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# Some Properties of $E$ -Bisimple Semigroups

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## Abstract

We give necessary and sufficient conditions for an  $E$ -bisimple semigroup  $S^*$  to be the homomorphic image of an  $E$ -bisimple semigroup  $S$ . We determine the translational hull of an  $E$ -bisimple semigroup and use this result to determine all ideal extensions of an  $E$ -bisimple semigroup by completely 0-simple semigroup. Finally, we determine all ideal extensions of a Brandt semigroup with finite set of idempotents by a simple ( $E$ -bisimple) semigroup with zero appended.

Let  $E$  be a band (idempotent semigroup). The collection  $E(R)$  of  $R$ -classes of  $E$  may be partially ordered by the following rule. If  $R_1, R_2 \in E(R)$ ,  $R_1 < R_2$  if and only if  $e < f$  for all  $e \in R_1$  and  $f \in R_2$  ( $e \leq f$  if and only if  $ef = fe = e$ ). If  $E(R)$ , under this order, is order isomorphic to  $I^0$ , the non-negative integers, under the reverse of the usual order,  $E$  is called a naturally ordered band. A bisimple semigroup whose idempotents form a naturally ordered band is termed an  $E$ -bisimple semigroup. In [27], Warne showed that  $S$  is an  $E$ -bisimple semigroup if and only if  $S = (I^0 \times \{0\}) \times (G \times P) \cup ((I^0 \times N) \times (G \times K))$ , where  $G$  is a group,  $N$  is the natural numbers, and  $P$  and  $K$  are sets under the multiplication  $((n, k), (g, p))((r, s), (h, q)) = ((n + r - t, k + s - t), (g\theta^{r-t}h\theta^{k-t}, x))$  where juxtaposition denotes multiplication in  $G$ ,  $\theta$  is an endomorphism of  $G$  ( $\theta^0$  is the identity automorphism),  $t = \min(r, k)$ ,  $x = q$  or  $p(h\theta^{k-r-1}\gamma)$  according to whether  $r \geq k$  or  $k > r$  and  $\gamma$  is a homomorphism of  $G$  into  $G_K$ , the symmetric group on  $K$ .

In section 1, we give necessary and sufficient conditions for an  $E$ -bisimple semigroup  $S^*$  to be the homomorphic image of an  $E$ -bisimple semigroup  $S$  (Theorem 1.5). Theorem 1.5 generalizes the corresponding result of Munn and Reilly [10] from bisimple  $\omega$ -semigroups ( $E$ -bisimple semigroups whose idempotents form a semilattice) to arbitrary  $E$ -bisimple semigroups. Theorem 1.5 also generalizes Warne's isomorphism theorem for  $E$ -bisimple semigroups [27, Theorem 2.1]. In section 2, we determine the translational hull of an  $E$ -bisimple semigroup (Theorem 2.1) and we use Theorem 2.1 to determine all ideal extensions of an  $E$ -bisimple semigroup by a completely 0-simple semigroup (Theorem 2.3). Finally, we determine all ideal extensions of a Brandt semigroup with finite set of idempotents by a simple ( $E$ -bisimple) semigroup with zero appended (Theorem 2.5).

If  $S$  is a semigroup,  $E(S)$  will denote the set of idempotents of  $S$ . If  $\alpha$  is an order type,  $\alpha^*$  denotes with the converse order. We term  $S$  an  $\alpha$ -semigroup if  $E(S)$  with its usual order has order type  $\alpha^*$ . The structure of  $\omega$ -bisimple inverse semigroups was given by Reilly [12] and Warne [17]. The structure of  $\omega^n$ -bisimple inverse semigroups was given by Warne [20]. The structure of  $I$ -bisimple inverse semigroups and  $\omega^n I$ -bisimple inverse semigroups was given by Warne in [21] and [26] respectively. The structure of  $\omega$ -inverse semigroups was given by Munn [8] and the structure of  $I$ -inverse semigroups was given by Warne [24]. For another approach to structure theory see [28]. Various properties of these semigroups (i.e. the determination of homomorphisms, congruences, ideal extensions, study of the lattice of congruences) have been investigated, for example, by Baird [1, 2], Bonzini

and Cherubini [3-5], Munn [9], Munn and Reilly [10], Petrich [11], Scheiblich [13], and Warne [19, 21-23, 25, 26.]

Unlike the semigroups in the paragraph above, the  $E$ -bisimple semigroups are not inverse semigroups nor are they  $\mathcal{H}$ -compatible. Nevertheless, the structure theorem is of sufficient simplicity to yield a homomorphism theory and an ideal extension theory. A determination of the congruence relations and an investigation of the lattice of congruences will be the subject of future papers.

We will use the following basic definitions, concepts, and notation of [7]: Green's relations ( $\mathcal{R}$ ,  $\mathcal{L}$ ,  $\mathcal{H}$ , and  $\mathcal{D}$ ),  $R$ -class, regular semigroup, simple semigroup, bisimple semigroup, inverse semigroup, equivalent definitions of inverse semigroup, right zero semigroup, idempotent, natural (usual) partial order of idempotents, semilattice, band, completely 0-simple semigroup, Rees representation  $M^0(G; F, \Lambda; Q)$  of a completely 0-simple semigroup, Brandt semigroup, Rees representation  $M^0(G; \Lambda, \Lambda; \Delta)$  of a Brandt semigroup, right (left) translation, inner right (left) translation, linked left and right translation, translational hull, partial homomorphism, ideal extension (we will often call an ideal extension just an extension), weakly reductive semigroup, full symmetric inverse semigroup, and determination of Green's relations on a completely 0-simple semigroup, adjunction of a zero element to semigroup  $S$  (for brevity, we say  $S$  with a zero appended).

Let  $\rho$  be a congruence relation on a semigroup  $S$  ( $\rho$  will also denote the natural homomorphism of  $S$  onto  $S/\rho$ ). If  $a \in S$ ,  $a\rho$  will denote the  $\rho$ -class of  $S$  containing

a. If  $S/\rho$  is a type  $A$  semigroup,  $\rho$  is called a type  $A$  semigroup congruence on  $S$ . For example, if  $S/\rho$  is an inverse semigroup,  $\rho$  is called an inverse semigroup congruence on  $S$ . A congruence  $\rho$  is termed an idempotent separating if  $e\rho f$  ( $e, f \in E(S)$ ) implies  $e = f$ . A semigroup  $S$  is termed  $\mathcal{H}$ -compatible if  $\mathcal{H}$  is a congruence relation on  $S$ .

## 1 Homomorphism of $E$ -bisimple semigroup.

In this section, we give necessary and sufficient conditions for the  $E$ -bisimple semigroup  $S^*$  to be the homomorphic image of the  $E$ -bisimple semigroup  $S$  (Theorem 1.5).

First we will state the structure theorem for  $E$ -bisimple semigroups in more convenient form (Theorem 1.1), and, for an  $E$ -bisimple semigroup  $S$ , we will give the determination of  $\mathcal{R}$ ,  $\mathcal{L}$ ,  $E(S)$ , and the usual order on  $E(S)$  ( $e \leq f$  if and only if  $ef = fe = e$ ).

(Lemma 1.2). These results will be used frequently and often without explicit mention.

**Theorem 1.1** (Warne [27, Theorem 1.1]).  *$S$  is an  $E$ -bisimple semigroup if and only if  $S = ((I^0 \times \{0\}) \times (G \times P)) \cup ((I^0 \times N) \times (G \times K))$  where  $G$  is a group and  $P$  and  $K$  are sets under the multiplication*

$$((n, k), (g, p))((r, s), (h, q)) = \begin{cases} ((n + r - k, s), (g\theta^{r-k}h, q)) & \text{if } r \geq k \\ ((n, k + s - r), (g(h\theta^{k-r}), p(h\theta^{k-r-1}\gamma))) & \text{if } k > r \end{cases}$$

where  $\theta$  is an endomorphism on  $G$  and  $\gamma$  is a homomorphism of  $G$  into  $G_K$ , the symmetric group on  $K$ .

We will write  $S = (G, P, K, \theta, \gamma)$ .

**Lemma 1.2** (Warne, [27, Lemma 2.1.]). *Let  $S = (G, P, K, \theta, \gamma)$  be an  $E$ -bisimple semigroup. Then (a)  $((n, k), (g, p))\mathcal{R}((r, s), (h, q))$  if and only if  $n = r$ , (b)  $((n, k), (g, p))\mathcal{L}((r, s), (h, q))$  if and only if  $k = s$  and  $p = q$ , (c)  $ES = \{((0, 0), (e, p)) : p \in P\} \cup \{((n, n), (e, q)) : q \in K\}$  where  $e$  is the identity of  $G$ , (d)  $((k, k), (e, p)) < ((r, r), (e, q))$  if and only if  $k > r$ .*

Let  $E_0 = \{((0, 0), (e, p)) : p \in P\}$  and, for  $i > 0$ ,  $E_i = \{((i, i), (e, q)) : q \in K\}$ . Thus,  $E(S) = \cup(E_i : i \in I^0)$ . Note, each  $E_i$  is a right zero semigroup and if  $f \in E_i$  and  $t \in E_j$ , then  $f < t$  if and only if  $i > j$ . Furthermore,  $E_i E_j \subseteq E_{\max(i, j)}$ . These facts will be utilized without explicit mention.

We will need a determination of the smallest inverse semigroup congruence on an  $E$ -bisimple semigroup (Lemma 1.3).

Let  $S$  be an  $E$ -bisimple semigroup. Then,  $((n, k), (g, p))\rho((r, s), (h, q))$  if  $n = r$ ,  $k = s$ , and  $g = h$ .

**Lemma 1.3** *Let  $S$  be an  $E$ -bisimple semigroup. Then,  $\rho$  is the smallest inverse semigroup congruence on  $S$ .*

*Proof.* By a routine calculation,  $\rho$  is a congruence relation on  $S$ . It is easily checked that  $a^2\rho a$  implies  $a \in E(S)$ . If  $e \in E_i$  and  $f \in E_i$ ,  $ef\rho fe$ . If  $e \in E_i$

and  $f \in E_j$  with  $i \neq j$ , then  $ef = fe$ . So, in either case,  $e\rho f\rho = f\rho e\rho$ . Hence,  $S/\rho$  is an inverse semigroup ( $S/\rho$  is a regular semigroup whose idempotents commute). Let  $\lambda$  be an inverse semigroup congruence on  $S$ . We next show  $\rho \subseteq \lambda$ . Let  $((n, k), (g, p)), ((n, k), (g, q)) \in \rho$ . Since  $((k, k), (e, p))\mathcal{R}((k, k), (e, q)), ((k, k), (e, p))\lambda\mathcal{R}((k, k), (e, q))\lambda$ . Hence,  $((k, k), (e, p))\lambda((k, k), (e, q))$ . Thus,  $((n, k), (g, p))\lambda = ((n, k), (g, p))((k, k), (e, p))\lambda = ((n, k), (g, p))((k, k), (e, q))\lambda = ((n, k), (g, q))\lambda$ .

The next proposition exhibits the difficulties encountered in generalizing the isomorphism theorem for  $E$ -bisimple semigroups to a homomorphism theorem. Such difficulties were not encountered in the case of bisimple  $\omega$ -semigroups.

**Proposition 1.4** *Let  $S = (G, P, K, \theta, \gamma)$  and  $S^* = (G^*, P^*, K^*, \theta^*, \gamma^*)$  be  $E$ -bisimple semigroups with  $E(S) = \cup(E_i : i \in I^0)$  and  $E(S^*) = \cup(E_i^* : i \in I^0)$ . Let  $\phi$  be a homomorphism of  $S$  onto  $S^*$ . Then,  $E_i\phi = E_i^*$  for all  $i \in I^0$ .*

*Proof.* Let  $\phi$  be a homomorphism of  $S$  onto  $S^*$ . Thus, if  $((k, k), (e, p)) \in E(S^*)$ , there exists  $((r, s), (g, t)) \in S$  such that  $((r, s), (g, t))\phi = ((k, k), (e, p))$ . Since  $((r, s), (g, t))\mathcal{L}((s, s), (e, t)), ((k, k), (e, p))\mathcal{L}((s, s), (e, t))\phi$ . Thus,  $((s, s), (e, t))\phi = ((k, k), (e, p))$ . Hence,  $\tilde{\phi} = \phi|E(S)$  defines a homomorphism of  $E(S)$  onto  $E(S^*)$ . We will show  $E_i\phi = E_i^*$  for all  $i \in I^0$ . First, we will show that  $E_0\phi = E_0^*$ . Clearly,  $E_0\tilde{\phi} \subseteq E_k^*$ , for some  $k \in I^0$ . Suppose  $k > 0$ . Thus, if  $e_0^* \in E_0^*$ ,  $e_0^* = e_t\phi$  for some  $e_t \in E_t$  with  $t > 0$ . Hence,  $e_0e_t = e_t$  if  $e_0 \in E_0$ . Thus,  $e_0\tilde{\phi}e_0^* = e_0^*$  which contradicts the fact  $k > 0$ . So,  $E_0\tilde{\phi} \subseteq E_0^*$ . Suppose  $f \in E_0^* - E_0\tilde{\phi}$ . Then,  $f = e_r\tilde{\phi}$  for some  $e_r \in E_r$  with  $r > 0$ . Since  $e_r e_0 = e_r$ ,  $f(e_0\tilde{\phi}) = f$  and, thus,  $e_0\tilde{\phi} = f$ , a contradiction. Hence,  $E_0\tilde{\phi} = E_0^*$ . Next, we assume  $E_i\tilde{\phi} = E_i^*$  for  $0 \leq i \leq n$ . We

will show  $E_{n+1}\tilde{\phi} = E_{n+1}^*$ . First, we show  $E_{n+1}\tilde{\phi} \subseteq E_{n+1}^*$ . We note  $E_{n+1}\phi \subseteq E_k^*$  for some  $k \in I^0$ . First, we assume  $k > n + 1$ . If  $e_{n+1}^* \in E_{n+1}^*$ , then  $e_{n+1}^* = e_t\tilde{\phi}$  for some  $e_t \in E_t$  with  $t > n + 1$ . Thus,  $e_t e_{n+1} = e_t$ . Hence,  $e_{n+1}^*(e_{n+1}\tilde{\phi}) = e_{n+1}^*$ . So,  $n + 1 \geq k$ , a contradiction. Thus,  $k \leq n + 1$ . Suppose  $k < n + 1$ . Hence,  $E_k\tilde{\phi} = E_k^*$ . Let  $e_{n+1} \in E_{n+1}$  and  $e_k \in E_k$ . Since  $e_{n+1}e_k = e_{n+1}$ ,  $e_{n+1}\tilde{\phi}e_k\tilde{\phi} = e_{n+1}\tilde{\phi}$ . Hence,  $e_{n+1}\tilde{\phi} = e_k\tilde{\phi}$ . Thus,  $E_{n+1}\tilde{\phi} = \tilde{E}_k\phi = E_k^*$ . Suppose  $|E_k^*| > 1$ . Let  $g_1, g_2 \in E_k^*$  with  $g_1 \neq g_2$ . Hence there exists  $e_{n+1} \in E_{n+1}$  and  $e_k \in E_k$  such that  $e_{n+1}\tilde{\phi} = g_1$  and  $e_k\tilde{\phi} = g_2$ . Since  $e_{n+1}e_k = e_k e_{n+1}$ ,  $g_1 g_2 = g_2 g_1$ . Thus,  $g_1 = g_2$ , a contradiction. Thus, in the case  $|E_k^*| > 1$ ,  $k = n + 1$  and  $E_{n+1}\tilde{\phi} \subseteq E_{n+1}^*$ . Next, we assume that  $|E_k^*| = 1$ . Thus, if  $E_k^* = \{e_k^*\}$ ,  $e_{n+1}\tilde{\phi} = e_k\tilde{\phi} = e_k^*$  for  $e_{n+1} \in E_{n+1}$  and  $e_k \in E_k$ . For  $|E_k^*| = 1$ , we have two possibilities  $|K^*| = 1$  or  $|P^*| = 1$  and  $|K^*| > 1$ . We first consider the case  $|K^*| = 1$ . Let  $e_0^* \in E_0^*$  and choose  $e_0 \in E_0$  such that  $e_0\tilde{\phi} = e_0^*$ . Thus,  $(e_0 S e_0)\phi = e_0^* S^* e_0^*$ . Using [14, corollary 1.3],  $e_0 S e_0$  is an  $E$ -bisimple semigroup with  $E(e_0 S e_0) = \{e_0\} \cup (\cup(E_j : j \in N))$  and  $e_0^* S^* e_0^*$  is a bisimple  $\omega$ -semigroup (a bisimple semigroup  $T$  such that  $E(T) = \{f_j : j \in I^0\}$  with  $f_j < f_k$  if and only if  $j > k$ ). Let  $\phi_1 = \phi|_{e_0 S e_0}$ . Let  $\lambda$  denote the smallest inverse semigroup congruence on  $e_0 S e_0$ . Thus,  $\lambda \subseteq \ker \phi_1$  ( $(a, b) \in \ker \phi_1$  if and only if  $a\phi_1 = b\phi_1$ ). Hence,  $(a\lambda)\bar{\phi}_1 = a\phi_1$  ( $a \in e_0 S e_0$ ) defines a homomorphism of  $e_0 S e_0/\lambda$  onto  $e_0^* S^* e_0^*$ . Thus,  $(e_{n+1}\lambda)\bar{\phi}_1 = (e_k\lambda)\bar{\phi}_1$  and, hence  $(e_{n+1}\lambda, e_k\lambda) \in \ker \bar{\phi}_1$ . Using Lemma 3,  $Ese/\lambda$  is a bisimple  $\omega$ -semigroup. Hence, by [21, Theorem 4.1] or [11], every congruence relation on  $eSe/\lambda$  is a group congruence or an idempotent separating congruence. Thus,  $(e_{n+1}, e_k) \in \lambda$  which contradicts Lemma 3. Thus,



$k = n + 1$  and  $E_{n+1}\tilde{\phi} \subseteq E_{n+1}^*$  in the case  $|E_k^*| = 1$  and  $|K^*| = 1$ . Finally, we consider the case  $|P^*| = 1$  and  $|K^*| > 1$ . If  $k > 0$ ,  $|E_k^*| > 1$ . So, we may assume  $k = 0$ . Thus,  $E_{n+1}\tilde{\phi} = \{e_0^*\}$ . We first show that  $E_1\tilde{\phi} = \{e_0^*\}$ . If  $n = 0$ , we are finished. So, we assume  $n > 0$ . Thus, if  $e_1 \in E_1$  and  $e_{n+1} \in E_{n+1}$ ,  $e_{n+1}e_1 = e_{n+1}$  and, hence,  $e_{n+1}\tilde{\phi}e_1\tilde{\phi} = e_{n+1}\tilde{\phi}$ . Thus,  $e_0^*(e_1\tilde{\phi}) = e_0^*$ . Hence,  $e_1\tilde{\phi} = e_0^*$ . Thus,  $E_1\tilde{\phi} = \{e_0^*\}$ . Since  $|P^*| = 1$ , let  $P^* = \{q\}$ . Let  $p \in K$  and  $p_1 \in P$ , and let  $e(e^*)$  denote the identity of  $G(G^*)$ . Then,  $((0, 1), (e, p))\mathcal{R}((0, 0), (e, p_1))$  implies  $((0, 1), (e, p))\phi\mathcal{R}((0, 0), (e, p_1))\phi = ((0, 0), (e^*, q))$ . Furthermore,  $((0, 1), (e, p))\mathcal{L}((1, 1), (e, p))$  implies  $((0, 1), (e, p))\phi\mathcal{L}((1, 1), (e, p))\phi = ((0, 0), (e^*, q))$ . Thus,  $((0, 1), (e, p))\phi = ((0, 0), (g, q))$  for some  $g \in G^*$ . Hence, for  $n \in N$ ,  $((0, n), (e, p))\phi = ((0, 0), (g^n, q))$ . Let  $((n, 0), (e, p_1))\phi = ((x, y), (h, t))$ . Since  $((0, n), (e, p))((n, 0), (e, p_1)) = ((0, 0), (e, p_1))$ ,  $((0, 0), (g^n, q))((x, y), (h, t)) = ((0, 0), (e^*, q))$ . Hence,  $x = y = 0$ ,  $t = q$  and  $g^n h = e^*$ . Thus,  $((n, 0), (e, p_1))\tilde{\phi} = ((0, 0), (g^{-n}, q))$ . Hence,  $((n, n), (e, p))\tilde{\phi} = ((0, 0), (e^*, q)) = e_0^*$ . Thus,  $E_n\tilde{\phi} = \{e_0^*\}$  for all  $n \in I^0$ . This contradicts the fact that  $\tilde{\phi}$  maps  $E$  onto  $E^*$ . Thus in the case  $|E_k^*| = 1$ ,  $|P^*| = 1$  and  $|K^*| > 1$ , we have shown  $k = n + 1$  and, thus,  $E_{n+1}\tilde{\phi} \subseteq E_{n+1}^*$ . So, this completes the proof that  $E_{n+1}\tilde{\phi} \subseteq E_{n+1}^*$ . We next show that  $E_{n+1}\tilde{\phi} = E_{n+1}^*$ . Suppose  $f \in E_{n+1}^* - E_{n+1}\tilde{\phi}$ . Hence,  $f = e_r\tilde{\phi}$  where  $e_r \in E_r$  with  $r > n + 1$ . Thus,  $e_r e_{n+1} = e_r$ . Hence,  $f(e_{n+1}\tilde{\phi}) = f$ . Thus,  $f = e_{n+1}\tilde{\phi}$ , a contradiction. Hence,  $E_{n+1}\tilde{\phi} = E_{n+1}^*$ .

**Theorem 1.5** *The  $E$ -bisimple semigroup  $S^* = (G^*, P^*, K^*, \theta^*, \gamma^*)$  is a homomorphic image of the  $E$ -bisimple semigroup  $S = (G, P, K, \theta, \gamma)$  if and only if there exists a homomorphism  $\phi$  of  $G$  onto  $G^*$ , a surjection  $\varphi$  of  $P$  onto  $P^*$ , a surjection  $\psi$  of*

$K$  onto  $K^*$ , and  $q \in G^*$  such that  $\phi\theta^* = \theta\phi C_q$  where  $gC_q = q^{-1}gq$  for  $g \in G$  and  $(h\gamma)\psi = \psi(h\phi\gamma^*)$  for all  $h \in G$ .

Proof. Let  $\tilde{\phi}$  be a homomorphism of  $S$  onto  $S^*$ . Using Proposition 1.4,  $((k, k), (e, p))\tilde{\phi} = ((k, k), (e^*, p\psi_k))$  where  $\psi_0$  is a surjection of  $P$  onto  $P^*$  and  $\psi_k (k > 0)$  is a surjection of  $K$  onto  $K^*$ . The remainder of the proof is the same as the proof of [27, Theorem 2.1] with “isomorphism” replaced by “homomorphism” and “bijection” replaced by “surjection” and so will be omitted.

## 2 Extensions of $E$ -Bisimple Semigroups

In this section, we determine the translational hull of an  $E$ -bisimple semigroup (Theorem 2.1) and use this result to find all ideal extensions of an  $E$ -bisimple semigroup by a completely 0-simple semigroup (Theorem 2.3). We also determine all ideal extensions of a Brandt semigroup with finite number of idempotents by a simple ( $E$ -bisimple) semigroup (with zero appended) (Theorem 2.5).

**Theorem 2.1** *Let  $S = (G, P, K, \theta, \gamma)$  be an  $E$ -bisimple semigroup, and let  $W = \{(g, \delta) : g \in G \text{ and } \delta : P \rightarrow P \text{ is a nonconstant mapping}\}$ . Let  $\bar{S}$  be the translational hull of  $S$ . Then,  $\bar{S} = S \cup W$  under the multiplication*

$$(g_1, \delta_1) \cdot (g_2, \delta_2) = \begin{cases} (g_1g_2, \delta_1\circ\delta_2) & \text{if } \delta_1\circ\delta_2 \text{ is not constant} \\ (0, 0), (g_1g_2, k) & \text{if } (p)\delta_1\circ\delta_2 = k \text{ for all } p \in P \end{cases}$$

$$((a, b), (h, q)) \cdot (g, \delta) = \begin{cases} ((a, b), (h(g\theta^b), q(g\theta^{b-1}\gamma))) & \text{if } b > 0 \\ ((a, 0), (hg, q\delta)) & \text{if } b = 0 \end{cases}$$

$$(g, \delta) \cdot ((a, b), (h, q)) = ((a, b), ((g\theta^a)h, q))$$

where juxtaposition denotes the multiplication in  $G$  and  $o$  denotes iteration of mappings,

and  $v_1 \cdot v_2 = v_1 v_2$  where  $v_1, v_2 \in S$  and juxtaposition denotes the multiplication of  $S$ .

Proof. First note that if  $e_0 \in E_0$ , then  $e_0$  is a left identity of  $S$ . We will use this fact without explicit mention. Let  $\rho$  be any right translation of  $S$  and define  $\phi(p) = ((0, 0), (e, p))$  for  $p \in P$ . Then,

$$((n, k), (g, q))\rho = \begin{cases} ((n, k), (g, q))\phi(z) & \text{for any } z \in P \quad \text{if } k > 0 \\ ((n, k), (g, q))\phi(q) & \text{if } k = 0. \end{cases}$$

Let  $\rho = \rho_\phi$ . Conversely, we show  $\rho_\phi$  is a right translation of  $S$ . Let  $x = ((n, k), (g, p))$  and  $y = ((r, s), (h, q))$ . If  $s > 0$  and  $s + k - \min(k, r) > 0$ ,  $(xy)\rho_\phi = (xy)\phi(z) = x(y\phi(z)) = x(y\rho_\phi)$  where  $z$  is any element of  $S$ . If  $s = 0$  and  $k - \min(k, r) > 0$ ,  $(xy)\rho_\phi = (xy)\phi(q) = x(y\phi(q)) = x(y\rho_\phi)$ . If  $s = 0 = k - \min(k, r)$ ,  $(xy)\rho_\phi = (xy)\phi(q) = x(y\phi(q)) = x(y\rho_\phi)$ . So,  $\rho_\phi$  is a right translation of  $S$ . Let  $\lambda$  be any left translation of  $S$ . (Thus, if  $e_0 \in E_0$   $a\lambda = (e_0 a)\lambda = (e_0 \lambda)a$  for all  $a \in S$ . So,  $\lambda$  is the inner left translation  $\lambda_{e_0 \lambda}$ . Next, let  $\rho_\phi$  be a right translation of  $S$  which is linked to some left translation of  $S$ . We will show that  $\rho_\phi$  is an inner right translation of  $S$  or  $\phi(p) = ((0, 0), (g_\phi, p\delta_\phi))$  where  $\delta_\phi : P \rightarrow P$  is a nonconstant mapping. Suppose  $\rho_\phi$  is linked with  $\lambda_t$ . Let  $q \in P$ ,  $e_0 = ((0, 0), (e, q))$ , and  $f_0 \in E_0$ . Then,  $(e_0 \rho_\phi)f_0 =$

$e_0(f_0\lambda_t)$ . So,  $\phi(q)f_0 = tf_0$  for all  $q \in P$  and all  $f_0 \in E_0$ . So, if  $q, u \in P$ ,  $\phi(q)f_0 = \phi(u)f_0$  for all  $f_0 \in E_0$ . Let  $\phi(q) = ((a_1, b_1), (x_1, y_1))$  and  $\phi(v) = ((a_2, b_2), (x_2, y_2))$ . Thus  $((a_1, b_1), (x_1, y_1))((0, 0), (e, p)) = ((a_2, b_2), (x_2, y_2))((0, 0), (e, p))$  for all  $p \in P$ . Hence,  $a_1 = a_2$  and  $b_1 = b_2$ . If  $b_1 > 0$ ,  $x_1 = x_2$  and  $y_1 = y_2$ . Thus,  $\phi$  is a constant function, and, hence  $\rho_\phi$  is an inner right translation. If  $b_1 = 0$ ,  $x_1 = x_2$ . By the definition of  $\rho_\phi$ ,  $((0, 1), (e, p))\phi(q) = ((0, 1), (e, p))\phi(v)$  for all  $p \in P$ . So,  $((0, 1), (e, p))((a_1, 0), (x_1, y_1)) = ((0, 1), (e, p))((a_1, 0), (x_1, y_2))$  for all  $p \in P$ . If  $a_1 > 0$ ,  $y_1 = y_2$ . Thus, in this case,  $\phi$  is a constant mapping, and hence,  $\rho_\phi$  is an inner right translation. If  $a_1 = 0$ , either  $\rho_\phi$  is an inner right translation or we may write  $\phi(p) = ((0, 0), (g_\phi, p\delta_\phi))$  where  $\delta_\phi : P \rightarrow P$  is a nonconstant mapping. Hence, a right translation  $\rho$  of  $S$  that is linked with some left translation of  $S$  is either an inner right translation or  $\rho = \rho_{(g, \delta)}$  where  $(g, \delta)(p) = ((0, 0), (g, p\delta))$  for  $p \in P$  where  $\delta : P \rightarrow P$  is a nonconstant mapping. Let  $W' = \{\rho_{(g, \delta)} : g \in G \text{ and } \delta : P \rightarrow P \text{ is a nonconstant mapping}\}$ . Let  $\rho_{(g_1, \delta_1)}, \rho_{(g_2, \delta_2)} \in W'$ . Let  $e_0 = ((0, 0), (e, p)) \in E_0$ . Then,  $e_0\rho_{(g_1, \delta_1)}\rho_{(g_2, \delta_2)} = (e_0((0, 0), (g_1, p\delta_1)))\rho_{(g_2, \delta_2)} = ((0, 0), (g_1, p\delta_1))((0, 0), (g_2, p(\delta_1\circ\delta_2))) = ((0, 0), (g_1g_2, p(\delta_1\circ\delta_2)))$ . If  $p(\delta_1\circ\delta_2) = k$  for  $p \in P$ ,  $e_0\rho_{(g_1, \delta_1)}\rho_{(g_2, \delta_2)} = e_0\rho_{((0, 0), (g_1g_2, k))}$ . If  $\delta_1\circ\delta_2$  is a nonconstant mapping, then  $e_0\rho_{(g_1, \delta_1)}\rho_{(g_2, \delta_2)} = e_0\rho_{(g_1g_2, \delta_1\circ\delta_2)}$ . Using the fact that for  $x \in S$ , there exists  $e_0 \in E_0$  such that  $xe_0 = x$ , we obtain

$$\rho_{(g_1, \delta_1)}\rho_{(g_2, \delta_2)} = \begin{cases} \rho_{(g_1g_2, \delta_1\circ\delta_2)} & \text{if } \delta_1\circ\delta_2 \text{ is not a constant map} \\ \rho_{((0, 0), (g_1g_2, k))} & \text{if } p(\delta_1\circ\delta_2) = k \text{ for all } p \in P \end{cases} \quad (2.1)$$

Similarly,

$$\rho_{((a_1, b_1), (h_1, q_1))} \rho_{(g_2, \delta_2)} = \begin{cases} \rho_{((a_1, b_1), (h_1(g_2 \theta^{b_1}), q_1(g_2 \theta^{b_1-1} \gamma)))} & \text{if } b_1 > 0 \\ \rho_{((a_1, b_1), (h_1 g_2, q_1 \delta_2))} & \text{if } b_1 = 0 \end{cases} \quad (2.2)$$

and

$$\rho_{(g_1, \delta_1)} \rho_{((a_2, b_2), (h_2, q_2))} = \rho_{((a_2, b_2), ((g_1 \theta^{a_2}) h_2, q_2))}. \quad (2.3)$$

Clearly,

$$\rho_{v_1} \rho_{v_2} = \rho_{v_1 v_2} \text{ for } v_1, v_2 \in S. \quad (2.4)$$

Note, if  $(\lambda_s, \rho_v) \in \bar{S}$ ,  $(e_0 \rho_v) e_0 = e_0 (e_0 \lambda_s)$  implies  $v e_0 = s e_0$ . Hence,  $\lambda_v = \lambda_s$  and, thus,  $(\lambda_s, \rho_v) = (\lambda_v, \rho_v)$ .

We will show that  $T : \bar{S} \rightarrow S \cup W$  ( $W = \{(g, \delta) : g \in G \text{ and } \delta : P \rightarrow P, \text{ a nonconstant map}\}$ ) defined by  $T(\lambda_v, \rho_v) = v$  and  $T(\lambda_s, \rho_{(g, \delta)}) = (g, \delta)$  defines an isomorphism from  $\bar{S}$  onto  $S \cup W$  under the multiplication given in the statement of the theorem.

It is easily checked  $T$  is well defined. Using 2.1 – 2.4,  $T$  is a homomorphism. We show  $T$  maps  $S$  onto  $S \cup W$ . If  $v \in S$ ,  $T(\lambda_v, \rho_v) = v$ . Suppose,  $(g, \delta) \in W$ . Let  $s = ((0, 0), (g, k))$ . Then,  $(g, \delta)(q)((0, 0), (e, p)) = ((0, 0), (g, \delta q))((0, 0), (e, p)) = ((0, 0), (g, k))((0, 0), (e, p)) = s((0, 0), (e, p))$  for all  $p, q \in P$ . Thus, if  $y \in S$ ,  $(g, \delta)(q)y = sy$  for all  $q \in P$ . Hence,  $(x \rho_{(g, \delta)})y = x(y \lambda_s)$  for  $x \in S$ . Thus,  $(\lambda_s, \rho_{(g, \delta)}) \in \bar{S}$  and  $T(\lambda_s, \rho_{(g, \delta)}) = (g, \delta)$ .

Next, we show  $T$  is one-to-one. Suppose  $T(\lambda_{s_1}, \rho_{(g_1, \delta_1)}) = T(\lambda_{s_2}, \rho_{(g_2, \delta_2)})$ . Thus,  $(g_1, \delta_1) = (g_2, \delta_2)$ . So,  $s_1 f_0 = (g_1, \delta_1)(p) f_0 = (g_2, \delta_2)(p) f_0 = s_2 f_0$  for all  $p \in P$

and  $f_0 \in E_0$ . Hence,  $\lambda_{g_1} = \lambda_{g_2}$ . Thus,  $(\lambda_{g_1}, \rho_{(g_1, \delta_1)}) = (\lambda_{g_2}, \rho_{(g_2, \delta_2)})$ . Clearly,  $T(\lambda_v, \rho_v) = T(\lambda_t, \rho_t)$  implies  $(\lambda_v, \rho_v) = (\lambda_t, \rho_t)$ .

The next lemma will be utilized in the determination of the extensions of  $E$ -bisimple semigroup by a completely 0-simple semigroup (Theorem 2.3).

**Lemma 2.2** *Let  $S = (G, K, P, \theta, \gamma)$  be an  $E$ -bisimple semigroup and let  $T = M^0(H; F, \Lambda, Q)$  be a completely 0-simple semigroup. Let the following mappings be given*

$$\psi : F \rightarrow I^0, \xi : \Lambda \rightarrow I^0, \tau : \Lambda \rightarrow P \cup K, \alpha : F \rightarrow G \text{ and } \beta : \Lambda \rightarrow G$$

*and  $\eta$  be a homomorphism of  $H$  into  $G$  such that  $q_{m_i} \neq 0$  implies  $m\xi = i\psi$  and  $(m\beta)(i\alpha) = q_{m_i}\eta$ . Then  $\varphi$  defined on  $T^*$  (i.e.,  $T \setminus \{0\}$ ) by*

$$(a; i, m)\varphi = ((i\psi, m\xi), (i\alpha\eta m\beta, m\tau))$$

*is a partial homomorphism of  $T^*$  into  $S$ . Conversely, every partial homomorphism of  $T^*$  into  $S$  is obtained in this fashion.*

**Proof.** Using the conditions of the theorem, it is easily checked that  $\varphi$  defines a partial homomorphism of  $T^*$  into  $S$ .

Conversely, let  $\varphi$  be a partial homomorphism of  $T^*$  into  $S$ . Using the determination of  $R$  and  $L$  for  $T$  and Lemma 1.2 ((a) and (b)), we may write

$$(a; i, m)\varphi = ((i\psi, m\xi), (g, m\tau))$$

where  $g \in G$  and  $\psi : F \rightarrow I^0$ ,  $\xi : \Lambda \rightarrow I^0$ , and  $\tau : \Lambda \rightarrow P \cup K$  are mappings. Since  $(q_{mi}^{-1}; i, m) \in E(T)$  if  $q_{mi} \neq 0$ ,  $(q_{mi}^{-1}; i, m)\varphi \in E(S)$  and, hence, using Lemma 1.2 (c),  $m\xi = i\psi$ . By proof of [7, Theorem 3.5], we may assume  $1 \in F \cap \Lambda$  and  $q_{11} = e$ , the identity of  $H$ . Define the mappings  $\alpha$  of  $F$  into  $G$  and  $\beta$  of  $\Lambda$  into  $G$  as follows:

$$(e; i, 1)\varphi = ((i\psi, 1\xi), (i\alpha, 1\tau)) \text{ and } (e; 1, m)\varphi = ((1\psi, m\xi), (m\beta, m\tau)).$$

Define the mapping  $\eta$  of  $H$  into  $G$  by  $(a; 1, 1)\varphi = ((1\psi, 1\xi), (a\eta, 1\tau))$ . Since  $(a; 1, 1)\varphi(b; 1, 1)\varphi = (ab; 1, 1)\varphi$  for  $a, b \in H$ , one easily shows that  $\eta$  defines a homomorphism of  $H$  into  $G$ . Since  $(e; 1, m)\varphi(e; i, 1)\varphi = (q_{mi}; 1, 1)\varphi$  if  $q_{mi} \neq 0$ , we obtain  $m\beta i\alpha = q_{mi}\eta$ . Finally,  $(a; i, m)\varphi = (e; i, 1)\varphi(a; 1, 1)\varphi(e; 1, m)\varphi = ((i\psi, m\tau), i\alpha(a\eta)m\beta)$ .

In order to describe the extensions of an  $E$ -bisimple semigroup by a completely 0-simple semigroup, we will need the following concept (see [15]).

If  $V$  is an extension of a semigroup  $S$  by a semigroup  $T$  (with zero), then we say  $V$  is determined by a partial homomorphism if there exists a partial homomorphism  $\pi : T^* \rightarrow S$  such that for all  $A, B \in T^*$ ,  $c, d \in S$ ,

$$A \circ B = \begin{cases} AB & \text{if } AB \neq 0 \text{ ( in } T) \\ A\pi B\pi & \text{if } AB = 0 \text{ ( in } T). \end{cases}$$

$A \circ c = (A\pi)c$ ;  $c \circ A = c(A\pi)$ ,  $c \circ d = cd$  where  $\circ$  denotes the operation in  $V$ . [15].

**Theorem 2.3** *Let  $S = (G, P, K, \theta, \gamma)$  be an  $E$ -bisimple semigroup and let  $T = M^0(H; F, \Lambda; Q)$  be a completely 0-simple semigroup. Let  $i \rightarrow u_i$  and  $m \rightarrow v_m$  be mappings of  $F$  into  $G$  and  $\Lambda$  into  $G$ , respectively and let  $\omega$  be a homomorphism of*

$H$  into  $G$  such that  $q_{mj}\omega = v_m u_j$  if  $q_{mj} \neq 0$ . Let  $Y$  denote the set of nonconstant mappings of  $P$  into  $P$ . Let  $a \rightarrow \gamma_a$ ,  $i \rightarrow \alpha_i$ , and  $m \rightarrow \beta_m$  be mappings of  $H$  into  $Y$ ,  $F$  into  $Y$ , and  $\Lambda$  into  $Y$  respectively such that  $\gamma_a \gamma_b = \gamma_{ab}$  for all  $a, b \in H$ ,  $\alpha_i \gamma_a \beta_m \in Y$  for all  $i \in F$ ,  $a \in H$ , and  $m \in \Lambda$ ,  $\beta_m \alpha_j = \gamma_{q_{mj}}$  if  $q_{mj} \neq 0$ , and  $\beta_m \alpha_j$  is a constant mapping  $P$  into  $P$  if  $q_{mj} = 0$ .

Let  $V = S \cup T^*$  under the multiplication "o" defined as follows:

$$(a; i, m) o (b; j, n) = \begin{cases} (a; i, m)(b; j, n) & \text{if } q_{mj} \neq 0 \\ ((0, 0), (u_i(a\omega)v_m u_j(b\omega)v_n, k_0)) & \text{if } q_{mj} = 0 \text{ where } k_0 = \ell_0 \gamma_b \beta_n \\ & \text{where } p\beta_m \alpha_j = \ell_0 \text{ for all } p \in P. \end{cases} \quad (2.1)$$

$$(a, i, m) o ((n, k), (g, p)) = ((n, k), ((u_i(a\omega)v_m)\theta^n g, p)) \quad (2.2)$$

$$((n, k), (g, p)) o (a, i, m) = \begin{cases} ((n, k), (g((u_i(a\omega)v_m)\theta^k), p((u_i(a\omega)v_m)\theta^{k-1}\gamma))) & \text{if } k > 0 \\ ((n, 0), (g(u_i(a\omega)v_m), p\alpha_i \gamma_a \beta_m)) & \text{if } k = 0 \end{cases} \quad (2.3)$$

$$((n, k), (g, p)) o ((r, s), (h, q)) = ((n, k), (g, p))((r, s), (h, q)) \quad (2.4)$$

where juxtaposition denotes the multiplications of  $T$  and  $S$ . Then  $(V, o)$  is an extension of  $S$  by  $T$ .

Conversely, every extension of  $S$  by  $T$  is determined in the above manner or is given by a partial homomorphism and hence in this case an explicit multiplication is given by Lemma 2.2.

Proof. We first establish the converse. Let  $(V, o)$  be an extension of  $S$  by  $T$  and let  $\bar{S}$  be the translational hull of  $S$ . Since  $S$  is a weakly reductive semigroup (as  $e_0 \in E_0$  is a left identity of  $S$ ), there exists an extension  $(\bar{V}, o)$  of  $\bar{S}$  by  $T$



such that  $(V, o)$  is a subsemigroup of  $(\bar{V}, o)$  by [7, Theorem 4.20: see also [6]]. Using Theorem 2.1,  $(e_G, \delta_i)$  where  $e_G$  is the identity of  $G$  and  $\delta_i$  is the identity of mapping of  $P$  onto  $P$ , is the identity of  $\bar{S}$ . Thus, using [7, Theorem 4.19],  $(\bar{V}, o)$  is determined by a partial homomorphism  $\pi$  from  $T^*$  into  $\bar{S}$ . Since partial homomorphisms map  $\mathcal{D}$ -classes into  $\mathcal{D}$ -classes and since  $T^*$  is a single  $\mathcal{D}$ -class and  $S$  is a single  $\mathcal{D}$ -class of  $\bar{S}$ , either  $T^*\pi \subseteq S$  or  $T^*\pi \subseteq \bar{S} \setminus S$ . So, if  $T^*\pi \subseteq S$ , then  $(V, o)$  is given by the partial homomorphism  $\pi$  and the multiplication “ $o$ ” of  $V$  is determined by employing Lemma 2.2. But, if  $T^*\pi \subseteq \bar{S} \setminus S = W = G \times Y$  (see Theorem 2.1), we may write  $(a; i, m)\pi = ((a; i, m)\varphi, (a; i, m)\psi)$  where  $\varphi$  is a mapping of  $T^*$  into  $G$  and  $\psi$  is a mapping of  $T^*$  into  $Y$ . It is easily seen that  $\varphi$  defines a partial homomorphism of  $T^*$  into  $G$ . We may assume that  $1 \in F \cap \Lambda$  and  $q_{11} = e$ , the identity of  $H$  (use proof of [7, Theorem 3.5]). Let  $(e; i, 1)\varphi = u_i$  for  $i \in F$ , let  $(e; 1, m)\varphi = v_m$  for  $m \in \Lambda$ , and let  $(a; 1, 1)\varphi = a\omega$  for  $a \in H$ . It is easily checked that  $\omega$  defines a homomorphism of  $H$  into  $G$  and that  $i \rightarrow u_i$  and  $m \rightarrow v_m$  are mappings of  $F$  into  $G$  and  $\Lambda$  into  $G$  such that  $q_{mi}\omega = v_m u_i$  if  $q_{mi} \neq 0$  and  $(a; i, m)\varphi = u_i(a\omega)v_m$  (see also [7, Theorem 4.22]). Let  $(a; 1, 1)\psi = \gamma_a$  for  $a \in H$ , let  $(e; i, 1)\psi = \alpha_i$   $i \in F$ , and let  $(e; 1, m)\psi = \beta_m$  for  $m \in \Lambda$ . Then,  $a \rightarrow \gamma_a$ ,  $i \rightarrow \alpha_i$ , and  $m \rightarrow \beta_m$  define mappings of  $H$  into  $Y$ ,  $F$  into  $Y$ , and  $\Lambda$  into  $Y$ , respectively. Since  $(a; 1, 1)\psi(b; 1, 1)\psi = (ab; 1, 1)\psi$ ,  $\gamma_a \gamma_b = \gamma_{ab}$  for all  $a, b \in H$ . Since  $(a; i, m)\psi = (e; i, 1)\psi(a; 1, 1)\psi(e; 1, m)\psi = \alpha_i \gamma_a \beta_m$ ,  $\alpha_i \gamma_a \beta_m \in Y$  for all  $i \in F$ ,  $a \in H$ , and  $m \in \Lambda$ . If  $q_{mj} \neq 0$   $\beta_m \alpha_j = (e; 1, m)\psi(e; j, 1)\psi = (q_{mj}; 1, 1)\psi = \gamma_{q_{mj}}$ . If  $q_{mj} = 0$ ,  $(e; 1, m)o(e; j, 1) = (e; 1, m)\pi(e; j, 1)\pi \in S$ . Thus, using Theorem 2.1,

$\beta_m \alpha_j$  must be a constant mapping of  $P$  into  $P$ . Furthermore,

$$(a; i, m)\pi = (u_i(a\omega)v_m, \alpha_i \gamma_a \beta_m). \quad (2.5)$$

We next establish (2.1) - (2.4). Let  $(a; i, m), (b; j, n) \in T^*$ . Thus, if  $q_{mj} = 0$ , using (2.5) and Theorem 2.1, we obtain

$$\begin{aligned} (a; i, m)o(b; j, n) &= (a; i, m)\pi(b; j, n)\pi \\ &= (u_i(a\omega)v_m, \alpha_i \gamma_a \beta_m)(u_j(b\omega)v_n, \alpha_j \gamma_b \beta_n) \\ &= ((0, 0), (u_i(a\omega)v_m u_j(b\omega)v_n, k_0)) \end{aligned}$$

where  $p\alpha_i \gamma_a \beta_m \alpha_j \gamma_b \beta_n = \ell_0 \gamma_b \beta_m = k_0$  for all  $p \in P$  where  $q\beta_m \alpha_j = \ell_0$  for all  $q \in P$ . So, the second part of (2.1) has been established. The first part of (2.1) and (2.4) are valid by the definition of extension. Use (2.5), the definition of extension by partial homomorphism  $\pi$ , and Theorem 2.1 to establish (2.2) and (2.3).

We now establish the direct part of the theorem. Let  $i \rightarrow u_i, m \rightarrow v_m, a \rightarrow \gamma_a, i \rightarrow \alpha_i$ , and  $m \rightarrow \beta_m$  and  $\omega$  be as in the statement of the theorem. We will show that  $V = S \cup T^*$  under the multiplication (2.1) - (2.4) is an extension of  $S$  by  $T$ . First, define  $\pi : T^* \rightarrow S$  by  $(a; i, m)\pi = (u_i(a\omega)v_m, \alpha_i \gamma_a \beta_m)$  (note  $\alpha_i \gamma_a \beta_m \in Y$ ). We will show that  $\pi$  defines a partial homomorphism of  $T^*$  into  $S$ . Let  $(a; i, m), (b; j, n) \in T^*$  with  $q_{mj} \neq 0$ . Then,

$$\begin{aligned} ((a; i, m)(b; j, n))\pi &= (aq_{mj}b; i, n)\pi \\ &= (u_i(aq_{mj}b)\omega v_n, \alpha_i \gamma_a q_{mj} \beta_n) \\ &= (u_i a \omega v_m u_j b \omega v_n, \alpha_i \gamma_a \gamma_{q_{mj}} \gamma_b \beta_n) \end{aligned}$$

$$\begin{aligned}
&= (u_i a \omega v_m u_j b \omega v_n, \alpha_i \gamma_a \beta_m \alpha_j \gamma_b \beta_n) \\
&= (u_i (a \omega) v_m, \alpha_i \gamma_a \beta_m) (u_j (b \omega) v_n, \alpha_j \gamma_b \beta_n) \\
&= (a; i, m) \pi (b; j, n) \pi.
\end{aligned}$$

Thus, by [7, Theorem 4.19],  $\pi$  determines an extension  $(\bar{V}, o)$  of  $\bar{S}$  by  $T$ . Using [7, Theorem 4.20],  $(V, o)(V = T^* \cup S)$  is an extension of  $S$  by  $T$  if and only if  $(a; i, m) o (b; j, n) \in S$  if  $q_{mj} = 0$ . Using the condition  $\beta_m \alpha_j$  is a constant mapping if  $q_{mj} = 0$ , for  $q_{mj} = 0$ ,  $(a; i, m) o (b; j, n) = (a; i, m) \pi (b; j, n) \pi = ((0, 0), (u_i (a \omega) v_m u_j b \omega v_n, k_0))$  where  $p \alpha_i \gamma_a \beta_m \alpha_j \gamma_b \beta_n = \ell_0 \gamma_b \beta_n = k_0$  for all  $p \in P$  where  $q \beta_m \alpha_j = \ell_0$  for all  $q \in P$ . So,  $(V, o)$  is an extension of  $S$  by  $T$ . We have already shown that  $(a; i, m) o (b; j, n)$  for  $q_{mj} = 0$  is given by the second part of (2.1). The first part of (2.1) and (2.4) are valid since  $(V, o)$  is an extension of  $S$  by  $T$ . It is easily checked that  $(a; i, m) o ((n, k), (g, p)) = (a; i, m) \pi ((n, k), (g, p))$  and  $((n, k), (g, p)) o (a; i, m) = ((n, k), (g, p)) ((a; i, m) \pi)$  are given by (2.2) and (2.3) respectively.

The following proposition will be used in the determination of the extensions of a Brandt semigroup by an  $E$ -bisimple semigroup (with zero appended) (Theorem 2.5).

**Proposition 2.4** *Let  $S = (G, P, K, \theta, \gamma)$  be an  $E$ -bisimple semigroup and let  $H$  be a group. Let  $t \in H$  and let  $\eta$  be a homomorphism of  $G$  into  $H$  such that  $g\eta = t(g\theta)\eta t^{-1}$  for all  $g \in G$ . Then,*

$$((n, k), (g, p))\varphi = t^n g \eta t^{-k}$$

defines a homomorphism of  $S$  into  $H$ .

Conversely, every homomorphism of  $S$  into  $H$  is determined in the above manner.

Proof. Let  $e_G(e_H)$  denote the identity of  $G(H)$ . Let  $\varphi$  be a homomorphism of  $S$  into  $H$ . Fix elements  $p_0 \in P$  and  $q_0 \in K$  and suppose  $((1, 0), (e_G, p_0))\varphi = t$  and  $((0, 1), (e_G, q_0))\varphi = v$ . It is easily established that  $((n, 0), (e_G, p_0))\varphi = t^n$  and  $((0, n), (e_G, q_0))\varphi = v^n$  for all  $n \in N$ . Moreover, for any  $p \in P$ ,  $q \in K$ , and  $n \in N$ ,  $((n, 0), (e_G, p))\varphi = (((n, 0), (e_G, p_0))((0, 0), (e_G, p)))\varphi = ((n, 0), (e_G, p_0))\varphi e_H = t^n$  and  $((0, n), (e_G, q))\varphi = ((0, n), (e_G, q_0))\varphi((n, n), (e_G, q))\varphi = v^n e_H = v^n$ . Since  $tv = (((1, 0), (e_G, p_0))((0, 1), (e_G, q_0)))\varphi = ((1, 1), (e_G, q_0))\varphi = e_H$ ,  $v = t^{-1}$ . Thus,  $((0, n), (e_G, q))\varphi = t^{-n}$  for  $q \in K$  and  $n \in N$ . Next, define a mapping  $\eta$  from  $G$  to  $H$  by  $g\eta = ((0, 0), (g, \ell_0))\varphi$  where  $g \in G$  and  $\ell_0$  is a fixed element of  $P$ . As above,  $((0, 0), (g, p))\varphi = g\eta$  for any  $p \in P$ . It is easily seen that  $g_1\eta g_2\eta = (g_1 g_2)\eta$  for  $g_1, g_2 \in G$ . Furthermore,  $((n, k), (g, p))\varphi = ((n, 0), (e_G, p_0))\varphi((0, 0), (g, p_0))\varphi((0, k), (e_G, p))\varphi = t^n g\eta t^{-k}$ . Finally, since  $((0, 1), (e_G, p))((0, 0), (g, q)) = ((0, 1), (g\theta, p(g\gamma)))$ ,  $t^{-1}g\eta = ((0, 1), (e_G, p))\varphi((0, 0), (g, q))\varphi = ((0, 1), (g\theta, p(g\gamma)))\varphi = (g\theta)\eta t^{-1}$  or  $g\eta = t(g\theta)\eta t^{-1}$  for  $g \in G$ .

Next, we establish the direct part of the theorem. Since  $(g\eta)t = t(g\theta)\eta$ ,  $(g\eta)t^2 = t(g\theta)\eta t = ttg\theta^2\eta = t^2(g\theta^2)\eta$ . Proceeding by induction,  $(g\eta)t^n = t^n(g\theta^n)\eta$  for all  $n \in N$ . We will show  $\varphi$  is a homomorphism of  $S$  into  $H$ . Let  $((n, k), (g, p)), ((r, s), (f, q)) \in S$

with  $r > k$ . We will use the fact that  $t^{r-k}(g\theta^{r-k})\eta = (g\eta)t^{r-k}$ .

$$\begin{aligned}
(((n, k), (g, p))((r, s), (f, q)))\varphi &= ((n + r - k, s), (g\theta^{r-k}f, q))\varphi \\
&= t^{n+r-k}(g\theta^{r-k}f)\eta t^{-s} \\
&= t^n t^{r-k}(g\theta^{r-k}\eta) f \eta t^{-s} \\
&= t^n (g\eta) t^{r-k} f \eta t^{-s} \\
&= t^n (g\eta) t^{-k} t^r (f\eta) t^{-s} \\
&= ((n, k), (g, p))\varphi((r, s), (f, q))\varphi
\end{aligned}$$

The case  $k > r$  is similar. We use the fact that  $(f\theta^{k-r})\eta t^{-(k-r)} = t^{-(k-r)}f\eta$ . The case  $r = k$  is straightforward.

**Theorem 2.5** *Let  $S = M^0(H; I, I; \Delta)$ , where  $I$  is a finite set, be a Brandt semigroup and let  $T^*$  be a simple semigroup. Let  $V$  be an extension of  $S$  by  $T$ . Then, there exists a homomorphism  $w : A \rightarrow w_A$  of  $T^*$  into  $U_X$ , the full symmetric group on some  $r$  element subset  $X$  of  $I$ . For each  $A \in T^*$ , there exists a mapping  $\psi_A$  of  $X$  into  $H$  such that*

1.  $i\psi_A(iw_A\psi_B) = i\psi_{AB}$  for all  $i \in X$ . The products in  $V$  are given by

2.  $AoB = AB$

3.  $(a; i, m)oA = \begin{cases} (a(m\psi_A); i, mw_A) & \text{if } m \in X \\ 0, \text{ the zero of } S & \text{if } m \notin X \end{cases}$

4.  $Ao(a; i, m) = \begin{cases} ((iw_A^{-1}\psi_A)a; iw_A^{-1}, m) & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases}$   
 $Ao0 = 0oA = 0.$

Conversely, let  $S$  be a Brandt semigroup and  $T^*$  be a simple semigroup such that  $T \cap S = \square$ . If we are given the mappings  $w$  and  $\psi_A$  described above and define product  $\circ$  in the class sum of  $S$  and  $T^*$  by (2) - (4) above, then  $V$  is an extension of  $S$  by  $T$ . In the special case  $T^* = (G, P, K, \theta, \gamma)$ , an  $E$ -bisimple semigroup, the homomorphism  $w$  is explicitly given by Proposition 2.4.

Proof. Let  $V$  be an extension of  $S$  by  $T$ . Using [16, Theorem 1] or [18, Theorem], there exists a homomorphism  $w : A \rightarrow w_A$  of  $T^*$  into  $\mathcal{I}_I$  the full symmetric inverse semigroup on  $I$  and, for each  $A \in T^*$ , a mapping  $\psi_A$  of  $s_A$ , the domain of  $w_A$ , into  $H$  such that (1) is valid with  $X = s_{AB}$ , (2) is valid, (3) is valid with  $X = s_A$ , and (4) is valid with  $X = t_A$ , the range of  $w_A$ .  $T^*w$  is a finite simple inverse semigroup and hence a group [7]. Let  $w_E (E \in T^*)$  denote the identity of  $T^*w$ . Thus,  $w_E$  is the identity mapping on some (finite) subset  $X$  of  $I$ . Hence, using [16, Lemma 1],  $s_A = t_A = X$  for each  $A \in T^*$  and  $T^*w$  is a subgroup of  $U_X$ . The converse is a consequence of [16, Theorem 1].

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