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# Generalized vector quasi-equilibrium problems with applications

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**Abstract**

In this paper, we consider the generalized vector quasi-equilibrium problem with or without involving $\Phi$-condensing maps and prove the existence of its solution by using known fixed point and maximal element theorems. As applications of our results, we derive some existence results for a solution to the vector quasi-optimization problem for nondifferentiable functions and vector quasi-saddle point problem.


*Keywords:* Generalized vector quasi-equilibrium problem; Vector quasi-optimization problem; Regular $C$-quasi-saddle point; $\Phi$-condensing maps; Subdifferential

## 1. Introduction

Let $X$ and $Y$ be real topological vector spaces and $K$ a nonempty subset of $X$. Let $C$ be an ordered cone in $Y$, that is, a closed and convex cone in $Y$ with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$

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denotes the topological interior of $C$. For a given vector-valued bifunction $F : K \times K \to Y$, the *vector equilibrium problem* (for short, VEP) is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \notin -\operatorname{int} C, \quad \text{for all } y \in K,$$

which is a unified model of several known problems, for instance, vector variational and variational-like inequality problems, vector complementarity problem, vector optimization problem and vector saddle point problem; See, for example, [1,9,12,13,15,16,18,22,28] and references therein. For a more comprehensive bibliography on vector equilibrium problems, vector variational and variational-like inequality problems and their generalizations, we refer to a recent volume [15] edited by F. Giannessi.

For a more generalized form of (VEP) which includes vector quasi-variational inequality problem (for short, VQVIP), vector quasi-optimization problem (for short, VQOP) and vector quasi-saddle point problem (for short, VQSPP) as special cases, we let $A : K \to 2^K$ be a multivalued map with nonempty values, where $2^K$ denotes the family of subsets of $K$, then we consider the following problem:

$$\text{Find } \bar{x} \in K \text{ such that}$$
$$\bar{x} \in A(\bar{x}):\ F(\bar{x}, y) \notin -\operatorname{int} C, \quad \text{for all } y \in A(\bar{x}).$$

It is known as *vector quasi-equilibrium problem* (for short, VQEP) and introduced in [7]. Some existence results for a solution to (VQEP) and consequently for (VQVIP), (VQOP) and (VQSPP) have been established in [7].

In [19] (respectively, [7]) (VVIP) (respectively, VQVIP) is used as a tool to solve (VOP) (respectively, VQOP) for differentiable (in some sense) vector-valued functions. The (VOP) for nondifferentiable vector-valued functions can be solved by using generalized vector variational inequality problems. For further details, we refer to [6] and references therein. To obtain a more general problem which contains (VEP) and generalized vector variational inequality problems as special cases, we consider the function $F$ to be multivalued, that is, $F : K \times K \to 2^Y \setminus \{\emptyset\}$ and in this case, (VEP) can be generalized in the following way:

$$\text{Find } \bar{x} \in K \text{ such that}$$
$$F(\bar{x}, y) \nsubseteq -\operatorname{int} C, \quad \text{for all } y \in K.$$

It is called *generalized vector equilibrium problem* (for short, GVEP) and it has been studied by many authors; See, for example, [2,4,5,17] and references therein. For other possible ways to generalize (VEP), we refer to [3,14,23,24,27].

In this paper, we consider the following problem which is a unified format of all above mentioned problems:

$$\text{Find } \bar{x} \in K \text{ such that}$$
$$\bar{x} \in A(\bar{x}):\ F(\bar{x}, y) \nsubseteq -\operatorname{int} C, \quad \text{for all } y \in A(\bar{x}).$$

We shall call it *generalized vector quasi-equilibrium problem* (for short, GVQEP). For a more general form of (GVQEP), we replace the ordered cone $C$ by a "moving cone". More precisely, we consider a multivalued map $C : K \to 2^Y$ such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with $\operatorname{int} C(x) \neq \emptyset$, then the (GVQEP) can be written in the following form:

$$\text{(GVQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \bar{x} \in A(\bar{x})\colon\ F(\bar{x}, y) \nsubseteq -\operatorname{int} C(\bar{x}), \quad \text{for all } y \in A(\bar{x}). \end{cases}$$

The main motivation of this paper is to establish some existence results for a solution to (GVQEP) with or without involving $\Phi$-condensing maps. In the next section, we recall some definitions, notations and results which will be used in the sequel. By using fixed point and maximal element theorems, some existence results for a solution to (GVQEP) are established in Section 3. The last section deals with applications of results of Section 3 to derive some existence results for a solution to the vector quasi-optimization problem for nondifferentiable functions and vector quasi-saddle point problem.

## 2. Preliminaries

Let $T : X \to 2^Y$ be a multivalued map. The *graph of* $T$, denoted by $\mathcal{G}(T)$, is

$$\mathcal{G}(T) = \{(x, z) \in X \times Y\colon\ x \in X,\ z \in T(x)\}.$$

The *inverse* $T^{-1}$ of $T$ is the multivalued map from $\mathcal{R}(T)$, the range of $T$, to $X$ defined by

$$x \in T^{-1}(y) \quad \text{if and only if} \quad y \in T(x).$$

A multivalued map $T : X \to 2^Y$ is said to be *upper semicontinuous at* $x_0 \in X$ [8] if $T(x_0)$ is compact and, for any open set $V$ in $Y$ containing $T(x_0)$, there exists an open neighborhood $U$ of $x_0$ in $X$ such that $T(x) \subseteq V$ for all $x \in U$.

$T$ is called *upper semicontinuous on* $X$ [8] if it is upper semicontinuous at each point of $X$.

The multivalued map $T$ is said to be *closed* [8] if, its graph is closed in $X \times Y$.

**Lemma 2.1** [8]. *Let $T : X \to 2^Y$ be an upper semicontinuous on $X$ and $D$ a compact subset of $X$. Then $T(D)$ is compact.*

**Lemma 2.2** [8]. *If a multivalued map $T : X \to 2^Y$ is upper semicontinuous on $X$ then it is closed.*

Let $E$ be a Hausdorff topological vector space and $L$ a lattice with least element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^E \to L$ is called a *measure of noncompactness* [25,26] provided that the following conditions hold for any $A, B \in 2^E$:

(i) $\Phi(A) = \mathbf{0}$ if and only if $A$ is precompact (i.e., it is relatively compact).
(ii) $\Phi(\overline{\operatorname{co}}\, A) = \Phi(A)$, where $\overline{\operatorname{co}}\, A$ denotes the closed convex hull of $A$.
(iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows from (iii) that if $A \subseteq B$, then $\Phi(A) \leqslant \Phi(B)$.

Let $\Phi : 2^E \to L$ be a measure of noncompactness on $E$ and $D \subseteq E$. A multivalued map $T : D \to 2^E$ is called *$\Phi$-condensing* [25,26] provided that if $A \subseteq D$ with $\Phi(T(A)) \geqslant \Phi(A)$ then $A$ is relatively compact.

Note that every multivalued map defined on a compact set is necessarily $\Phi$-condensing. If $E$ is locally convex, then a compact multivalued map (i.e., $T(D)$ is precompact) is $\Phi$-condensing for any measure of noncompactness $\Phi$. Obviously, if $T : D \to 2^E$ is $\Phi$-condensing and if $T' : D \to 2^E$ satisfies $T'(x) \subseteq T(x)$ for all $x \in D$, then $T'$ is also $\Phi$-condensing.

Let $K$ be a subset of a topological vector space $X$ such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence (in the sense that $K_n \subseteq K_{n+1}$) of nonempty compact sets. A sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ is said to be *escaping from $K$* (relative to $\{K_n\}_{n=1}^{\infty}$) ([10, p. 34]) if for each $n = 1, 2, \ldots$, there exists $m > 0$ such that $x_k \notin K_n$ for all $k \geqslant m$.

The following fixed point and maximal element theorems will be used to prove the main results of this paper.

**Theorem 2.1** [20]. *Let $K$ be a nonempty closed convex subset of a Hausdorff topological vector space $X$ and $S, T : K \to 2^K$ be multivalued maps such that for each $x \in K$, $\operatorname{co} S(x) \subseteq T(x)$ and $K = \bigcup\{\operatorname{int}_K S^{-1}(y)\colon\ y \in K\}$. If $T$ is $\Phi$-condensing, then $T$ has a fixed point.*

**Theorem 2.2** [29]. *Let $K$ be a subset of a topological vector space $X$ such that*

$$K = \bigcup_{n=1}^{\infty} K_n,$$

*where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of $K$. Assume that the multivalued map $S : K \to 2^K$ satisfies the following conditions*:

(i) *For each $x \in K$, $S^{-1}(x) \cap K_n$ is open in $K_n$ for all $n = 1, 2, \ldots$.*
(ii) *For each $x \in K$, $x \notin \operatorname{co} S(x)$.*
(iii) *For each sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ with $x_n \in K_n$ for all $n = 1, 2, \ldots$, which is escaping from $K$ relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \mathbb{N}$ and $y_n \in K_n$ such that $y_n \in \operatorname{co} S(x_n) \cap K_n$.*

*Then there exists $\hat{x} \in K$ such that $S(\hat{x}) = \emptyset$.*

## 3. Existence results

Throughout this section, unless otherwise specified, we shall assume that $Y$ is a real topological vector space and $C : K \to 2^Y$ is a multivalued map such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with $\operatorname{int} C(x) \neq \emptyset$. We denote by $\mathcal{F}(K)$ the family of multivalued maps from $K \times K$ to $2^Y \setminus \{\emptyset\}$.

**Theorem 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hausdorff topological vector space $X$ and, let $A : K \to 2^K$ be a $\Phi$-condensing multivalued map such that for* *each $x \in K$, $A(x)$ is nonempty and convex, $A^{-1}(y)$ is open in $K$ for each $y \in K$ and the set $\mathcal{F} := \{x \in K\colon\ x \in A(x)\}$ is closed. Assume that $F, G \in \mathcal{F}(K)$ satisfy the following conditions*:

(i) *For each $x \in K$, $G(x,x) \nsubseteq \operatorname{int} C(x)$.*
(ii) *For each $x \in K$, the set $\{y \in K\colon\ F(x,y) \subseteq -\operatorname{int} C(x)\}$ is convex.*
(iii) *For each $y \in K$, the set $\{x \in K\colon\ F(x,y) \nsubseteq -\operatorname{int} C(x)\}$ is closed in $K$.*
(iv) *For all $x, y \in K$,*

$$G(y,x) \nsubseteq \operatorname{int} C(x) \quad \textit{implies} \quad F(x,y) \nsubseteq -\operatorname{int} C(x).$$

*Then the* (*GVQEP*) *has a solution.*

**Proof.** From condition (ii), the multivalued map $P : K \to 2^K$ defined by

$$P(x) = \big\{y \in K\colon\ F(x,y) \subseteq -\operatorname{int} C(x)\big\} \quad \text{for all } x \in K,$$

is convex valued and, from condition (iii), the complement of $P^{-1}(y)$ in $K$, denoted by $[P^{-1}(y)]^c$,

$$\big[P^{-1}(y)\big]^c = \big\{x \in K\colon\ F(x,y) \nsubseteq -\operatorname{int} C(x)\big\}$$

is closed in $K$ for each $y \in K$. Therefore, for each $y \in K$, $P^{-1}(y)$ is open in $K$.

For each $x \in K$, we define multivalued maps $S, T : K \to 2^K$ by

$$S(x) = \begin{cases} A(x) \cap P(x) & \text{if } x \in \mathcal{F}, \\ A(x) & \text{if } x \in K \setminus \mathcal{F}, \end{cases}$$

and

$$T(x) = \begin{cases} A(x) \cap Q(x) & \text{if } x \in \mathcal{F}, \\ A(x) & \text{if } x \in K \setminus \mathcal{F}, \end{cases}$$

where $Q : K \to 2^K$ is a multivalued map defined as

$$Q(x) = \big\{y \in K\colon\ G(y,x) \subseteq \operatorname{int} C(x)\big\} \quad \text{for all } x \in K.$$

Then for each $x \in K$, $S(x)$ is convex since $A(x)$ and $P(x)$ are convex. Therefore, from condition (iv), we have $\operatorname{co} S(x) \subseteq T(x)$ for all $x \in K$.

Since for each $y \in K$, $A^{-1}(y)$ and $P^{-1}(y)$ are open in $K$,

$$S^{-1}(y) = \big(A^{-1}(y) \cap P^{-1}(y)\big) \cup \big((K \setminus \mathcal{F}) \cap A^{-1}(y)\big)$$

(see, for example, the proof of Lemma 2.3 in [11]) and $K \setminus \mathcal{F}$ is open in $K$, we have $S^{-1}(y)$ is open in $K$. Now assume that for each $x \in \mathcal{F}$, $A(x) \cap P(x) \neq \emptyset$. Then for each $x \in K$, $S(x) \neq \emptyset$ and therefore

$$K = \bigcup_{y \in K} S^{-1}(y) = \bigcup_{y \in K} \operatorname{int}_K S^{-1}(y).$$

Since for each $x \in K$, $T(x) \subseteq A(x)$ and $A$ is $\Phi$-condensing, we have that $T$ is also $\Phi$-condensing.

Thus by Theorem 2.1, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$. From the definition of $\mathcal{F}$ and $T$, we have $\{x \in K\colon x \in T(x)\} \subseteq \mathcal{F}$. Therefore, $\hat{x} \in \mathcal{F}$ and $\hat{x} \in A(\hat{x}) \cap Q(\hat{x})$ and, in particular, $G(\hat{x}, \hat{x}) \subseteq \operatorname{int} C(\hat{x})$, a contradiction of (i). Hence there exists $\bar{x} \in \mathcal{F}$ such that $A(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is,

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad F(\bar{x}, y) \nsubseteq -\operatorname{int} C(\bar{x}) \quad \text{for all } y \in A(\bar{x}).$$

This completes the proof. □

The next corollary results by taking $G(x, y) = -F(y, x)$ in the previous theorem.

**Corollary 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hausdorff topological vector space $X$ and, let $A : K \to 2^K$ be a $\Phi$-condensing multivalued map such that for each $x \in K$, $A(x)$ is nonempty and convex, $A^{-1}(y)$ is open in $K$ for each $y \in K$ and the set $\mathcal{F} := \{x \in K\colon x \in A(x)\}$ is closed. Assume that $F \in \mathcal{F}(K)$ satisfies the following conditions*:

(i) *For each $x \in K$, $F(x, x) \nsubseteq -\operatorname{int} C(x)$.*
(ii) *For each $x \in K$, the set $\{y \in K\colon F(x, y) \subseteq -\operatorname{int} C(x)\}$ is convex.*
(iii) *For each $y \in K$, the set $\{x \in K\colon F(x, y) \nsubseteq -\operatorname{int} C(x)\}$ is closed in $K$.*

*Then the* (*GVQEP*) *has a solution.*

When $A$ is not necessarily $\Phi$-condensing, then we have the following result.

**Theorem 3.2.** *Let $K$ be a subset of a topological vector space $X$* (*not necessarily, Hausdorff*) *such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of $K$. Let $A : K \to 2^K$ be a multivalued map such that for all $x \in K$, $A(x)$ is nonempty and convex, $A^{-1}(y)$ is compactly open in $K$ for any $y \in K$ and the set $\mathcal{F} := \{x \in K\colon x \in A(x)\}$ is closed. Assume that $F \in \mathcal{F}(K)$ satisfies the following conditions*:

(i) *For each $x \in K$, $F(x, x) \nsubseteq -\operatorname{int} C(x)$.*
(ii) *For each $x \in K$, the set $\{y \in K\colon F(x, y) \subseteq -\operatorname{int} C(x)\}$ is convex.*
(iii) *For each $y \in K$, the set $\{x \in K\colon F(x, y) \nsubseteq -\operatorname{int} C(x)\}$ is compactly closed in $K$.*
(iv) *For each sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ with $x_n \in K_n$ for all $n = 1, 2, \ldots$, which is escaping from $K$ relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \mathbb{N}$ and $y_n \in K_n$ such that $y_n \in A(x_n)$ and $F(x_n, y_n) \subseteq -\operatorname{int} C(x_n)$.*

*Then the* (*GVQEP*) *has a solution.*

**Proof.** Let $P, \mathcal{F}$ and $S$ be the same as defined in the proof of Theorem 3.1. Then for each $x \in K$, $S(x)$ is convex. Suppose that there exists $\hat{x} \in K$ such that $\hat{x} \in S(\hat{x})$. If $\hat{x} \in \mathcal{F}$, then $\hat{x} \in A(\hat{x}) \cap P(\hat{x})$ and thus $F(\hat{x}, \hat{x}) \subseteq -\operatorname{int} C(\hat{x})$, a contradiction of (i). If $\hat{x} \notin \mathcal{F}$, then $S(\hat{x}) = A(\hat{x})$ which implies $\hat{x} \in A(\hat{x})$, a contradiction. Hence for all $x \in K$, $x \notin S(x) = \operatorname{co} S(x)$.

By (iii), for each $y \in K$, $P^{-1}(y)$ is compactly open in $K$. Following the argument of the proof of Theorem 3.1, we have for each $y \in K$, $S^{-1}(y)$ is compactly open in $K$. Condition (iv) implies condition (iii) of Theorem 2.2. Therefore, by Theorem 2.2, there exists $\bar{x} \in K$ such that $S(\bar{x}) = \emptyset$. Since for each $x \in K$, $A(x)$ is nonempty, we have $\bar{x} \in A(\bar{x})$ such that $A(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is,

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad F(\bar{x}, y) \nsubseteq -\operatorname{int} C(\bar{x}) \text{ for all } y \in A(\bar{x}).$$

This completes the proof. □

**Definition 3.1** [5]. A map $F \in \mathcal{F}(K)$ is called $C_x$-quasiconvex—like if, for all $x, y_1, y_2 \in K$ and $\alpha \in [0, 1]$, we have either $F(x, \alpha y_1 + (1-\alpha)y_2) \subseteq F(x, y_1) - C(x)$ or $F(x, \alpha y_1 + (1-\alpha)y_2) \subseteq F(x, y_2) - C(x)$.

**Remark 3.1.** (a) If $F \in \mathcal{F}(K)$ is $C_x$-quasiconvex-like, then the set $\{y \in K\colon F(x, y) \subseteq -\operatorname{int} C(x)\}$ is convex, for each $x \in K$ (see, for example, the proof of Theorem 2.1 in [5]).

(b) If the multivalued map $W : K \to 2^Y$ defined by $W(x) = Y \setminus \{-\operatorname{int} C(x)\}$ for all $x \in K$, is closed on $K$ (respectively, on each compact subset of $K$) and for each $y \in K$, $F(\cdot, y)$ is upper semicontinuous on $K$ (respectively, on each compact subset of $K$), then condition (iii) of Theorem 3.1 and Corollary 3.1 (respectively, Theorem 3.2) is satisfied; See, for example, the proof of Theorem 2.1 in [5].

**Remark 3.2.** Theorem 3.2 along with Remark 3.1 is an extension of Theorem 3.2 in [7] to multivalued map $F \in \mathcal{F}(K)$ and "moving cone".

## 4. Applications

Throughout this section, unless otherwise specified, we consider the case where $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $K \subseteq X$ and for all $x \in K$, $C(x) = C$ a proper, closed, pointed and convex cone with $\operatorname{int} C \neq \emptyset$.

Let $\varphi : K \to Y$ be a vector-valued function. We consider the following *vector quasi-optimization problem* (for short, VQOP):

$$\min \varphi(x) \quad \text{subject to } x \in A(x),$$

where $A : K \to 2^K \setminus \{\emptyset\}$ is a multivalued map.

By $\bar{x} \in K$ is a solution of (VQOP), we mean

$$\bar{x} \in A(\bar{x})\colon\ \varphi(y) - \varphi(\bar{x}) \notin -\operatorname{int} C, \quad \text{for all } y \in A(\bar{x}).$$

Let $\varphi$ be a function from a nonempty convex subset $K \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. We recall that $\varphi$ is said to be *convex* on $K$ if, for every $x, y \in K$, $\lambda \in (0, 1)$, we have

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) - \varphi\big(\lambda x + (1-\lambda)y\big) \in C.$$

Following [21], we define the subdifferential of a convex function $\varphi$ at $x_0 \in K$, denoted by $\partial\varphi(x_0)$, as

$$\partial\varphi(x_0) = \big\{u \in L(\mathbb{R}^n, \mathbb{R}^m)\colon\ \varphi(x) - \varphi(x_0) - \langle u, x - x_0\rangle \in C,\ \forall x \in K\big\},$$

where $L(\mathbb{R}^n, \mathbb{R}^m)$ and $\langle u, x\rangle$ denote the space of linear continuous functions from $\mathbb{R}^n$ into $\mathbb{R}^m$ and the evaluation of $u \in L(\mathbb{R}^n, \mathbb{R}^m)$ at $x \in \mathbb{R}^n$, respectively.

Now we present some applications of Corollary 3.1 and Theorem 3.2.

**Theorem 4.1.** *Let $K$ be a nonempty closed convex subset of $\mathbb{R}^n$ and $A$ be the same as in Theorem* 3.1. *Let $\varphi : K \to \mathbb{R}^m$ be a convex function such that $\partial\varphi$ is upper semicontinuous on $K$ and for each $x \in K$, $\partial\varphi(x)$ is nonempty and convex. Then the* (*VQOP*) *has a solution.*

**Proof.** For all $x, y \in K$, we set

$$F(x, y) = \bigl\langle \partial\varphi(x), y - x\bigr\rangle = \bigl\{\langle u, y - x\rangle\colon\ u \in \partial\varphi(x)\bigr\}.$$

It is easy to verify conditions (i) and (ii) of Corollary 3.1. So, we shall verify only condition (iii) of Corollary 3.1, that is, for each $y \in K$, the following set is closed in $K$:

$$\begin{aligned} B &= \bigl\{x \in K\colon\ \langle \partial\varphi(x), y - x\rangle \nsubseteq -\operatorname{int} C\bigr\} \\ &= \bigl\{x \in K\colon\ \exists u \in \partial\varphi(x) \text{ s.t. } \langle u, y - x\rangle \notin -\operatorname{int} C\bigr\}. \end{aligned}$$

Let $\{x_n\}$ be a sequence in $B$ such that $x_n \to \tilde{x} \in K$. Then

$$\exists u_n \in \partial\varphi(x_n) \text{ s.t. } \langle u_n, y - x_n\rangle \notin -\operatorname{int} C, \quad \forall n.$$

This implies that

$$\langle u_n, y - x_n\rangle \in Y \setminus \{-\operatorname{int} C\}, \quad \forall n.$$

Let $\mathcal{I} = \{x_n\} \cup \{\tilde{x}\}$. Then $\mathcal{I}$ is compact and by Lemma 2.1, $\partial\varphi(\mathcal{I})$ is also compact and so $u_n \in \partial\varphi(\mathcal{I})$. Therefore $\{u_n\}$ has a convergent subsequence with limit $\tilde{u}$, say. Without loss of generality, we may assume that $u_n \to \tilde{u}$. Then by upper semicontinuity of $\partial\varphi$ and Lemma 2.2, we have $\tilde{u} \in \partial\varphi(\tilde{x})$. Since the pairing $\langle\cdot,\cdot\rangle$ is continuous and $Y \setminus \{-\operatorname{int} C\}$ is closed, we have

$$\langle u_n, y - x_n\rangle \to \langle \tilde{u}, y - \tilde{x}\rangle \in Y \setminus \{-\operatorname{int} C\}$$

and hence $\langle \tilde{u}, y - \tilde{x}\rangle \notin -\operatorname{int} C$. Therefore, $\tilde{x} \in B$ and thus $B$ is closed in $K$.

By Corollary 3.1, there exists $\bar{x} \in K$ such that

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad \bigl\langle \partial\varphi(\bar{x}), y - \bar{x}\bigr\rangle \nsubseteq -\operatorname{int} C \text{ for all } y \in A(\bar{x}),$$

that is,

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad \forall y \in A(\bar{x}),\ \exists \bar{u} \in \partial\varphi(\bar{x})\colon\ \langle \bar{u}, y - \bar{x}\rangle \notin -\operatorname{int} C. \tag{1}$$

Since $\bar{u} \in \partial\varphi(\bar{x})$, we have

$$\varphi(y) - \varphi(\bar{x}) - \langle \bar{u}, y - \bar{x}\rangle \in C. \tag{2}$$

Combining (1) and (2), we get

$$\varphi(y) - \varphi(\bar{x}) \notin -\operatorname{int} C \quad \text{for all } y \in A(\bar{x}),$$

since $a \notin -\operatorname{int} C$ and $b - a \in C \Rightarrow b \notin -\operatorname{int} C$. Hence $\bar{x} \in K$ is a solution of (VQOP). □

**Theorem 4.2.** *Let $K$ be a convex subset of $\mathbb{R}^n$ such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of $K$. Let $A$ be the same as in Theorem* 3.2 *and $\varphi : K \to \mathbb{R}^m$ be a convex function such that $\partial\varphi$ is upper semicontinuous on $K$ and for each $x \in K$, $\partial\varphi(x)$ is nonempty and convex. Assume that for each sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ with $x_n \in K_n$ for all $n = 1, 2, \ldots,$ which is escaping from $K$ relative to $\{K_n\}_{n=1}^{\infty}$, there exist $n \in \mathbb{N}$ and $y_n \in K_n$ such that $y_n \in A(x_n)$ and $\langle \partial\varphi(x_n), y_n - x_n \rangle \subseteq -\operatorname{int} C$. Then the* (*VQOP*) *has a solution.*

**Proof.** It follows from the proof of Theorem 4.1 by using Theorem 3.2. □

**Remark 4.1.** Theorem 4.12 in [21] provides that $\partial\varphi(x)$ is a nonempty compact convex set if $x \in \operatorname{int} K$, $K$ is a convex set and $\varphi$ is a convex function on $K$; Lemma 4.3 in [21] provides that $\partial\varphi : K \to L(\mathbb{R}^n, \mathbb{R}^m)$ is closed if $\varphi$ is convex and continuous on $K$.

Now we consider (VQOP) in the setting of infinite dimensional spaces, that is, $K$ is a nonempty closed convex subset of a real Hausdorff topological vector space $X$ and $Y$ is a real topological vector space with a pointed, proper, closed and convex cone such that $\operatorname{int} C \neq \emptyset$.

A function $\varphi : K \to Y$ is called *$C$-upper semicontinuous on $K$* [9] if, for all $\alpha \in Y$, the (upper level) set

$$U(\alpha) = \bigl\{x \in K:\ \varphi(x) - \alpha \notin -\operatorname{int} C\bigr\}$$

is closed in $K$.

$\varphi$ is called *$C$-lower semicontinuous on $K$* [9] if, $-\varphi$ is $C$-upper semicontinuous on $K$.

$\varphi$ is called *$C$-quasiconvex* (respectively, *$C$-quasiconcave*) [9] if, for all $\alpha \in Y$, the set

$$\bigl\{x \in K:\ \varphi(x) - \alpha \in -C\bigr\} \quad (\text{respectively, } \{x \in K:\ \varphi(x) + \alpha \in C\})$$

is convex.

If $\varphi$ is $C$-quasiconvex (respectively, $C$-quasiconcave), then the set $\{x \in K:\ \varphi(x) \in -\operatorname{int} C\}$ (respectively, $\{x \in K:\ \varphi(x) \in \operatorname{int} C\}$) is also convex; See, for example, [9].

**Theorem 4.3.** *Let $K, X, Y$ and $A$ be the same as in Corollary* 3.1. *If $\varphi : K \to Y$ is $C$-quasiconvex and $C$-lower semicontinuous on $K$, then the* (*VQOP*) *has a solution.*

**Proof.** Consider $F$ as a single-valued map, that is, $F : K \times K \to Y$ in Corollary 3.1 and define $F(x, y) = \varphi(y) - \varphi(x)$ for all $x, y \in K$. Then from Corollary 3.1, we get the result. □

We close this section by giving another application of Corollary 3.1 to the vector quasi-saddle point problem.

Let $X_i$, $i = 1, 2$, be real topological vector spaces and $K_i \subseteq X_i$, $i = 1, 2$, nonempty convex sets. Let $A : K = K_1 \times K_2 \to 2^K$ be a multivalued map defined as $A(x_1, x_2) = A_1(x_1) \times A_2(x_2)$ for all $x_1 \in K_1$ and $x_2 \in K_2$, where $A_i : K_i \to 2^{K_i} \setminus \{\emptyset\}$, $i = 1, 2$, are multivalued maps, $\phi : K \to Y$ a vector-valued function, and $Y$ and $C$ be the same as above. Then $\bar{x} = (\bar{x}_1, \bar{x}_2) \in K_1 \times K_2$ is called

(i) *regular $C$-saddle point of $\phi$* [28] if,

$$\phi(y_1, \bar{x}_2) - \phi(\bar{x}_1, y_2) \notin -\operatorname{int} C, \quad \text{for all } (y_1, y_2) \in K_1 \times K_2;$$

(ii) *regular $C$-quasi-saddle point of $\phi$* [7] if,

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad \phi(y_1, \bar{x}_2) - \phi(\bar{x}_1, y_2) \notin -\operatorname{int} C, \\ \text{for all } (y_1, y_2) \in A(\bar{x}).$$

From Corollary 3.1, we derive the following existence result for a regular $C$-quasi-saddle point of $\phi$.

**Theorem 4.4.** *Let $X_i$, $i = 1, 2$, be real Hausdorff topological vector spaces, $K_i \subseteq X_i$, $i = 1, 2$ nonempty closed convex sets and $A_i : K_i \to 2^{K_i}$, $i = 1, 2$, multivalued maps. Let $A : K = K_1 \times K_2 \to 2^K$ be a $\Phi$-condensing multivalued map defined as $A(x_1, x_2) = A_1(x_1) \times A_2(x_2)$ for all $x_1 \in K_1$ and $x_2 \in K_2$ such that for each $x \in K$, $A(x)$ is nonempty and convex, $A^{-1}(y)$ is open in $K$ for all $y \in K$ and the set $\mathcal{F} := \{x = (x_1, x_2) \in K_1 \times K_2\colon x \in A(x)\}$ is closed. Let $\phi : K_1 \times K_2 \to Y$ be a vector-valued function such that*

(i) *for each $x_2 \in K_2$, the function $y_1 \mapsto \phi(y_1, x_2)$ is $C$-quasiconvex and $C$-lower semicontinuous on $K$;*
(ii) *for each $x_1 \in K_1$, the function $y_2 \mapsto \phi(x_1, y_2)$ is $C$-quasiconcave and $C$-upper semicontinuous on $K$.*

*Then there exists a regular $C$-quasi-saddle point for $\phi$.*

**Proof.** Consider $F$ as a single-valued map, that is, $F : K \times K \to Y$ in Corollary 3.1 and define

$$F(x, y) = \phi(y_1, x_2) - \phi(x_1, y_2)$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in K_1 \times K_2$. Then from Corollary 3.1, we get the conclusion. □

## References


[1] Q.H. Ansari, Vector equilibrium problems and vector variational inequalities, in: F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Kluwer, Dordrecht, 2000, pp. 1–16.

[2] Q.H. Ansari, I.V. Konnov, J.-C. Yao, On generalized vector equilibrium problems, Nonlinear Anal. 47 (2001) 543–554.

[3] Q.H. Ansari, W. Oettli, D. Schläger, A generalization of vectorial equilibria, Math. Methods Oper. Res. 46 (1997) 147–152.

[4] Q.H. Ansari, A.H. Siddiqi, S.Y. Wu, Existence and duality of generalized vector equilibrium problems, J. Math. Anal. Appl. 259 (2001) 115–126.

[5] Q.H. Ansari, J.-C. Yao, An existence result for the generalized vector equilibrium problem, Appl. Math. Lett. 12 (8) (1999) 53–56.

[6] Q.H. Ansari, J.-C. Yao, Nondifferentiable and nonconvex vector optimization problems, J. Optim. Theory Appl. 106 (2000) 487–500.
[7] Q.H. Ansari, J.-C. Yao, On vector quasi-equilibrium problems, in: A. Maugeri, F. Giannessi (Eds.), Variational Problems and Equilibria, Kluwer, Dordrecht, in press.
[8] C. Berge, Topological Spaces, Oliver & Boyd, Edinburgh, 1963.
[9] M. Bianchi, N. Hadjisavvas, S. Schaible, Vector equilibrium problems with generalized monotone bifunctions, J. Optim. Theory Appl. 92 (1997) 527–542.
[10] K.C. Border, Fixed Point Theorems with Applications to Economics and Game Theory, Cambridge University Press, Cambridge, 1985.
[11] X.P. Ding, Existence of solutions for quasi-equilibrium problems in noncompact topological spaces, Comput. Math. Appl. 39 (2000) 13–21.
[12] Fabián Flores-Bazán, Fernando Flores-Bazán, Vector equilibrium problems under asymptotic analysis, J. Global Optim., to appear.
[13] J. Fu, Simultaneous vector variational inequalities and vector implicit complementarity problems, J. Optim. Theory Appl. 93 (1997) 141–151.
[14] P.G. Georgiev, T. Tanaka, Vector-valued set-valued variants of Ky Fan's inequality, J. Nonlinear Convex Anal. 1 (2000) 245–254.
[15] F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, Kluwer, Dordrecht, 2000.
[16] N. Hadjisavvas, S. Schaible, From scalar to vector equilibrium problems in the quasi monotone case, J. Optim. Theory Appl. 96 (1998) 297–309.
[17] I.V. Konnov, J.-C. Yao, Existence of solutions for generalized vector equilibrium problems, J. Math. Anal. Appl. 233 (1999) 328–335.
[18] G.M. Lee, D.S. Kim, B.S. Lee, On noncooperative vector equilibrium, Indian J. Pure Appl. Math. 27 (1996) 735–739.
[19] G.M. Lee, D.S. Kim, B.S. Lee, N.D. Yen, Vector variational inequality as a tool for studying vector optimization problems, Nonlinear Anal. 34 (1998) 745–765.
[20] L.-J. Lin, S. Park, Z.-T. Yu, Remarks on fixed points, maximal elements, and equilibria of generalized games, J. Math. Anal. Appl. 233 (1999) 581–596.
[21] D.T. Luc, N.X. Tan, P.N. Tinh, Convex vector functions and their subdifferential, Acta Math. Vietnam. 23 (1998) 107–127.
[22] W. Oettli, A remark on vector valued equilibria and generalized monotonicity, Acta Math. Vietnam. 22 (1997) 213–221.
[23] W. Oettli, D. Schläger, Generalized vectorial equilibria and generalized monotonicity, in: M. Brokate, A.H. Siddiqi (Eds.), Functional Analysis with Current Applications in Science, Technology and Industry, in: Pitman Research Notes in Mathematics Series, Vol. 377, Longman, Essex, 1998, pp. 145–154.
[24] W. Oettli, D. Schläger, Existence of equilibria for monotone multivalued mappings, Math. Methods Oper. Res. 48 (1998) 219–228.
[25] W.V. Petryshyn, P.M. Fitzpatrick, Fixed point theorems of multivalued noncompact inward maps, J. Math. Anal. Appl. 46 (1974) 756–767.
[26] W.V. Petryshyn, P.M. Fitzpatrick, Fixed point theorems of multivalued noncompact acyclic mappings, Pacific J. Math. 54 (1974) 17–23.
[27] W. Song, Vector equilibrium problems with set-valued mappings, in: F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Kluwer, Dordrecht, 2000, pp. 403–422.
[28] N.X. Tan, P.N. Tinh, On the existence of equilibrium points of vector functions, Numer. Funct. Anal. Optim. 19 (1998) 141–156.
[29] G.X.-Z. Yuan, G. Isac, K.-K. Tan, J. Yu, The study of minimax inequalities, abstract economics and applications to variational inequalities and Nash equilibria, Acta Appl. Math. 54 (1998) 135–166.