in Algebraic and Geometric Methods in Nonlinear Control Theory, D. Reidel Publishing Company, (Eds.) M. Fliess and M. Hazewinkel, pp3-31.

18. E. D. Sontag and Y. Wang, 1990, 'Input/output equations and realizability', in Progress in Systems and Control Theory, Realization and Modelling in Systems theory, Birkhauser, pp125-132.

19. R. Hirschorn, 1979, 'Invertibility of nonlinear systems', SIAM J. Control

Opt., 17,

20. E. T. Whitaker, G. N. Watson, 1927, **A course of modern analysis,** Cambridge at the University Press. The fourth edition, p128-132.

21. S. P. Banks, A. Moser, D. McCaffrey, 1994, 'Lie Series and the Realization Problem', Research Report No.562, ACSE Dpt. University of Sheffield, U.K.