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Abstract

We will consider the use of block elimination for the calculation of generalized turning and bifurcation points for two point B.V.P's. It will be shown that such algorithm will reduce the amount of work required interms of LU-factorizations to minimal. Since the discretization error of the approximated solution has an asymptotic expansion interms of even powers of h (the step size). This will lead to the use of some type of extrapolation to produce more accurate results. Finite differences will be used to discretize the two point B.V.P's.

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1 Introduction

The problem under consideration is a two point boundary value problem of the form

$$\bar{F}(z,\lambda) = \begin{cases} z''(x) + f(x, z(x), \lambda) = 0, \ a \leq x \leq b \\ G(z(a), z(b)) = 0, \end{cases} \tag{1.1}$$

where $\bar{F}(z,\lambda)$ is a nonlinear operator from $D \subset Z \times R \longrightarrow Y$ where Z and Y are two Banach spaces, λ is a real parameter and D is an open domain in $Z \times R$. When discretized by for example finite differences, it will produce a family of nonlinear operators

$$F(z,\lambda) = 0. \tag{1.2}$$

For any given λ, one may be able to characterize the full solution set

$$S_\lambda = \{z \in Z; F(z,\lambda) = 0\}. \tag{1.3}$$

If the Frechet derivative $F_z^0 \equiv F_z(z_0, \lambda_0)$ has range all of Y, then S_λ will form a smooth manifold, which vary smoothly in λ, of dimension

$$i = dimNull(F_z^0) - Def(F_z^0), \tag{1.4}$$

where i is the Fredholm index of F_z^0, see Keller[11], and $Def(F_z^0) \equiv codimRange(F_z^0)$. While if the range of F_z^0 is a proper subset of Y, then (z_0, λ_0) will be a singular point and the numerical computation of such points will not be straight forward but rather complicated. The latter case is what we are interested in.

We will assume:

$$Null(F_z^0) = Span\{\phi_0\}$$

and

$$Range(F_z^0) = \{y \in Y; \psi_0^T y = 0\}. \tag{1.5}$$

If (z_0, λ_0) is a simple turning point, then F_λ^0 is not in the range of F_z^0 and the Fredholm index i is zero since $Null(F_z^0)$ is spanned by one vector and $Def(F_z^0) = 1$ and $Def(F_z^0, F_\lambda) = 0$. While if (z_0, λ_0) is a bifurcation

point, then F_λ^0 is in the range of F_z^0 which will mean that $i = 1$. We will also assume that

$$\psi_0^T f_{zz}^0 \phi_0 \phi_0 \neq 0, \tag{1.6}$$

in the simple turning point case and

$$det \begin{pmatrix} \psi_0^T F_{zz}^0 \phi_0 \phi_0 & \psi_0^T F_{zz}^0 \phi_0 \phi_1 \\ \psi_0^T F_{zz}^0 \phi_0 \phi_1 & \psi_0^T F_{zz}^0 \phi_1 \phi_1 \end{pmatrix} \neq 0, \tag{1.7}$$

in the simple bifurcation case, see Crandall and Rabinwitz[8]. This no more than the Morse condition on the Hessian of the reduced mapping obtained through Lyaponov-Schmidt reduction, where $\phi_0, \phi_1 \in Z$ and $\psi_0 \in Z^*$ span the null spaces of F_z^0 and F_z^{0*} respctively. The condition (1.6) and (1.7) are needed to gaurantee the existence of a unique solution in each case.

To find λ_0 exactly (which could represent a critical load or temperature) and in the simple turning point case, the under determined system $F(z, \lambda) = 0$ can be augmented with an equation $g(z, \lambda) = 0$ that enforces the singularity of F_z^0; that is, to solve the extended system

$$F(z, \lambda) = 0$$

$$g(z, \lambda) = 0. \tag{1.8}$$

In the simple bifurcation case, the singularity will be unfolded as $F(z, \lambda) + \gamma r = 0$ where r is not in the range of (F_z, F_λ) and $g(z, \lambda)$ will have two components (the procedure of finding g explicitly in both cases will be discussed in section 2). Perturbed bifurcation points and cubic turning points will have the form $F(z, \lambda, \mu) = 0$, which depends on two parameters. In the case of perturbed bifurcation points, the problem will be computing a generalized simple turning point without any modification. In the cubic turning point case, the problem will be computing a simple turning point of the extended system (1.8), thus one have to extend (1.8) again to obtain $\tilde{g}$ to correspond to $\tilde{F}(y, \mu)$ in the usual way (to be explained in section 2) to obtain the system

$$\tilde{F}(y, \mu) = \begin{cases} F(z, \lambda, \mu) = 0 \\ g(z, \lambda, \mu) = 0 \end{cases}$$

$$\tilde{g}(z, \lambda, \mu) = 0, \tag{1.9}$$

where $y = (z, \lambda)$, see Attili[1]. This system is a nonsingular characterization of cusp points. More on the terminology used here and on the nonsingularity of the Jacobian of (1.9) can be found in Jepson and Spence[10], Attili[1] and Spence and Werner[16].

remark 1.1 : The cubic turning point (z_0, λ_0, μ_0) of $F(z, \lambda, \mu) = 0$ with respect to λ and $\mu = \mu_0$ fixed corresponds to a simple turning point (z_0, λ_0, μ_0) of the extended system (1.8) with respect to μ. This justifies the idea of the system in (1.9).

The main concern of this paper is to make use of the special structure of $F_z(z, \lambda)$ (tridiagonal matrix), which is the result of discretization by finite differences. This will be through the use of block elimination which will lead to saving alot in the number of operations needed to carry out each iteration of the Newton's method. The details of such saving will be in section 3. It will be shown that once the LU-factorization of $F_z(z, \lambda)$ is computed, only back substitutions are needed to solve the systems involved. Also, it was shown in Attili[4] that the discretization error of the approximate solution of the extended system has an asymptotic expansion in even powers of h (the step size). This will justify the use of some type of extrapolation, inparticular, Richardson extrapolation. Section 2 will contain the characterization of the singular points. In section 3, we will present the numerical details. Finally some numerical experementation will be presented in section 4.

2 Characterization of Simple Singularities

To numerically calculate the singular points, one will be practically finding the intersection of the solution manifold $F^{-1}(0)$ and the singular manifold, Attili[1,2]. The general problem will then be the characterization of the singular manifold. For that reason, choose $r(z, \lambda) \in Z_2$ and $T(z, \lambda) \in Z_1 \times R^{i+1}$, where Z_1 and Z_2 are Banach spaces, to be completely continous and continously differentiabl in Z such that if (z_0, λ_0) is a singular point then $r_0 \equiv r(z_0, \lambda_0) \notin Range(F_z^0)$ and $T^T(z_0, \lambda_0)v = 0, F_z v = 0$ imply $v = 0$. Then we will have the following lemma~

Lemma 2.1 *The linear operator*

$$A(z, \lambda) = \begin{bmatrix} F_z & r(z, \lambda) \\ T^T(z, \lambda) & 0 \end{bmatrix}$$

where $A : Z_1 \times R \longrightarrow Z_2 \times R^{i+1}$, *has a bounded inverse for* (z, λ) *in the neighborhood of* (z_0, λ_0), *where* Z_1 *and* Z_2 *are Banach spaces. Moreover there exist smooth functions* $v(z,\lambda) \in Z_1 \times R^{i+1}, g(z,\lambda) \in R^{i+1}$ *and* $u(z,\lambda) \in Z_2^*$ *satisfying*

$$F_z v = -r g^T, \; u^* F_z = -g^T T^T$$
$$T^T v = I_{i+1}, \; u^* r = 1. \tag{2.1}$$

The proof is straight forward and more can be found in Attili[1,2] and Griewank and Reddien[9]. Note that in the turning point case $i = 0$ and in the simple bifurcation point case $i = 1$. It follows from the above lemma that (z, λ) is in the singular set if and only if $g(z, \lambda) = 0$, the columns of v spans the null space of F_z and

$$g^T(z, \lambda) = -u^* F_z v. \tag{2.2}$$

Differentiation of (2.2) and using (2.1) will lead to

$$(g^T)' = -u^* F_z' v + u^* r' g^T - g^T T' v, \tag{2.3}$$

where the prime denotes differentiation with respect to z or λ. If r and T are chosen to be constants or (z, λ) is in the singular set; that is, $g(z, \lambda) = 0$, then (2.3) simplifies to

$$(g^T)' = -u^* F_z' v \tag{2.4}$$

or componentwise,

$$g_j' \approx -u^* [F_z(z_0 + \epsilon v_j, \lambda) - F_z(z_0, \lambda)]/\epsilon \tag{2.5}$$

which approximates g' with only $i+1$ extra evaluations of F_z. The solution of the augmented system

$$F(z, \lambda) = 0$$
$$g(z, \lambda) = 0, \tag{2.6}$$

will produce the intersection of the manifolds $F^{-1}(0)$ and the singular manifold. The system (2.6) will be regular if and only if

$$H_0 = \nabla_z g(z_0, \lambda_0) \phi_0$$

is nonsingular. Such condition is nothing more than (1.6) and (1.7) in the simple turning and simple bifurcation points respectevely. The framwork discussed above can be extended to characterize bifurcation points with higher rank deficiency, Attili[3] and Rabier and Reddien[14], and also to characterize symmetry breaking bifurcation points, Attili[2].

3 Numerical Details

We will present the details of carrying out the computation of cubic turning points since they are more involved than other singular points for which the computation is very similar. To do so, one needs to solve the system in (1.9); that is,

$$\begin{aligned} F(z,\lambda,\mu) &= 0 \\ g(z,\lambda,\mu) &= 0 \\ \tilde{g}(z,\lambda,\mu) &= 0. \end{aligned} \tag{3.1}$$

Note that the Jacobian of this system has a bounded inverse as was shown by Attili[1]. From laemma 2.1, the determining systems for g and $\tilde{g}$ will be respectively,

$$\begin{bmatrix} F_z & R \\ T^T & 0 \end{bmatrix} \begin{bmatrix} v \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.2}$$

and

$$\begin{bmatrix} F_z & F_\lambda & \tilde{R} \\ g_z & g_\lambda & \\ \tilde{T}^T & & 0 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.3}$$

where $T, \tilde{T}, R$ and $\tilde{R}$ are to be chosen as was described in lemma 2.1. To find u, the system to be solved will be

$$[u \quad g] \begin{bmatrix} F_z & R \\ T^T & 0 \end{bmatrix} = [0 \quad 1]. \tag{3.4}$$

Looking back at the systems (3.2),(3.3) and (3.4) and the Jacobian of (3.1), one realizes that F_z is the main block of the matrices involved. This gives rise to the idea of deflated block-elimination which requires one LU-factorization of F_z and then back substitutions will only be needed to carry out the solutions of the previous 4−systems. In addition to that, one can make use of the special structure of F_z. This is important since the LU-factorization requires $O(N^3)$ operations while the back substitution requires only $O(N^2)$ operations. To apply the Richardson extrapolation, $3-4$ different step sizes h are needed. This will mean solving more systems with the minimum number of LU-factorizations. Not only this but we will show

that even the calculation of $g_z, g_\lambda, g_\mu, \tilde{g}_z, \tilde{g}_\lambda$ and $\tilde{g}_\mu$ will require the solution of systems with F_z being the main block.

Let us start with the block-elimination algorithm, Chan[6,7] and Keller[12]. To solve

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ \gamma \end{bmatrix} \tag{3.5}$$

where A is an $n \times n$ matrix, b, c^T are column and row vectors respectively and d a scalar.

Algorithm 3.1:

(i) Solve

$$Av = b$$

$$Aw = f$$

(ii) Compute

$$y = (\gamma - c^T w)/(d - c^T v)$$

$$x = w - yv.$$

This algorithm will take care of the systems in (3.2) and, after slight modification (3.4) using $A \equiv F_z$. To solve (3.1) and (3.3), we have to expand step(i) in algorithm 3.1 since b and c^T are not just one column and one row vectors as before. For example, to solve the system (3.3) and with $\tilde{R} = (\tilde{R}_1, \tilde{R}_2)^T$ where $\tilde{R}_1$ is an $n \times 1$ vector and $\tilde{R}_2$ a scalar, we will have the following algorithm. Algorithm 3.2:

(i) Solve

$$(a)\ F_z y_1 = F_\lambda$$

$$(b)\ F_z y_2 = \tilde{R}_1,$$

then

$$x_{12} = (\tilde{R}_2 - g_z^T y_2)/(g_\lambda - g_z^T y_1)$$

$$x_{11} = y_2 - x_{12} y_1$$

and

$$x_1 = (x_{11}\ x_{12}).$$

(ii) Solve

$$(a)\ F_z \tilde{y}_1 = F_\lambda$$

$$(b)\ F_z \tilde{y}_2 = 0,$$

then

$$x_{22} = (-g_z^T \tilde{y}_2)/(g_\lambda - g_z^T \tilde{y}_1)$$

$$x_{21} = \tilde{y}_2 - x_{22}\tilde{y}_1$$

and

$$x_2 = (x_{21} \quad x_{22})^T$$

(iii) Compute

$$\tilde{g} = (1 - \tilde{T}^T x_2)/(-\tilde{T}^T x_1)$$

and

$$\tilde{v} = (x_2 - \tilde{g}x_1).$$

For more details on the above algorithm, see Shehadeh[15]. Note that the systems in steps $(i-a)$ and $(ii-a)$ are identical; that is, $y_1 = \tilde{y}_1$. Thus there is no need to repeat the back-substitution to find $\tilde{y}_1$ once y_1 is computed. This means three back-substitutions will do the job.

It should be noted here that using the block-elimination may be unstable since F_z will be nearly singular. When numerically tested, however, it was fairly reliable and it only fails when F_z is very singular at which point the accuracy is usually high enough to stop the iterations. Still to avoid the difficulties which arise when F_z is singular or nearly singular, consider the expanded form of (3.5); that is,

$$Ax + by = f$$

$$c^T x + dy = \gamma. \tag{3.6}$$

Since ψ_0^T is a left null vector of A, we will have

$$y = \psi_0^T f/\psi_0^T b \tag{3.7}$$

where $u^T = \psi_0^T$ at the singular point. Here (3.7) is well defined since $\psi_0^T b \neq 0 (u^T R = 1)$. If (3.7) is substituted in (3.6), we obtain

$$Ax = f - b(u^T f/u^T b). \tag{3.8}$$

The right hand side of (3.8) is in the range of A which means it is solvable and the general solution is

$$x = x_p + \gamma_0 \phi_0 \tag{3.9}$$

where x_p is any particular solution and $\gamma_0 \in R$. With this and since $c^T \phi_0 \neq 0$, we will have the following theorem

Theorem 3.1 *The system* (3.6) *has a unique solution given by*

$$y = \psi_0^T f / \psi_0^T b, \quad x = x_p + \gamma_0 \phi_0$$

where γ_0 *is given by*

$$\gamma_0 = (\gamma - dy - c^T x_p)/c^T \phi_0$$

and ψ_0 *and* ϕ_0 *are the left and right null vectors of* $A = F_z$ *at the singular point.*

To carry out the Newton's iterations, one also needs the partials of g and $\tilde{g}$ with respect to z, λ and μ. The maindifficulty in doing so is that the gradient of g and $\tilde{g}$, however they are defined, depends on second derivatives of f which may be costly to evaluate.

Consider $g = -u^T F_z v$ as defined in lemma 2.1. Considering the special case $R = F_\lambda$ $(F_\lambda \notin Range(F_z))$ and differentiating g with respect to z, we obtain

$$g_z = -u_z^T F_z v - u^T F_{zz} v - u^T F_z v_z. \tag{3.10}$$

From the determining systems (3.2) and (3.4) for g, u and v, we will have $F_z v = -g F_\lambda, u^T F_z = -g T^T, T^T v_z = 0$ and $u^T F_{z\lambda} = -u_\lambda^T F_\lambda$ which when substituted in (3.10) leads to

$$g_z = -u^T F_{z\lambda} g - u^T F_{zz} v. \tag{3.11}$$

Using the same approach one obtains

$$g_\lambda = -u^T F_{\lambda\lambda} g - u^T F_{z\lambda} v. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$(g_z \;\; g_\lambda) = (-u^T F_{zz} - u^T F_{z\lambda} \qquad - u^T F_{z\lambda} - u^T F_{\lambda\lambda}) \begin{pmatrix} v \\ g \end{pmatrix} \tag{3.13}$$

or equivalently

$$\nabla_{z\lambda}\, g = -u^T (\nabla_{z\lambda}^2 F) \begin{pmatrix} v \\ g \end{pmatrix}. \tag{3.14}$$

Equation (3.14) gives a way of evaluating g_z and g_λ.

To evaluate the other partials of g and $\tilde{g}$, we consider the special cases of having $R = F_\lambda$ in the determining system for g and $\tilde{R} = (0\ 1)^T$ and $\tilde{T}^T = (T^T\ 0)$ in the determining system for $\tilde{g}$; that is, respectively (3.2) and (3.3) becoms

$$\begin{aligned} &(i)\ \ F_z v + F_\lambda g = 0 \\ &(ii)\ \ T^T v = 1 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &(i)\ \ F_z \tilde{v} + \delta F_\lambda = 0 \\ &(ii)\ \ g_z \tilde{v} + \delta g_\lambda + \tilde{g} = 0 \\ &(iii)\ \ T^T \tilde{v} = 1. \end{aligned} \tag{3.16}$$

Comparing (3.15) and (3.16), we conclude that $\tilde{v} = v, g = \delta$ and $\tilde{g} = -g_z v - g g_\lambda$. This will imply that

$$\begin{aligned} &(i)\ \ \tilde{g}_z = -g_z v_z - g_{zz} v - g_z g_\lambda - g g_{z\lambda} \\ &(ii)\ \ \tilde{g}_\lambda = -g_z v_\lambda - g_{z\lambda} v - g_\lambda g_\lambda - g g_{\lambda\lambda} \\ &(iii)\ \ \tilde{g}_\mu = -g_z v_\mu - g_{z\mu} v - g_\mu g_\lambda - g g_{\lambda\mu}. \end{aligned} \tag{3.17}$$

Let us consider the computation of $(3.17 - i)$ only since other partials can be done in a similar way. To do so one needs v_z and g_{zz} or an approximation to them since we already computed g_z, v, g_λ and g. Differentiating (3.15) with respect to z we obtian

$$\begin{aligned} &F_z v_z + F_{zz} v + g_z F_\lambda + g F_{z\lambda} = 0 \\ &T^T v_z = 0, \end{aligned} \tag{3.18}$$

or equivalently

$$\begin{bmatrix} F_z & F_\lambda \\ T^T & 0 \end{bmatrix} \begin{bmatrix} v_z \\ g_z \end{bmatrix} = \begin{bmatrix} -F_{zz} v - g F_{z\lambda} \\ 0 \end{bmatrix}. \tag{3.19}$$

Similarly, to calculate g_{zz} one obtains from (3.18) the system

$$\begin{bmatrix} F_z & F_\lambda \\ T^T & 0 \end{bmatrix} \begin{bmatrix} v_{zz} \\ g_{zz} \end{bmatrix} = \begin{bmatrix} -R_1 \\ 0 \end{bmatrix}, \tag{3.20}$$

where $\dot{R}_1 = 2F_{zz}v_z + F_{zzz}v + 2g_zF_{z\lambda} + \dot{g}F_{zz\lambda}$. The matrix on the left hand side of (3.19) or (3.20) is the same one used in (3.15), which means we can compute the value of g_z while solving for v_z. Similar arguments will be needed and similar systems will be solved to calculate all partials of g and $\tilde{g}$ in which we make use of the LU-factorization of F_z. This means that 6-systems which involve the same matrix in (3.19) and (3.20) are needed to calculate the partials of g and $\tilde{g}$. As a result, one can solve 10 systems each iteration using one LU-factorization of F_z. It should be noted that one less system is needed when the singular point is of the simple turning point or simple bifurcation point type. Other than that the details are exactly the same and are omitted.

4 Numerical Examples

The following examples were used for numerical experimentation

(1) We solved the one-dimensional nonlinear problem

$$y'' + \lambda e^y = 0 \text{ on the interval } [0,1]$$

With the boundary conditions $y(1) = y(0) = 0$ which has a simple turning point at the critical parameter, $\lambda = 3.513807$. With the initial guess $\lambda = 3.4$ and using the extended system (3.1.5) The results for the fifth iteration with $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}$ and $\frac{1}{24}$ using block elimination are given in Table 1.

Table 1

h	g	λ
$\frac{1}{3}$	0.143051E-05	3.31092
$\frac{1}{6}$	-0.722452E-05	3.46261
$\frac{1}{12}$	0.560958E-05	3.50110
$\frac{1}{24}$	0.471423E-05	3.51062

Using the results in Table 1, we applied the Richardson extrapolation and obtained $\lambda = 3.51378$. The results of the application of the extrapolation are given in Table 2.

Table 2.

h	Number of Extrapolation			
	0	1	2	3
$\frac{1}{3}$	3.31092			
		3.51318		
$\frac{1}{6}$	3.46261		3.51398	
		3.51393		3.51378
$\frac{1}{12}$	3.50110		3.51378	
		3.51379		
$\frac{1}{24}$	3.51062			

It is clear from Table 1 that $g \to 0$ for the various values of h as expected. The CPU time is equal to 0.040 sec.

Repeating the same calculations for $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}$ and $\frac{1}{24}$ but without using the block-elimination this time. The results were almost the same and are given in Table 3.

Table 3

h	g	λ
$\frac{1}{3}$	0.1172142E-03	3.31091
$\frac{1}{6}$	.395983E-05	3.46261
$\frac{1}{12}$	.171911E-05	3.50110
$\frac{1}{24}$	-.404437E-05	3.51062

Again applying Richardson extrapolation on the approximate values of λ_h the results are given in Table 4.

Table 4.

h	Number of Extrapolation			
	0	1	2	3
$\frac{1}{3}$	3.31092			
		3.51318		
$\frac{1}{6}$	3.46261		3.51398	
		3.51393		3.51378
$\frac{1}{12}$	3.50110		3.51378	
		3.51379		
$\frac{1}{24}$	3.51062			

The CPU time is equal to 0.240 sec. It is clear that the CPU time used with block elimination is $\frac{1}{6}$ of the time used without block elimination.

We also solved the system with $h = \frac{1}{50}$ using block elimination. With the same initial guess $\lambda = 3.4$, we obtain the solution $\lambda = 3.51295$ and $g = .98533E - 04$. The CPU time was 0.190 sec. This shows that with such a large system, we were not able to obtain the same accuracy as was obtained in Table 3, also the time required is 4 times that needed with Richardson extrapolation.

(2) We consider the two parameter problem

$$y'' + \lambda \exp\left(\frac{y}{1+\mu y}\right) = 0, \qquad y(0) = y(1) = 0,$$

which has a cubic turning point at $\lambda = 5.22949$ and $\mu = 0.2457804$, See Attili [1]. With the initial guesses $\lambda = 5.0$ and $\mu = 0.2$, and using the extended system (3.1.14), the results after four iterations with $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}$ and $\frac{1}{24}$ using block elimination are given in Table 5.

Table 5

h	g	$\tilde{g}$	λ	μ
$\frac{1}{3}$	.107766E-03	.419806E-02	4.87464	.250123
$\frac{1}{6}$	.201616E-05	0.133458E-04	5.14327	.246672
$\frac{1}{12}$	.230550E-04	.591536E-05	5.20793	.245981
$\frac{1}{24}$	-.454786E-04	.603387E-04	5.22398	.245823

Similarly applying Richardson extrapolation for the results in Table 5 we obtained $\lambda = 5.22931$ and $\mu = .245772$. It is also clear from Table 5 that $g \to 0$ and $\tilde{g} \to 0$ for various values of h as expected. The CPU time required is 0.160 Sec.

Repeating the same calculations for $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}$ and $\frac{1}{24}$ but without using the block elimination. The results were very close to what we obtained in Table 5. Applying Richardson extrapolation on the approximate values of λ_h and μ_h we obtained $\lambda = 5.22938$ and $\mu = .245776$ The CPU time is 1.35 sec. Then the CPU time used with block elimination is less than $\frac{1}{8}$ the time used without block elimination.

(3) We consider the two point boundary value problem

$$x''(s) - p(\lambda)X''(s) + \pi^2\lambda f(x(s) - p(\lambda)X(s)) = 0$$

$x(0) = x(1) = 0$,
where $p(\lambda) = \lambda^4 \exp(-\lambda/2)$, $f(z) = z^2 + z$ and $X(s) = s(1-s)\exp(s)$, which has a simple bifurcation point at $\lambda = 1$. With the initial guess $\lambda = .9$ and $\gamma = .01$. With $h = \frac{1}{36}$ the result for the fifth iteration using block elimination is $\lambda = 1.00025$. As expected $\gamma \to 0$ and both components of g did the same. The CPU time required is .08 sec.

Repeating the same calculations for the same h but without using block elimination, the results we obtained were very similar to the ones with block elimination. The CPU time required is .83 sec. This means that the CPU time used with block elimination is less than $\frac{1}{10}$ the time used without block elimination.

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