MATH102 -072 FINAL

1. Show that volume of the solid generated by revolving the region enclosed by the curve $y = \sin(x^2)$ and the x-axis over the interval $\left[0, \sqrt{\pi}\right]$ about the y-axis is 2π

Hint: Use shell method V= $\int_0^{\sqrt{\pi}} 2\pi x f(x)dx = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)dx$ (Let $x^2 = u$)

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2. The volume obtained by rotating the region bounded by

$y = x^4, y = 1$ about y = 1 is equal to $\frac{64}{45}\pi$

Pont of intersection x=0, and x = 1

Hint: Volume using Disc Method V= $2\int_0^1 \pi r^2 dx = 2\int_0^1 \pi\left(1 - x^4\right)^2 dx$......

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3. The area of the region enclosed by the graphs of

$y^2 = x$ and y = x - 2 is equal to $\int_{-1}^{2} \left(2 + y - y^2\right)dy$

Hint: Point of intersection $y^2 = y + 2 \Rightarrow (y-2)(y+1) = 0 \Rightarrow y = 2, y = -1$

Area = $\int_{-1}^{2} \left((2+y) - y^2\right)dy$

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4. $\int \frac{dx}{\sqrt{6x - x^2}} = \sin^{-1}\frac{x-3}{3} + C$.

Hint: $\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - 9 + 6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x-3)^2}}$, Let x-3 = 3 sin$\theta$

Complete yourself…

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5. If f' is continuous function, f(1) = 3, and $\int_0^3 x f'(1 + x^2)dx = 4$, then f(10)=11

Hint: = Let $u = 1 + x^2 \Rightarrow xdx = du/2, u = 1(x=0), u = 10(x=3)$

$\int_0^3 x f'(1+x^2)dx = \int_1^{10} \frac{du}{2} f'(u) = \frac{1}{2}\int_1^{10} f'(u)du = \frac{1}{2}\left(f(10) - f(1)\right) = 4$

You can find f(10)

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6. $\int_0^{\frac{\pi^2}{4}} \cos\sqrt{x}\ dx = \pi - 2$

Hint: $\int_0^{\frac{\pi^2}{4}} \cos\sqrt{x}\ dx = \int_0^{\pi/2} \cos u\ du.2u\ \left(u = \sqrt{x} => u^2 = x => 2udu = dx\right)$

Now you can integrate by parts

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7. The graph of the function f(x) is given in the figure. Show that

$\int_0^7 f(x)dx = 5 - 2\pi$

Hint: [Since the graph was not available on the sheet – you can leave this question]

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8. $\int_0^1 \frac{1}{1+e^{-x}}\ dx = \ln\frac{(e+1)}{2}$

Hint: $\int_0^1 \frac{1}{1+e^{-x}}\ dx = \int_0^1 \frac{e^x}{e^x+1}\ dx$ (Let $\mathrm{e}^x + 1 = u => \mathrm{e}^x dx = du$)

$\int_2^{e+1} \frac{du}{u} = \left[\ln|u|\right]_2^{e+1}$ =ln (e+1) - ln 2

9. $\int_4^8 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2\ dx = 32 + \ln 2$

Hint :. $\int_4^8 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2\ dx = \int_4^8 \left(x + 2 + \frac{1}{x}\right)\ dx$ .. Complete it..

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10. Show that $\lim_{n->\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} = \frac{\pi}{4}$

Hint: $\Delta x = \frac{1}{n}, a = 0, b = 1, x_i = i\Delta x ==> \int_0^1 \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_0^1$

11. The area of the surface of the solid obtained by rotating the curve $y = \sqrt{1+e^x},\ 0 \le x \le 1$ about the x-axis is equal to $\pi(e+1)$

Hint: Surface Area =

$$\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\ \text{where } y=\sqrt{1+e^x} \Longrightarrow dy/dx=\frac{e^x}{2\sqrt{1+e^x}}$$

$$\int_0^1 2\pi\sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}dx=\int_0^1 \pi\sqrt{(1+e^x)^2}\,dx$$

Complete yourself..

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12. The length of the curve $y=\ln(\cos x),\ 0\le x\le\frac{\pi}{3}$ is $\ln(2+\sqrt{3})$

Hint: $\int_0^{\pi/3}\sqrt{1+\left(y'\right)^2}\,dx=\int_0^{\pi/3}\sqrt{1+\frac{\sin^2 x}{\cos^2 x}}dx=\int_0^{\pi/3}\sec x\,dx=\left[\ln\left|\sec x+\tan x\right|\right]_0^{\pi/3}$

Complete yourself..

13. The series $\sum_{n=2}^{\infty}\frac{1}{n-\sqrt{n}}$ diverges by comparison test with $\sum_{n=2}^{\infty}\frac{1}{n}$

Hint: Choose

$$b_n=\sum_{n=2}^{\infty}\frac{1}{n} \Longrightarrow \sum_{n=2}^{\infty}\frac{1}{n-\sqrt{n}}>\sum_{n=2}^{\infty}\frac{1}{n} \Longrightarrow \textit{So by comparision test},\ \text{divergent.}$$

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14. $\int\frac{dx}{x^3-x}=\frac{1}{2}\ln\left|x^2-1\right|-\ln|x|+C$

Hint: $\int\frac{dx}{x^3-x}=\int\frac{dx}{x(x-1)(x+1)}$, use method of partial fractions..

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15. The series $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges by the integral test. .

Hint: (It is not p-series, ration test will not work,, test of divergence is not working) Apply Integral Test

$$\lim_{t-\infty}\int_2^t\frac{dx}{x\sqrt{\ln x}}=\lim_{t-\infty}\int_2^t\frac{du}{\sqrt{u}}=2\lim_{t-\infty}\left[\sqrt{\ln x}\right]_2^t=2\lim_{t-\infty}\left(\sqrt{\ln t}-\ln 2\right)=\infty$$

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16. . For the convergent alternating series $\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(k)^3}$, what is the smallest number of terms needed to guarantee that S_n is within $1\ x\ 10^{-8}$ of the actual sum S. (Ans 99)

$\left|S_n - S\right| \le 10^{-8} \Longrightarrow a_{n+1} = \frac{1}{(k+1)^4} \le 10^{-8} \Rightarrow 10^8 \le (k+1)^4 \Rightarrow 10^2 \le k+1$

K = 99

17. $\int \frac{2}{\left(x+\sqrt[3]{x}\right)} dx = 3\ln\left(x^{2/3}+1\right)+C$

Hint: $\int \frac{2}{\left(x+\sqrt[3]{x}\right)} dx = \int \frac{2.3u^2}{\left(u^3+u\right)} du$ ($u = \sqrt[3]{x} \Rightarrow u^3 = x$)

$6\int \frac{u}{\left(u^2+1\right)} du = 3 \ln\left|(u^2+1)\right| + C = 3 \ln\left|x^{\frac{2}{3}}+1\right| + C$ (Method of substitution will work)

18. The integral $\int_0^{\pi/2} \cos^2 x \sin 2x dx = \frac{1}{2}$

Hint:

$\int_0^{\pi/2} \cos^2 x \sin 2x\, dx = \int_0^{\pi/2} \cos^2 x.2\sin x.\cos x.dx = 2\int_0^{\pi/2} \cos^3 x.\sin x\, dx$ (let cosx = u)

Complete yourself..

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19. $\int \frac{1}{x^2\left(1+x^2\right)} dx = -\frac{1}{x} - \tan^{-1} x + C$

Hint: Use method of partial fractions

$\frac{1}{x^2\left(1+x^2\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{\left(1+x^2\right)} \Rightarrow Find$ A, B, C and D= $\frac{1}{x^2} - \frac{1}{\left(1+x^2\right)}$

$\int \frac{1}{x^2\left(1+x^2\right)} dx = \int \frac{1}{x^2} dx - \int \frac{1}{\left(1+x^2\right)} dx$

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20. The series $\sum_{n=2}^{+\infty} \frac{1}{n(n-1)}$ converges to 1.

Hint: Use method of telescoping to find sum

$\sum_{n=2}^{+\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty}\left(-\frac{1}{n}+\frac{1}{n-1}\right) = \left(-\frac{1}{2}+\frac{1}{1}\right)+\left(-\frac{1}{3}+\frac{1}{2}\right)+\left(-\frac{1}{4}+\frac{1}{3}\right)+.. = 1$

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21. If I = $\int_{-1}^{1} \sin(x^2)dx$, *then* show that $0 \le I \le 2$

Hint: The incorrect choices are $I = \infty, I = 0, I > 2, I \le 0$. The reason is the integral I = 2 $\int_0^1 \sin(x^2)dx$ is twice of the area of $\sin\left(x^2\right)$ from 0 to 1.

And area is less then the area of rectangle of size 2 X 1 (You can draw the graph of $\sin\left(x^2\right)$ from -1 to 1 and check. $0 \le \mathrm{I} \le 2$ is correct choice.

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22. $\int_0^\infty xe^{-x}dx = 1$

$$\int_0^\infty xe^{-x}dx = \lim_{t->\infty}\int_0^t xe^{-x}dx = \lim_{t->\infty}\left[\left[-xe^{-x}\right]_0^t + \int_0^t e^{-x}dx\right] = \lim_{t->\infty}(-te^{-t}) + \lim_{t->\infty}(-e^{-t}) + 1$$

$\lim_{t->\infty}(-te^{-t}) = 0$ (L'Hospital Rule), $\lim_{t->\infty}(-e^{-t}) = 0$

Answer = 1

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23. Using the power series of $\ln(1-x)$, then sum of the series

$\sum_{n=1}^\infty \frac{1}{n\ 3^n}$ is equal to $\ln(3/2)$

Hint:

$$\ln(1-x) = \int \frac{-1}{(1-x)}dx => -\ln(1-x) = \int \frac{1}{(1-x)}dx ==> \ln\left(\frac{1}{1-x}\right) = \int \sum_{n=0}^\infty x^n dx$$

$$\ln\frac{1}{1-x} = \sum_{n=0}^\infty \frac{x^{n+1}}{n+1} = \sum_{n=1}^\infty \frac{x^n}{n} \text{ (replace n with n-1)}$$

Let x=1/3➔ $\ln 3/2 = \sum_{n=1}^\infty \frac{1}{n3^n}$

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24. The sequence $\left\{\left(1+\frac{2}{n}\right)^n\right\}_{n=1}^\infty$ *converges to* e^2

Hint: $\lim_{n->\infty}\left(1+\frac{x}{n}\right)^n = e^x => If\ \ x = 2$ we get e^2

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25. The series $\sum_{k=0}^\infty \frac{(-1)^k k!}{e^k}$ is *divergent*

Hint: Use Ratio Test $\lim_{n->\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n->\infty}\left|\frac{k+1!}{e^{k+1}}.\frac{e^k}{k!}\right| = \lim_{n->\infty}\left|\frac{k+1}{e}\right| = \infty$ ==> *Divergent*

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26. The series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4\sqrt{n}}$ is *absolutely convergent*

Hint: This is convergent by Alternating Test. If we test for absolute convergence,

$\sum|a_n| = is\ convergent$ by p-test. ==> The series is absolutely convergent

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27. The first 5 terms of the Taylor series of the function f(x)= x ln(x) at x = 1 is given by

Hint:

$$f(x) = x\ln x,\ f^{'}(x) = \ln x + 1,\ f^{''}(x) = \frac{1}{x},\ f^{'''}(x) = -\frac{1}{x^2},\ f^{4}(x) = \frac{2}{x^3},\ f^{5}(x) = -\frac{2.3}{x^4}$$

Taylor Series at x = 1

=>

$$f(x) = \frac{f^0(1)}{0!}(x-1)^0 + \frac{f^1(1)}{1!}(x-1)^1 + \frac{f^2(1)}{2!}(x-1)^2 + \frac{f^3(1)}{3!}(x-1)^3 + \frac{f^4(1)}{4!}(x-1)^4$$

$$= (x-1)\ + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} - \frac{(x-1)^5}{5!}$$

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28. The interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^3 2^n}$ is $0 \le x \le 4$

Hint: Apply Ratio Test $\lim_{n->\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n->\infty}\frac{\left|(x-2)^{n+1}n^3 2^n\right|}{\left|(n+1)^3 2^{n+1}(x-2)^n\right|} = \lim_{n->\infty}\frac{|(x-2)|}{|2|} < 1$

0 < x <4. Test at x =0 and x=4. In both the cases, it is convergent.

$0 \le x \le 4$