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# Alpha, Beta – Compact Modules

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# $(\alpha, \beta)$–Compact Modules

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**Abstract**

We introduce a generalization of algebraic compactness which at the same time extends the notions of absolute purity and injectivity. Several characterizations for these injective properties are given without the use of injective or pure–injective hulls.

Throughout this article, all modules are left unital modules. A theory of ordinals is assumed where an ordinal $x = \{y < x : y \text{ is an ordinal }\}$, and where cardinals are initial ordinals. For notational convenience, we shall assign to all finite cardinals a unique symbol $\aleph_{-1}$, and for any set $I$, $|I|$ denotes the cardinality of $I$. In particular, $|I| = \aleph_{-1}$ simply means that $I$ is finite. If $M$ is an $R$–module, $M^I$ and $M^{(I)}$ denote respectively the direct product and the direct sum of $|I|$ copies of $M$. The following notation was introduced in [5]:

Let $M$ be an $R$–module. For each system $S$ of of linear equations

$$\sum_{k \in K} r_{jk} x_k = a_j \tag{1}$$

where $[r_{jk}]_{j \in J, k \in K}$ is a row–finite matrix over $R$ and $a_j \in M$, we denote by $S(M)$ the quotient $(M \oplus R^{(K)})/G$, where $G$ is the submodule of $M \oplus R^{(K)}$ generated by $(-a_j, (r_{jk})_{k \in K}) \quad (j \in J)$.

We shall use the following terminology.

**Definitions.** Let $M$ be an $R$–module and let $\alpha$ and $\beta$ be cardinals with $\alpha \leq \beta$ and $\beta$ infinite.

1. A system (1) of equations over $M$ is said to be $\alpha$*–solvable* if either (a) $\alpha$ is infinite and every subsystem of (1) consisting of less than $\alpha$ equations is solvable in $M$, or (b) $\alpha = \aleph_{-1}$ and (1) is compatible.

2. A submodule $N$ of $M$ is $\alpha$*–pure* in $M$ if every system of equations over $N$ which is $\alpha$–solvable in $M$ is also $\alpha$–solvable in $N$. $\aleph_0$–purity therefore coincides with the usual purity (in the sense of P.M. Cohn [2]), and clearly $N$ is $\aleph_{-1}$–pure in $M$ if and only if $N$ is a submodule of $M$.

3. $M$ is said to be $(\alpha, \beta)$*–compact* if every system of equations over $M$ which is $\alpha$–solvable in $M$ is $\beta$–solvable in $M$ also. If $M$ is $(\alpha, \gamma)$–compact for all $\gamma \geq \alpha$, we say that $M$ is $(\alpha, \infty)$*–compact.* On the other hand, $(\aleph_0, \beta^+)$–compactness is precisely $\beta$–compactness (see Fuchs [3] and Mycielski [7] for a universal algebraic approach). In particular, $M$ is $(\aleph_0, \infty)$–compact means $M$ is algebraically (or equationally) compact. Furthermore, $(\aleph_{-1}, \aleph_0)$–compactness and $(\aleph_{-1}, \infty)$–compactness coincide respectively with absolute purity and injectivity.

4. $M$ is $\beta$*–generated* if there exists a short exact sequence

$$0 \to N \to F \to M \to 0,$$

where $F$ is free and generated by less than $\beta$ elements. Also, if $N$ is generated by less than $\beta$ elements, we say that $M$ is $\beta$*–presented.*

**Remark.** Our notion of $(\alpha, \beta)$–compact modules extends that of $(m, n)$–pure injective abelian groups introduced by Megibben in [6]. The latter were characterized in [6, Theorem 4.5] as groups in which every $m$–solvable system of equations in $n$ unknowns ($m$ and $n$ infinite cardinals) is solvable. Since $Z$ is a countable ring, an abelian group is $(m,n)$–pure injective if and only if it is $(m,n^+)$–compact, where $n^+$ is the successor cardinal of $n$.

It is easy to see from the proof of [5, Proposition 1] that if $S$ is a system of equations over the $R$–module $M$ and if $f : M \to S(M)$ is the map $m \mapsto (\overline{m,0})$, then $S$ is compatible if and only if $f$ is a monomorphism, i.e. $S$ is $\aleph_{-1}$–solvable if and only if $M$ is $\aleph_{-1}$–pure in $S(M)$. Moreover, the same proof can be adapted to show that if $S$ is $\alpha$–solvable in $M$ for some infinite cardinal $\alpha$, then $M$ is $\alpha$–pure in $S(M)$ and, conversely, if this latter condition holds and $S_0$ is a subsystem of $S$ consisting of less than $\alpha$ equations, then $S_0$ is solvable in $S(M)$ and therefore in $M$ also. This proves the first statement of the following result. The second statement follows from the remark preceding [5, Theorem 1] and is included here for completeness.

**Proposition 1.** *Let $M$ be an $R$–module, let $S$ be a system of equations over $M$ and let $\alpha$ be an infinite cardinal. Then*

*(i) $S$ is $\alpha$–solvable in $M$ if and only if $M$ is $\alpha$–pure in $S(M)$.*

*(ii) $S$ is solvable in $M$ if and only if $M$ is a direct summand of $S(M)$.*

The following is a characterization of $\beta$–presented modules.

**Proposition 2.** *An $R$–module $P$ is $\beta$–presented for some infinite cardinal $\beta$ if and only if there exists an $R$–module $M$ and a compatible system of less than $\beta$ equations over $M$ such that $P$ and $S(M)/M$ are isomorphic.*

Proof. Suppose that first $P$ is $\beta$–presented and let

$$0 \to N \to R^{(K)} \to P \to 0$$

be an exact sequence where $|K| < \beta$ and $N$ is generated by $(r_{jk})_{k\in K} \in R^{(K)} \quad (j \in J, \quad |J| < \beta)$, say. Put $M = 0$, and consider the system $S : \sum_{k\in K} r_{jk}x_k = 0 \quad (j \in J)$. Clearly this system is compatible in $M$ and $S(M)/M \cong R^{(K)}/N \cong P$. Conversely, assume that $P \cong S(M)/M$ for some $R$–module $M$ and a compatible system $S$ over $M$,

$$\sum_{k\in K} r_{jk}x_k = a_j \qquad (j \in J, \quad |J| < \beta).$$

It is clear that we can suppose $|K| < \beta$. Now, let $H$ be the submodule of $R^{(K)}$ generated by $(r_{jk})_{k\in K} \quad (j \in J)$, and let $G$ be the submodule of $M \oplus R^{(K)}$ generated by $(-a_j, (r_{jk})_{k\in K}) \quad (j \in J)$. We have the following isomorphisms

$$P \cong S(M)/M \cong ((M \oplus R^{(K)})/G)/(((M+O)+G)/G) \cong (M \oplus R^{(K)})/(M \oplus H) \cong R^{(K)}/H.$$

Since $R^{(K)}/H$ is $\beta$–presented, so too is $P$.

**Proposition 3.** *Let $\alpha$ and $\beta$ be cardinals with $\beta$ infinite and let $M$ be an $\alpha$–pure submodule of an $R$–module $A$ such that $A/M$ is $\beta$–presented. Then there exists an $\alpha$–solvable system $S$ consisting of less than $\beta$ equations over $M$ such that $A$ and $S(M)$ are isomorphic.*

Proof. Since $A/M$ is $\beta$–presented, there exists an exact sequence

$$0 \to N \to R^{(K)} \xrightarrow{\pi} A/M \to 0$$

where $|K| < \beta$ and $N = \ker \pi$ is generated by $\{b_j\}_{j\in J}$ for some $J$ with $|J| < \beta$. Let $\{e_k\}_{k\in K}$ be the standard basis of $R^{(K)}$, and put $\pi(e_k) = a_k + M \quad (a_k \in A)$. Define an $R$–homomorphism $h : R^{(K)} \to A$ by $h(e_k) = a_k \quad (h \in K)$. Now there exist $r_{jk} \quad (j \in J, \quad k \in K)$ in $R$ such that $b_j = \sum_{k\in K} r_{jk}e_k \quad (j \in J)$. Clearly, $h(b_j) \in M$ for each $j \in J$ and $\sum_{k\in K} r_{jk}a_k = h(b_j)$. Hence the system $S$ given by $\sum_{k\in K} r_{jk}x_k = h(b_j) \quad (j \in J)$, is solvable in $A$, and therefore $\alpha$–solvable in $M$. Define now an $R$–homomorphism $g : M \oplus R^{(K)} \to A$ by $g(m, b) = m + h(b) \quad (m \in M, \quad b \in R^{(K)})$. Then $g$ is onto. For, if $a \in A$, then there exists $b \in R^{(K)}$ such that $\pi(b) = a + M$, and, observing that $a - h(b) \in M$, we obtain that $g(a - h(b), b) = a$. Finally, any element $(m, b)$ of $M \oplus R^{(K)}$ is in Ker $g$ precisely when $m = -\sum_{k\in K} s_k a_k$, $b = \sum_{k\in K} s_k e_k$ for some $s_k \in R \quad (k \in K)$. In this case $\pi(b) = \sum_{k\in K} s_k a_k = 0$ and there exists $t_j \in R \quad (j \in J)$ such that $b = \sum_j t_j b_j = \sum_k \sum_j t_j r_{jk} e_k$, which means $s_k = \sum_j t_j r_{jk} \quad (k \in K)$. Hence $(m, b) \in$ Ker $g$ if and only if $(m, b) = \sum_j t_k(-h(b_j), \quad (r_{jk})_{k\in K})$. Applying the first isomorphism theorem to $g$, we infer that $A \cong S(M)$.

**Remarks.**

1. Proposition 3 can easily be used to prove that if $M$ is $\beta$–pure in $A$ and $A/M$ is $\beta$–presented, then $M$ is a direct summand of $A$. This generalizes the familiar result stating that short pure–exact sequences with finitely presented third term split.

2. Let $R$ be left $\beta$–noetherian, i.e. every left ideal of $R$ is $\beta$–generated (so that $\aleph_0$–noetherian means noetherian in the usual sense). Then every $\beta$–generated $R$–module is $\beta$–presented. Using Remark 1, this implies that if $R$ is $\beta$–noetherian and $M$ is a submodule of $A$ where $A/M$ is $\beta$–generated, then $M$ is a direct summand of $A$ if and only if $M$ is $\beta$–pure in $A$ (cf [3] when $R$ is noetherian).

**Theorem 1.** *Let $M$ be an $R$–module and let $\alpha$ and $\beta$ be cardinals with $\alpha \leq \beta$ and $\beta$ infinite. The following statements are equivalent.*

*(i) $M$ is $(\alpha, \beta)$–compact.*

*(ii) $M$ is $\beta$–pure in every module in which it is $\alpha$–pure.*

*(iii) $M$ is a direct summand in every module $A$ in which it is $\alpha$–pure and such that $A/M$ is $\beta$–presented.*

*(iv) $M$ has the injective property with respect to every $\alpha$–pure exact sequence $0 \to A \to B \to C \to 0$ with $C$ $\beta$–presented.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). Follows from Remark 1 above.

(iii) $\Rightarrow$ (iv). Let $0 \to A \to B \to C \to 0$ be an $\alpha$–pure exact sequence of $R$-modules with $C$ $\beta$–presented and let $f \in \mathrm{Hom}_R(A, M)$. By Proposition 3, there exists an $\alpha$–solvable system $S$ consisting of less than $\beta$ equations over $M$ such that $\beta$ and $S(A)$ are isomorphic. Suppose $S$ is the system $\sum_{k \in K} r_{jk} x_k = a_j$, where $j \in J$, $|J| < \beta$ and $a_j \in A$. Denote by $S'$ the system over $M$ given by $\sum_{k \in K} r_{jk} x_k = f(a_j) \quad (j \in J)$. It is clear that $S'$ is $\alpha$–solvable in $M$ so that $S'(M)$ contains $M$ as an $\alpha$–pure submodule by Proposition 1. Also, by Proposition 2, $S'(M)/M$ is $\beta$–presented, which by hypothesis implies that $M$ is a direct summand of $S'(M)$. Therefore there is a projection $\pi : S'(M) \to M$, coinciding with the identity on $M$. Now, let $g : S(A) \to S'(M)$ be the map given by $g(\overline{a, p}) = (\overline{f(a), p})$ for any $a \in A$, $p \in R^{(K)}$. It is easy to check that $g$ is a well–defined homomorphism and that $\pi g$ is an extension of $f$ to $S(A)$. Since $B$ and $S(A)$ are isomorphic, we obtain (iv).

(iv) $\Rightarrow$ (i). Let $S$ be an $\alpha$–solvable system over $M$. To prove that $S$ is $\beta$–solvable, we may clearly assume that $S$ consists of less than $\beta$ equations, so that $M$ is $\alpha$–pure in $M$ and $S(M)/M$ is $\beta$–presented. Consequently, there exists an $R$–homomorphism $\pi : S(M) \to M$ extending the identity on $M$. This means that $M$ is a direct summand of $S(M)$ and so $M$ is $\beta$–pure in $S(M)$. By Proposition 1, $S$ is $\beta$–solvable in $M$.

**Remark.** In the particular case when $\alpha = \aleph_{-1}$ and $\beta = \aleph_0$, Theorem 1 yields the well–known characterization of absolutely pure modules: $M$ is absolutely pure if and only if $\mathrm{Ext}^1_R(C, M) = 0$ for all finitely presented modules. In the same vein, the familiar characterizations of both injective modules and of pure–injective (that is, algebraically or equationally compact) modules can similarly be generalized by using the proof of Theorem 1. This is given by the following.

**Theorem 2.** *Let $M$ be an $R$–module and let $\alpha$ be a cardinal. The following statements are equivalent.*

*(i) $M$ is $(\alpha, \infty)$–compact.*

*(ii) $M$ is a direct summand of every module in which it is $\alpha$–pure.*

*(iii) $M$ has the injective property with respect to every $\alpha$–pure exact sequence $0 \to A \to B \to C \to 0$. (cf Problem 27 in [4]).*

We note finally that by narrowing our attention to systems $S$ with one or finitely many unknowns, the arguments presented here can readily be applied so as to give a proof (which does not use the existence of compact extensions) of the equivalence of conditions (1), (3) and (4) on finitely pure–injectives in [1, Theorem 10].

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