11.9 Representation of Functions as Power Series

Certain functions can be expressed as power series e.g. sin x, cos x…

Like $f(x)=\frac{1}{1-x}=1+x+x^2+x^3+....=\sum_{n=0}^{\infty}x^n \quad |x|<1$

Other similar functions like $\frac{1}{1+x^2}, \frac{1}{2+x}$ which can be written from $\frac{1}{1-x}$

Write power series representation of $\frac{1}{1+x^2}, \frac{1}{2+x}$ and $\frac{x^3}{x+2}$

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n \quad \left(\frac{1}{1-x}=\sum_{n=0}^{\infty}(x)^n\right)$$

$\left|-x^2\right|<1 ==> x^2<1$ or $|x|<1$, interval of convergence (-1,1)

$$\frac{1}{2+x}=\frac{1}{2\left(1+\frac{x}{2}\right)}=\frac{1}{2\left(1-(-\frac{x}{2})\right)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

$|-x/2|<1 ==> |x|<2$, interval of convergence (-2,2)

$$\frac{x^3}{x+2}=x^3.\frac{1}{(x+2)}=x^3.\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

$|-x/2|<1 ==> |x|<2$, interval of convergence (-2,2)

Differentiation and Integration of Power Series

If a power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ **has radius of convergence R(| x - a|< R)**

and it's sum function is

$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$, then it is differentiable on (a- R, a - R)

(i) $f'(x)=c_1+2c_2(x-a)+...=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$

(ii) $\int f(x)=C+\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}$

Use of differentiation and integration of power series

Using Differentiation:

Example: Find the power series for $\frac{1}{(1-x)^2}$

Since $\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(1+x+x^2+x^3+....\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)$

$$\frac{1}{(1-x)^2}=1+2x+3x^2+....=\sum_{n=1}^{\infty}nx^{n-1}$$

Using Integration:

Example: Find the power series for $\tan^{-1}x$

Since $\tan^{-1}x=\int\frac{1}{1+x^2}=\int\left(1-x^2+x^4-...\right)dx=C+x-\frac{x^3}{3}+\frac{x^5}{5}-...$ and C=0 when x=0

(Read Example 6,7,8)

Ex1 (book-8): Find power series for $\frac{x}{4x+1}$

$f(x)=\frac{x}{4x+1}=x.\frac{1}{1-(-4x)}=x\sum_{n=0}^{\infty}(-4x)^n=\sum(-1)^n 2^{2n}x^{n+1}$. This series converges if $|-4x|<1$ or $|x|<1/4$ ➔ The interval of convergence (-1/4,1/4)

Ex2 (book-16): Find power series for $\frac{x^2}{(1-2x)^2}$

$\frac{1}{(1-2x)}=\sum_{n=0}^{\infty}(2x)^n$ => *differentiate* $\frac{2}{(1-2x)^2}=\sum_{n=0}^{\infty}2^n nx^{n-1}$

Replace n with n+1 $\frac{2}{(1-2x)^2}=\sum_{n=1}^{\infty}2^{n+1}(n+1)x^n$

$$\frac{x^2}{(1-2x)^2}=\frac{x^2}{2}\frac{2}{(1-2x)^2}=\frac{x^2}{2}\sum_{n=0}^{\infty}2^{n+1}(n+1)x^n=\sum_{n=0}^{\infty}2^n(n+1)x^{n+2}$$

$\frac{x^2}{(1-2x)^2}=\sum_{n=2}^{\infty}2^{n-2}(n-1)x^n$ (Replace n with n-2)

Also $|2x|<1$➔ or R = ½ (By Convergent Geometric Series Property)

Ex3 (book-30): Use power series to approximate the definite Integral to six decimal places. $\int_0^{0.5} \frac{dx}{1+x^6}$

Using the power series expansion of

$$\int_0^{0.5} \frac{1}{1-(-x^6)}dx = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n x^{6n} dx = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{6n+1}}{6n+1} \right]_0^{1/2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(6n+1)\left(2^{6n+1}\right)}$$

We can find the sum of few terms to get proper approximation.

Ex4 (book-38c) Find the sum of the series

(i) $\sum_{n=2}^{\infty} n(n-1)x^n, |x|<1$

We know $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$

$$\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \Rightarrow \frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n \text{ for } |x|<1$$

Therefore $\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$, for $|x|<1$

(ii) Find $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$

Put x= ½ in (i) $\sum_{n=2}^{\infty} n(n-1)(1/2)^n = \frac{2(1/2)^2}{(1-1/2)^3} = 4 \quad \left(\frac{2x^2}{(1-x)^3}\right)$