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Diffraction of SH-waves at a Mixed Interface

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Abstract

We study the diffraction of SH-type elastic waves propagating along with the plane interface between two dissimilar elastic half-spaces. It is assumed that the contact between the half-spaces is perfect except for a part of semi infinite extension where the contact is loosely coupled. We use the Wiener-Hopf technique to obtain the diffracted field in a closed form.

1 Introduction

The problem of diffraction of plane waves from different obstacles present in the media of propagation has attracted considerable interest in the past fifty years. Many workers have used the so-called Wiener-Hopf technique to tackle such problems. The technique is based upon integral equation formulation of the inherent problem together with the application of integral transforms, Liouville's theorem and complex integration. In 1952, D.S. Jones [6] proposed a modification due to which the mixed boundary value problem arising from diffraction of a plane wave from a plane could be directly solved without reducing it to an integral equation. An excellent treatment of Jones' method can be found in the book by Noble [8].

Since then many authors have considered scattering and diffraction problems using the Wiener-Hopf technique De Hoop [5] carried out an elegant study of diffraction of plane elastic waves by a semi-infinite plane.

Kazi [7], Asghar and Zaman [2], [3] studied the diffraction of horizontally polarized shear waves in a layered half-space by perfectly rigid or perfectly soft half planes of infinite or finite extention.

In this paper, we consider two dissimilar half-spaces in contact at an interface in such a way that part of the contact is perfect while at the remaining contact the two half-spaces are loosely coupled thus forming a crack at the interface. We study diffraction of a horizontally polarized shear wave travelling at the interface by this crack. We apply Jone's method to reduce the mixed boundary-value problem to the Wiener-Hopf equation and present the solution of the diffraction problem in a closed form.

2 The Incident Wave

We consider two dissimilar half-spaces occupying $y > 0$ and $y < 0$ having an interface at $y = 0$. The subscript 1 is used to denote quantities for the upper half-space $y > 0$ while 2 is used for the lower half-space $y < 0$. Thus μ_i, β_i, ρ_i respectively denote the rigidity, velocity of the shear wave and the density of the i medium $i = 1, 2$. It is assumed that the two half-spaces are in perfect contact for $x < 0$ and have a crack in between for $x \geq 0$. A horizontally polarized shear (SH-) wave is assumed to be travelling along the interface and is incident on the crack. This incident wave satisfies the equation

$$\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} = s_i^2 \frac{\partial^2 w_i}{\partial t^2} \ , \ i = 1, 2 \tag{1}$$

where $s_i = \dfrac{1}{\beta_i} = \sqrt{\dfrac{\rho_i}{\mu_i}}$ is the slowness of the SH-wave and w_i is the transverse velocity in the direction normal to the xy-plane in the *ith* medium. Since the wave is travelling along the x-axis, an appropriate form of the solution is $w_i = w_i(y, t)e^{ikx}$, where $k = k_1 + ik_2$ is assumed. We take k_2 to be small but positive to give the desired behavior at infinity. Equation (1) becomes

$$\frac{\partial^2 w_i}{\partial y^2} - k^2 w_i = s_i^2 \frac{\partial^2 w_i}{\partial t^2}, \ i = 1, 2 \tag{2}$$

Taking the Laplace transform in t defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-\alpha t}dt = F(\alpha), \tag{3}$$

we can employ the continuity of the displacement across $y = 0$.

We may write the solution in the form

$$\left.\begin{array}{l} W_1^{inc} = Ae^{-(s_1^2\alpha^2+k^2)^{1/2}y} \; e^{ikx}, \; y > 0 \\ W_2^{inc} = Ae^{+(s_2^2\alpha^2+k^2)^{1/2}y} \; e^{ikx}, \; y < 0 \end{array}\right\}, \tag{4}$$

where

W_1^{inc} is

The continuity of the shear stress σ_{yz} across $y = 0$ would imply

$$-\mu_1(s_1^2\,\alpha^2 + k^2)^{1/2} = \mu_2(s_2^2\,\alpha^2 + k^2)^{1/2} \tag{5}$$

3 Diffraction Problem

Due to the presence of crack at $y = 0, x \geq 0$, a diffracted wave will be produced which we would again denote by w_i. However, to avoid confusion the incident displacement field will be henceforth denoted by w_i^{inc}. We write the total displacement w_i^t as

$$w_i^t = w_i^{inc} + w_i.$$

The diffracted wave is to satisfy the same wave equation (1). The boundary conditions to be satisfied are

(a) At $y = 0, \quad -\infty < x < 0$

$$\left.\begin{array}{l} w_1 = w_2 \\ \mu_1 \frac{\partial w_1}{\partial y} = \mu_2 \frac{\partial w_2}{\partial y} \end{array}\right\} \qquad 6(a)$$

(b) At $y = 0^+, \quad x \geq 0.$

$$\left.\begin{array}{l} \frac{\partial w_1}{\partial y} = -\frac{\partial w_1^{inc}}{\partial y} \\ \text{At } y = 0^-, \quad x \geq 0 \\ \frac{\partial w_2}{\partial y} = -\frac{\partial w_2^{inc}}{\partial y} \end{array}\right\} \qquad 6(b)$$

In addition, certain edge conditions are assumed in order to ensure uniqueness of the solution (see Noble [8] for details).

4 Transformed Problem

We define the following double range and half range Laplace transforms in x :

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)e^{-px}dx \,, \tag{7a}$$

$$\hat{f}_+(p) = \int_0^{\infty} f(x)e^{-px}\,dx \,, \tag{7b}$$

$$\hat{f}_-(p) = \int_{-\infty}^{0} f(x)e^{-px}dx. \tag{7c}$$

Clearly,

$$\hat{f}(p) = \hat{f}_+(p) + \hat{f}_-(p). \tag{8}$$

If $p = \sigma + i\tau$ and $f(x) = O(e^{ax})$ as $x \to \infty$ then $\hat{f}_+(p)$ is analytic in the right half plane $\sigma > a$. If $f(x) = O(e^{bx})$ as $x \to -\infty$ then $\hat{f}_-(p)$

is analytic in the left half plane $\sigma < b$. Combining these we find that $\hat{f}(p)$ is analytic in the strip $a < \sigma < b$.

Applying the Laplace transform in time and double range Laplace transform in x, the wave equation transforms into

$$\frac{\partial^2 \hat{W}_i}{\partial y^2} + (p^2 - \alpha^2 s_i^2)\hat{W}_i = 0 \qquad i = 1,2. \tag{9}$$

(a) At $y = 0$

$$\left.\begin{aligned} \hat{W}_{1-} &= \hat{W}_{2-} \\ \mu_1 \hat{W}'_{1-} &= \mu_2 \hat{W}'_{2-} \end{aligned}\right\} \tag{9a}$$

(b)

$$\left.\begin{aligned} \hat{W}'_{1+} &= -\hat{W}'^{\,inc}_{1+} = \frac{-A(s_1^2\alpha^2+k^2)^{1/2}}{p-ik} \\ \hat{W}'_{2+} &= -\hat{W}'^{\,inc}_{2+} = \frac{A(s_2^2\alpha^2+k^2)^{1/2}}{p-ik} \end{aligned}\right\} \tag{9b}$$

Here $'$ denotes derivative with respect to y.

5 The Wiener-Hopf Equation

The solution of (9) may be written as

$$\left.\begin{aligned} \hat{W}_1(p,y,\alpha) &= E_1 e^{-\gamma_1 y} \quad , \; y > 0 \\ \hat{W}_2(p,y,\alpha) &= E_2 e^{\gamma_2 y} \quad , \; y < 0 \end{aligned}\right\} \tag{10}$$

where $\gamma_i = (\alpha^2 s_i^2 - p^2)^{1/2}$, $i = 1,2$.

Eliminating the constants E_1 and E_2,

$$\left.\begin{aligned} \hat{W}_1(p,0,\alpha) &= -\tfrac{1}{\gamma_1}\hat{W}'_1(p,0,\alpha)\,, \\ \hat{W}_2(p,0,\alpha) &= \tfrac{1}{\gamma_2}\hat{W}'_2(p,0,\alpha). \end{aligned}\right\} \tag{11}$$

Using the decomposition (8), we put (11) in the form

$$\left.\begin{aligned}\hat{W}_{1+} + \hat{W}_{1-} &= -\tfrac{1}{\gamma_1}(\hat{W}'_{1+} + \hat{W}'_{1-}), \\ \hat{W}_{2+} + \hat{W}_{2-} &= \tfrac{1}{\gamma_2}(\hat{W}'_{2+} + \hat{W}'_{2-})\end{aligned}\right\} \tag{12}$$

Using the transformed boundary conditions, the latter of the above pair of equations (12) can be written as

$$(\hat{W}_{1+} + \hat{W}_{1-}) + (\hat{W}_{2+} - \hat{W}_{1+}) = \frac{1}{\gamma_2}\left[\frac{\mu_1}{\mu_2}\hat{W}'_{1+} + \frac{\mu_1}{\mu_2}\hat{W}'_{1-}\right],$$

putting value of $\hat{W}_{1+} + \hat{W}_{1-}$ from the first equation in (12), we have

$$\begin{aligned}(\hat{W}_{2+} - \hat{W}_{1+}) &= \frac{\mu_1}{\mu_2\gamma_2}\left[\hat{W}'_{1+} + \hat{W}'_{1-}\right] + \frac{1}{\gamma_1}\left[\hat{W}'_{1+} + \hat{W}'_{1-}\right] \\ &= \frac{\mu_1\gamma_1 + \mu_2\gamma_2}{\mu_2\gamma_1\gamma_2}\left[\hat{W}'_{1+} + \hat{W}'_{1-}\right] \qquad (13)\end{aligned}$$

Writing

$$L(p,\alpha) = \frac{\mu_2\gamma_1\gamma_2}{\mu_1\gamma_1 + \mu_2\gamma_2} \tag{14}$$

and

$$\hat{V}_+ = \hat{W}_{2+} - \hat{W}_{1+}, \tag{15}$$

we can now write (13) with the aid of 9(b) as

$$L(p,\alpha)\hat{V}_+ = \frac{A(s_1^2\alpha^2 + k^2)^{1/2}}{p - ik} + \hat{W}'_{1-}. \tag{16}$$

Equation (16) is in the form of the so-called Wiener-Hopf equation.

6 Solution in the Transformed Plane

The function $L(p,\alpha)$ can be factored as

$$L(p,\alpha) = L_+(p,\alpha)L_-(p,\alpha) \tag{17}$$

(see Appendix 1), where $L_{\pm}(p,\alpha)$ are analytic in an apopropriate right (left) half plane.

We can thus write equation (16) as

$$L_{+}(p,\alpha)\hat{V}_{+} = \frac{A(s_1^2\alpha^2+k^2)^{1/2}}{L_{-}(p,\alpha)(p-ik)} + \frac{\hat{W}'_{1-}}{L_{-}(p,\alpha)} \tag{18}$$

The left hand side of (18) is analytic in the right uniformity while the right hand side is analytic in the left half-plane except for the first term which has a pole at $p = ik$. We decompose this term as(Appendix 2, B1)

$$\frac{(s_1^2\alpha^2+k^2)^{1/2}}{L_{-}(p,\alpha)(p-ik)} = R_{+}(p,\alpha) + R_{-}(p,\alpha)$$

The equation (18) can thus be written as

$$L_{+}(p,\alpha)\hat{V}_{+} - R_{+}(p,\alpha) = AR_{-}(p,\alpha) + \frac{\hat{W}'_{1-}}{L_{-}(p,\alpha)}. \tag{19}$$

The left hand side is now analytic in the right hand half complex p-plane while the right hand side is so in the left hand half-plane. Hence the equation (19) holds in the common strip of analyticity. Thus both sides are analytic continuation of each other and together define an entire function. By the Liouville's theorem this entire function is a constant. Following Georgiadis et al. [4] we infer that this constant is zero. We therefore obtain

$$\hat{W}'_{1-} = AR_{-}(p,\alpha)L_{-}(p,\alpha) \tag{20}$$

where $L_{-}(p,\alpha)$ and $R_{-}(p,\alpha,)$ are as given in (A7) and (B1).

We can therefore write

$$E_1 = -\frac{1}{\gamma_1}(\hat{W}'_{1+} + \hat{W}'_{1-}) = \frac{A(s_1^2\alpha^2+k^2)^{1/2}L_{-}(p,\alpha)}{\gamma_1 L_{-}(ik,\alpha)(p-ik)} \tag{21}$$

The solution in the transformed plane can then be written as

$$\hat{W}_1(p,y,\alpha) = \frac{-A}{\gamma_1}\,\frac{(s_1^2\alpha^2+k^2)^{1/2}L_-(p,\alpha)}{(p-ik)L_-(ik,\alpha)}\,e^{-\gamma_1 y}\ ,\ y>0 \tag{22}$$

Similarly, we can find

$$\hat{W}_2(p,y,\alpha) = \frac{A(s_2^2\alpha^2+k^2)^{1/2}L_-(p,\alpha)}{\gamma_2(p-ik)L_-(ik,\alpha)}e^{\gamma_2 y}\ ,\ y<0 \tag{23}$$

7 The Diffracted Wave

We can now apply the inversion integral for the two sided Laplace transform to equations (22) and (23) to obtain the diffracted field in the α-plane as

$$W_1(x,y,\alpha) = \frac{-A}{2\pi i}\int_{ic-\infty}^{ic+\infty}\frac{L_-(p,\alpha)(s_1^2\alpha^2+k^2)^{1/2}}{\gamma_1(p-ik)L_-(ik,\alpha)}\,e^{px-\gamma_1 y}\,dp\ ,\ y>0 \tag{24}$$

and

$$W_2(x,y,\alpha) = \frac{A}{2\pi i}\int_{ic-\infty}^{ic+\infty}\frac{L_-(p,\alpha)(s_2^2\alpha^2+k^2)^{1/2})}{\gamma_2(p-ik)L_-(ik,\alpha)}\,e^{px+\gamma_2 y}\,dp\ ,\ y<0. \tag{25}$$

If we draw the closed contour lying in the left half plane then both the integrals (24) and (25) have a pole at $p=ik$ lying inside the contour. We also have a branch point at $p=\alpha\,s_j$ for $j=1,2$ respectively in the case of (24) and (25). Let us write $W_j(x,y,p)=W_{j,1}(x,y,\alpha)+W_{j,2}(x,y,\alpha)$ where $W_{j,1}(x,y,\alpha)$ denotes the contribution from the pole $p=ik$ and $W_{j,2}(x,y,\alpha)$ denotes the contribution from the branch line integral for $j=1,2$.

The residue formula gives the pole contributions as

$$\left.\begin{aligned} W_{1,1}(x,y,\alpha) &= -Ae^{ikx-(\alpha^2 s_1^2+k^2)^{1/2}y}\ ,\ y>0 \\ W_{2,1}(x,y,\alpha) &= -Ae^{ikx-(\alpha^2 s_2^2+k^2)^{1/2}y}\ ,\ y\le 0. \end{aligned}\right\} \tag{26}$$

Comparing (26) with (4), we find that the contribution from the pole $p = ik$ cancels the incident wave exactly. This behavior agrees with similar situations in which one part of the diffracted wave cancels with the incident wave. This result could have been anticipated from the fact that the so-called 'shadow zone' in diffraction problems is caused by the incident wave cancelling one part of the diffracted wave (Achenbach [1] p.377.)

To calculate $W_{1,2}$ and $W_{2,2}$, we use an analysis similar to that presented in [4] and obtain these contributions in the form of the branch cut integrals

$$W_{1,2}(x,y,\alpha) = \frac{-A}{2\pi i}\left\{\int_{-\infty}^{-\alpha s_1} \frac{L_-(p,\alpha)(s_1^2\alpha^2+k^2)^{1/2}\, e^{px-\gamma_1 y}}{\gamma_1(p-ik)L_-(ik,\alpha)}\, dp \right.$$
$$\left. + \int_{-\alpha s_1}^{\infty} \frac{L_-(p,\alpha)(s_1^2\alpha^2+k^2) e^{px-\gamma_1 y}}{\gamma_1(p-ik)L_-(ik,\alpha)}\, dp\right\}. \tag{27}$$

and

$$W_{2,2}(x,y,\alpha) = \frac{-A}{2\pi i}\left\{\int_{-\infty}^{-\alpha s_2} \frac{L_-(p,\alpha)(s_2^2\alpha^2+k^2)^{1/2}\, e^{px+\gamma_2 y}}{\gamma_2(p-ik)L_-(ik,\alpha)}\, dp \right.$$

$$\left. + \int_{-\alpha s_2}^{-\infty} \frac{L_-(p,\alpha)(s_2^2\alpha^2+k^2) e^{px+\gamma_2 y}}{\gamma_2(p-ik)L_-(ik,\alpha)}\, dp\right\}. \tag{28}$$

In equations (27) and (28) the integral is along the branch cut joining $-\infty$ and the branch point $-\alpha s_j$, $j = 1,2$. The path being above the branch cut in the first integral and below the branch cut in the second integral of both of these equations. Employing the fact that the function $L(p,\alpha)$ takes complex conjugate values at the top and bottom of the cut $(-\infty, -\alpha s_1)$, we can write (27) as

$$W_{1,2}(x,y,\alpha) = \frac{-A}{\pi i}\int_{-\infty}^{-\alpha s_1} \frac{L_-(p,\alpha)(s_1^2\alpha^2+k^2)^{1/2}\; e^{px-\gamma_1 y}}{\gamma_1(p-ik)L_-(ik,\alpha)}\, dp,$$

and a similar expression for $W_{2,2}$.

8 Time-Harmonic Incident Wave

If we assume the incident wave to be time harmonic with $e^{i\omega t}$ as the time harmonic part, we may find, after suppressing the time factor,

$$w_j^{inc} = \begin{cases} e^{-(k^2-k_1^2)^{1/2}y+ikx} & , \; y > 0 \\ e^{-(k^2-k_2^2)^{1/2}y+ikx} & , \; y < 0 \end{cases}$$

where $k_j = \omega s_j, \;\; j = 1,2.$

We can now do without the Laplace transform in time and the Wiener-Hopf equation corresponding to (16) takes the form

$$L(p)\hat{V}_2 = \frac{-iA(k^2-k_1^2)^{1/2}}{p-ik} + \hat{w}'_{1-}$$

where

$$L(p) = \frac{\mu_1(p^2+k_1^2)^{1/2} + \mu_2(p^2+k_2^2)^{1/2}}{\mu_2(p^2+k_1^2)^{1/2}(p^2+k_2^2)^{1/2}} = L_+(p)L_-(p).$$

The solution in the transformed domain in this case is

$$\hat{w}_j(p,y) = \frac{A(k^2-k_j)^{1/2}}{(p^2+k_j^2)^{1/2}} \frac{L_-(p)}{(p-ik)L_-(ik)} e^{\{px \mp i(p^2+k_j)^{1/2}y\}},$$

$j = 1,2$, which upon inversion gives

$$w_j(x,y) = \frac{-A}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{L_-(p)(k^2-k_j^2)^{1/2}\, e^{px \mp i(p^2+k_j^2)^{1/2}y}}{(p^2+k_j^2)^{1/2}L_-(ik)(p-ik)}\, dp.$$

The contribution due to the pole $p = ik$ can again be seen to cancel the incident wave. The branch line integral similar to the above case can now be obtained to give the diffracted field in the two half-spaces.

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Appendix 1

We follow Georgiadis et al. and factorize $L(p,\alpha)$ as:

$$L(p,\alpha) = \frac{\mu_2\gamma_1\gamma_2}{\mu_1\gamma_1 + \mu_2\gamma_2} = \frac{\frac{\mu_2}{\mu_1}\gamma_2}{1 + \frac{\mu_2}{\mu_1}\frac{\gamma_2}{\gamma_1}}$$

$$= \frac{\frac{\mu_2}{\mu_1}(\alpha^2 s_1^2 - p^2)^{1/2}}{1 + \frac{\mu_2}{\mu_1}\left\{\frac{(\alpha^2 s_2^2 - p^2)}{(\alpha^2 s_1^2 - p^2)}\right\}^{1/2}} \tag{A1}$$

The numerator can be split by inspection while for the denominator we may use Noble's theorem C ([8], p 15). Note that $1+\frac{\mu_2}{\mu_1}\left\{\frac{\alpha^2 s_2^2 - p^2}{\alpha^2 s_1^2 - p^2}\right\}^{1/2} \longrightarrow 1 + \frac{\mu_2}{\mu_1}$ as $|p| \to \infty$. Hence

$$L(p,\alpha) = \frac{\frac{\mu_2}{\mu_1}(\alpha s_2 + p)^{1/2}(\alpha s_2 - p)^{1/2}}{1 + \frac{\mu_2}{\mu_1}} \exp\{-\ln[h(p)]\}, \tag{A2}$$

where

$$h(p) = \frac{1 + \frac{\mu_2}{\mu_1}\{(\alpha^2 s_2^2 - p^2)/(\alpha^2 s_1^2 - p^2)\}^{1/2}}{1 + \frac{\mu_2}{\mu_1}} \tag{A3}$$

so that $h(p) \to 1$ as , $|p| \to \infty$.

Hence

$$L_+(p,\alpha) = \frac{\frac{\mu_2}{\mu_1}(\alpha s_2 + p)^{1/2}}{1 + \frac{\mu_2}{\mu_1}} \exp\left\{\frac{1}{2\pi i}\int_{c_l} \frac{\ln[h(z)]dz}{z - p}\right\} \tag{A4}$$

and

$$L_-(p,\alpha) = (\alpha s_2 - p)^{1/2} \exp\left\{\frac{-1}{2\pi i}\int_{C_r} \ln\frac{[h(z)]dz}{z - p}\right\} \tag{A5}$$

The contours of integrations are described in Georgiadis et. al. [4]. After the desired manipulations we may arrive at

$$L_+(p,\alpha) = \frac{\mu_2}{\mu_1}\frac{(\alpha s_2 + p)^{1/2}}{1 + \frac{\mu_2}{\mu_1}} \exp\left\{\frac{-1}{\pi}\int_{-b_2}^{-b_1} \tan^{-1}\left[\frac{\mu_2}{\mu_1}\left(\frac{\alpha^2 s_2^2 - z^2}{z^2 - b_1^2}\right)\right]\frac{dz}{z - p}\right\}, \tag{A6}$$

$$L_-(p,\alpha) = (\alpha s_2 - p)^{1/2} \exp\left\{\frac{1}{\pi}\int_{b_1}^{b_2} \tan^{-1}\left[\frac{\mu_2}{\mu_1}\left(\frac{\alpha^2 s_2^2 - z^2}{z^2 - b_1^2}\right)\right]\frac{dz}{z-p}\right\}. \tag{A7}$$

Here b_1, and b_2 are the two branch points given by $b_i = \alpha s_i$, $i = 1, 2$. (A6) and (A7) give the required factorization.

Appendix 2

The sum splitting required in equation (19) can be performed by inspection by subtracting the contribution due to the pole at $p = ik$ lying in the left half-plane because $k = k_1 + ik_2$ where k_2 is taken to be positive. The mixed term on the R.H.S. may be written as

$$R(p,\alpha) = \frac{(s_1^2\alpha^2 + k^2)^{1/2}}{L_-(p,\alpha)(p-ik)}$$

$$= \frac{(s_1^2\alpha^2 + k^2)^{1/2}}{(p-ik)}\left[\frac{1}{L_-(p,\alpha)} - \frac{1}{L_-(ik,\alpha)}\right] + \frac{(s_1^2\alpha^2 + k^2)^{1/2}}{(p-ik)}\frac{1}{L_-(ik,\alpha)} \tag{B1}$$

$$= R_-(p,\alpha) + R_+(p,\alpha).$$