11.7 Strategy for Testing Series

Given the series $\sum a_n$, one use any test based on his experience. Given below is the recommended order:

1. If the series is of the form $\sum \frac{1}{n^p}$, it is p-series (if p>1 convergent ,otherwise it is divergent)
2. If it is geometric series with partial sum $s_n = \sum ar^{n-1}$ or $\sum ar^n$ and it converges if $|r| < 1$ and diverges otherwise.
3. If the series is similar to p-series or similar geometric, the Comparison Test can be used e.g. $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$, we can take $b_n = \sum \frac{\sqrt{n^3}}{3n^3} = \sum \frac{1}{3n^{3/2}}$. Here $\sum a_n < \sum b_n$ and $\sum b_n$ is convergent with p=3/2. Therefore a_n is convergent.
4. Apply divergence Test if $\lim_{n->\infty} a_b \neq 0$.
5. If the series is Alternating ($\sum(-1)^{n-1} b_n$ or $\sum(-1)^n b_n$, decreasing and nth term goes to 0), then it is convergent.
6. If the series has factorial , apply ratio test($\lim_{n->\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (*convergent* if L<1,divergent if L>1, if L=1 fails)
7. If the series has raised to power nth power, apply root test $\lim_{n->\infty} \sqrt{|a_n|} = L$ (convergent if L<1, divergent if L>1 , fails if L=1)
8. If the series is not geometric or p-series or telescoping of the terms cannot be done, apply integral test. /divergent/absolutely convergent.

Ex1(book-8): $\sum \frac{2^k k!}{(k+2)!}$. It can written as $\sum \frac{2^k \not{k!}}{(k+2)(k+1)\not{k!}} = \sum \frac{2^k}{(k+2)(k+1)}$

$$\lim_{k->\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k->\infty} \frac{\frac{2^{k+1}}{(k+2)(k+3)}}{\frac{2^k}{(k+1)(k+2)}} = \lim_{k->\infty} 2\frac{k+1}{k+3} = \lim_{k->\infty} 2\frac{(1+\frac{1}{k})}{(1+\frac{3}{k})} = 2$$ >1. It is **divergent**.

Ex2(book-18): $\sum_{n=2} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

This an alternating series with $b_k = \frac{1}{\sqrt{k}-1}$ (k>1)

1. This is decreasing sequence of positive numbers.
2. $\lim_{k->\infty} b_k = 0$

By alternating series test, it is **convergent**.

Ex3 (book-21): $\sum_{n=1} \frac{(-2)^{2n}}{n^n}$

Given series can be re-written as $\sum_{n=1} \left(\frac{4}{n}\right)^n$

Try root test: $\lim_{n->\infty} \sqrt[n]{|a_n|} = \lim_{n->\infty} \frac{4}{n} = 0 < 1$. It **is absolutely convergent**.

Ex4 (book-23): $\sum_{n=1} \tan\left(\frac{1}{n}\right)$

We can try limit comparison test where $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$

$$\lim_{n->\infty} \frac{a_n}{b_n} \equiv \lim_{x->\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x->\infty} \frac{\sec^2\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x->\infty} \sec^2\left(\frac{1}{x}\right) = 1 > 0$$

Since $\sum b_n$ is divergent (Harmonic Series), therefore $\sum a_n$ **is also divergent**.

Ex5 (book-31): $\sum_{k=1} \frac{5^k}{3^k + 4^k}$

We have $a_k = \frac{5^k}{3^k + 4^k}$, and $\lim_{k->\infty} a_k = \lim_{k->\infty} \frac{\left(\frac{5}{4}\right)^k}{1 + \left(\frac{3}{4}\right)^k} = \frac{\infty}{1+0} = \infty$

The series is divergent by divergent test.

Ex6 (book-32): $\sum_{n=1} \frac{(2n)^n}{n^{2n}}$

We can try root test $\lim_{n->\infty} \sqrt[n]{|a_n|} = \lim_{n->\infty} \frac{2n}{n^2} = \lim_{n->\infty} \frac{2}{n} = 0$

It is **absolutely convergent** by root test.

Ex7 (book-34): $\sum_{k=1} \frac{1}{n + n\cos^2 n}$

It seems only test that can be applied is Comparison Test.

We se that : $0 \le \cos^2 n \le 1 ==> 0 \le n\cos^2 n \le n ==> 0 \le n + n\cos^2 n \le n + n$

➔ $\frac{1}{n + n\cos^2 n} \ge \frac{1}{n+n} = \frac{1}{2n}$, Therefore

$\sum \frac{1}{n + n\cos^2 n} \ge \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ (This is divergent Harmonic Series)

Using The Comparison Test, the given series is divergent.