## 11.4 Comparison Tests

The comparison test is applied to know the convergence of $\sum a_n$ with the help of another series $\sum b_n$ :

Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series of **positive terms**

1. If $\sum b_n$ is convergent and $a_n \le b_n$ for all, then $\sum a_n$ **is also convergent**

   Example: Let $\sum a_n = \sum \frac{1}{2^n+1}$ and $\sum b_n = \sum \frac{1}{2^n}$ . Since $\sum b_n$ is convergent and $\frac{1}{2^n+1} < \frac{1}{2^n}$ → $\sum a_n$ is also convergent.

2. If $\sum b_n$ is divergent and $a_n \ge b_n$ for all, then $\sum a_n$ **is also divergent**

   Example: Let $\sum a_n = \sum \frac{\ln n}{n}$ and $\sum b_n = \sum \frac{1}{n}$ . Since $\sum b_n$ is divergent and $\frac{\ln n}{n} > \frac{1}{n}$ ( n>/3)→ $\sum a_n$ is also divergent.

Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series of **positive terms**

**If** $\lim_{n->\infty} \frac{a_n}{b_n} = c$ where c is finite and c>0 , then either both or convergent or both are divergent.

**Example:** $\sum a_n = \sum \frac{1}{2^n+1}$ and let $\sum b_n = \sum \frac{1}{2^n}$, $\lim_{n->\infty} \frac{a_n}{b_n} = \lim_{n->\infty} \frac{2^n}{2^n+1} = \lim_{n->\infty} \frac{1}{\left(1+\frac{1}{2^n}\right)} = 1 > 0$

Since $\sum b_n$ is convergent ➔ $\sum a_n$ is also convergent.

**Example:** $\sum a_n = \sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ and let $\sum b_n = \sum \frac{2n^2}{n^{5/2}} = \sum \frac{2}{n^{1/2}}$,

$\lim_{n->\infty} \frac{a_n}{b_n} = \lim_{n->\infty} \frac{n^2(2+3/n)}{n^{5/2}\sqrt{\frac{5}{n^5}+1}} . \frac{n^{1/2}}{2} = \frac{2+0}{2(\sqrt{0+1})} = 1$. Since $\sum b_n$ is divergent p-series, $\sum a_n$ is also divergent.

Determine the convergence/ divergence of

Ex1(Book-6): $\sum a_n = \sum_{n=2} \frac{1}{n-\sqrt{n}}$

Let $\sum b_n = \sum \frac{1}{n}$ , We can see that $a_n > b_n$ for n>2 . Since $\sum b_n$ is divergent (Divergent Harmonic Series) → $\sum a_n$ is divergent.

Ex2(Book-15): $\sum a_n = \sum_{n=1} \frac{2+(-1)^n}{n\sqrt{n}}$

Select $\sum b_n = \sum \frac{3}{n\sqrt{n}}$ . Since $\frac{2+(-1)^n}{n} \le \frac{3}{n\sqrt{n}}$ for all n, and $\sum b_n$ is convergent geometric series (|r|=1/3),, → $\sum a_n$ is convergent.

Ex2(Book-28): $\sum a_n = \sum_{n=1} \frac{2n^2+7n}{3^n(n^2+5n-1)}$

Select $\sum b_n = \sum \frac{1}{3^n}$.

Apply Limit Convergence Test $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+7n}{n^2+5n-1} = \lim_{n\to\infty} \left( \frac{2+7/n}{1+5/n-1/n^2} \right) = 2 > 0$

Since $\sum b_n$ is convergent geometric series (|r|=1/3), → $\sum a_n$ is convergent.

Ex2(Book-37): The meaning of the decimal representation of a number

$0.d_1d_2d_3d_{4....} = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + ....... = \sum \frac{d_n}{10^n}$

Show that the series always converges.

Let $\sum b_n = \frac{9}{10^n}$

The nth term $\frac{d_n}{10^n} \le \frac{9}{10^n}$ fro each n=1,2,3,4,...

Since $\sum b_n = \frac{9}{10^n}$ is convergent geometric series ($|r| = \frac{1}{10}$) → $\sum a_n = \sum \frac{d_n}{10^n}$ is convergent.