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Existence and Duality of Generalized Vector Equilibrium Problems[1]

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In this paper, we propose some dual formulations of generalized vector equilibrium problems. By using such dual formulations, we prove the existence of a solution to the generalized vector equilibrium problem under generalized pseudomonotonicity conditions. The results of this paper extend and generalize the results of I.V. Konnov and S. Schaible (2000, *J. Optim. Theory Appl.*, **104,** 395–408).

Key Words: generalized vector equilibrium problems; duality; C_x-quasiconvexity; explicitly $\delta(C_x)$-quasiconvexity; u-hemicontinuity; maximal pseudomonotonicity.

1. INTRODUCTION

Let K be a nonempty convex subset of a real topological vector space X and $f : K \times K \to \mathbb{R}$ be a bifunction such that $f(x, x) \geq 0$ for all $x \in K$. The scalar *equilibrium problem* is to

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0, \qquad \text{for all } y \in K, \tag{EP}$$

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which includes optimization, saddle point, fixed point, complementarity, and variational inequality problems; see, for example, [3, 7, 9–12, 17]. Recently, Konnov and Schaible [17] adopted the following rule to obtain the dual formulations of (EP):

> replace arguments of the underlying function and change the sign on the left-hand side of the inequality.

Then such a dual will satisfy the following fundamental duality property:

> the dual of the dual is the primal.

However, in addition to this property, it also satisfies some other properties similar to those in optimization, for example, the solution sets of dual problems coincide with the solution set of the primal problem under certain monotonicity and convexity conditions.

If we replace the range set $\mathbb{R}$ by a real topological vector space Y with a closed, convex, and solid cone P and consider a multivalued map $F: K \times K \to \Pi(Y)$, where $\Pi(Y)$ denotes the family of nonempty subsets of Y, then the inequality occurring in (EP) can be generalized as $F(x, y) \not\subseteq -\operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. For other possible ways to generalize inequality in (EP), we refer to [2, 22, 24]. Under these hypotheses, (EP) is known as the *generalized vector equilibrium problem* (in short, GVEP) which contains generalized vector implicit variational inequality problems and generalized vector variational and variational-like inequality problems as special cases; see, for example, [4] and references therein.

By extending the terminology of Konnov and Schaible [17], in this paper we give some dual formulations of the (GVEP) and prove that the solution sets of the dual problems coincide with the solution set of the primal (GVEP) under certain pseudomonotonicity assumptions. By using these dual formulations of the (GVEP), we also establish some existence results for a solution to the (GVEP). In the last two sections, we specialize the results of Section 3. The results of this paper extend and generalize the results of Konnov and Schaible [17].

2. FORMULATIONS AND PRELIMINARIES

Throughout this paper, unless otherwise specified, we assume that X and Y are real topological vector spaces (not necessarily Hausdorff) and K is a nonempty convex subset of X. We denote by $\mathscr{F}(K)$ the family of multivalued maps from $K \times K$ to $\Pi(Y)$. Let $C: K \to \Pi(Y)$ be a multivalued map such that for each $x \in K$, $C(x)$ is a closed and convex cone with $\operatorname{int} C(x) \neq \emptyset$, where $\operatorname{int} C(x)$ denotes the interior of $C(x)$. For a given multivalued map $F \in \mathscr{F}(K)$, we consider the following *generalized vector equilibrium problem*:

$$\text{Find } \bar{x} \in \text{K such that } F(\bar{x}, y) \nsubseteq -\operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K. \quad \text{(GVEP)}$$

We denote by K^p the solution set of (GVEP). This problem is considered and studied in [4, 6, 18, 23] and includes the generalized implicit vector variational inequality problem and generalized vector variational and variational-like inequality problems [15]. In the recent past, generalized vector variational and variational-like inequality problems are used as tools to solve vector optimization problems; see, for example, [5, 20].

When F is a single-valued map, (GVEP) is known as the *vector equilibrium problem* considered and studied in [1, 8, 14–16, 19, 21, 25].

Recently, Konnov and Yao [18] defined the dual form of (GVEP) in the following way:

$$\text{Find } \bar{x} \in K \text{ such that } F(y, \bar{x}) \nsubseteq \operatorname{int}(\bar{x}), \qquad \text{for all } y \in K. \quad \text{(DGVEP)}$$

The solution set of this problem will be denoted by K^d.

We extend the terminology of Konnov and Schaible [17] to obtain the dual of (GVEP) by the following rule:

> replace arguments of the underlying multivalued map and change the sign on the right-hand side of the inclusion.

Then clearly the dual of (DGVEP) is (GVEP).

We shall use the following concepts and a result in the sequel.

DEFINITION 2.1 [18]. Let $F \in \mathscr{F}(K)$ and $C: K \to \Pi(Y)$ be a multivalued map such that for each $x \in K$, $C(x)$ is a closed and convex cone in Y. F is called

(i) *C_x-quasiconvex* if, for all $x, y_1, y_2 \in K$ and $\alpha \in [0, 1]$, we have either

$$F(x, y_1) \subseteq F(x, y_\alpha) + C(x)$$

or

$$F(x, y_2) \subseteq F(x, y_\alpha) + C(x),$$

where $y_\alpha = \alpha y_1 + (1 - \alpha) y_2$;

(ii) *explicitly $\delta(C_x)$-quasiconvex* if, for all $y_1, y_2 \in K$ and $\alpha \in (0, 1)$, we have either

$$F(y_\alpha, y_1) \subseteq F(y_\alpha, y_\alpha) + C(y_1)$$

or

$$F(y_\alpha, y_2) \subseteq F(y_\alpha, y_\alpha) + C(y_1),$$

and, in case $F(y_\alpha, y_1) - F(y_\alpha, y_2) \subseteq \operatorname{int} C(y_1)$ for all $\alpha \in (0,1)$, we have

$$F(y_\alpha, y_1) \subseteq F(y_\alpha, y_\alpha) + \operatorname{int} C(y_1),$$

where $y_\alpha = \alpha y_1 + (1-\alpha) y_2$;

(iii) *pseudomonotone* if, for all $x, y \in K$,

$$F(x,y) \nsubseteq -\operatorname{int} C(x) \qquad \text{implies} \qquad F(y,x) \nsubseteq \operatorname{int} C(x).$$

For $B \subseteq X$, we denote by $\operatorname{conv}(B)$ and $\operatorname{cl} B$ the convex hull and the closure of B, respectively.

Let $T: X \to \Pi(Y)$ be a multivalued map. The *graph* of T, denoted by $\mathscr{G}(T)$, is

$$\mathscr{G}(T) = \{(x,u) \in X \times Y : x \in X, u \in T(x)\}.$$

DEFINITION 2.2. A multivalued map $T: K \to \Pi(Y)$ is called:

(i) *upper semicontinuous on* K if, for each $x_0 \in K$ and any open set V in Y containing $T(x_0)$, there exists an open neighborhood U of x_0 in K such that $T(x) \subseteq V$ for all $x \in U$;

(ii) *u-hemicontinuous* if, for any $x, y \in K$ and $\alpha \in (0,1)$, the multivalued map $\alpha \mapsto T(\alpha x + (1-\alpha)y)$ is upper semicontinuous at 0^+.

DEFINITION 2.3. A multivalued map $T: X \to \Pi(X)$ is called a *KKM-map* if, for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $\operatorname{conv}(\{x_1, x_2, \ldots, x_n\}) \subseteq \bigcup_{i=1}^n T(x_i)$.

We shall use the following well known Fan–KKM Lemma [13] which originally was proved in the setting of Hausdorff topological vector spaces, but it turned out to be true in general topological vector spaces.

LEMMA 2.1. *Let B be an arbitrary nonempty set in a topological vector space E and $T: B \to \Pi(E)$ be a KKM-map. If $T(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x \in B} T(x) \neq \emptyset$.*

3. GENERALIZED DUALITY

With the help of an operator Φ from $\mathscr{F}(K)$ into itself, we propose the following *dual generalized vector equilibrium problem*,

$$\text{find } \bar{x} \in K \text{ such that } \Phi(F(\bar{x}, y)) \nsubseteq -\operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K, \tag{DGVEP$_\Phi$}$$

and Φ is called the duality operator. In fact, the operator Φ is nothing but a set of fixed rules applied to (GVEP), rather than to a bifunction. We shall see that under certain conditions, the dual of (DGVEP$_\Phi$) is the primal (GVEP) and the solution set K_Φ^d of (DVEP$_\Phi$) coincides with the solution set K^p of the primal (GVEP).

For simplicity, we set

$$G(y,x) = -\Phi(F(x,y));$$

then (DGVEP$_\Phi$) can be written as the following problem:

$$\text{Find } \bar{x} \in K \text{ such that } G(y,\bar{x}) \not\subseteq \operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K. \quad (3.1)$$

Such a type of problem is considered by Konnov and Yao [18]. The set of all solutions of (3.1) will be denoted by K_G^d.

DEFINITION 3.1. Let $F, G \in \mathscr{F}(K)$ and $C\colon K \to \Pi(Y)$ be a multivalued map such that for each $x \in K$, $C(x)$ is a closed and convex cone with $\operatorname{int} C(x) \neq \emptyset$. F is called

(i) *G-pseudomonotone* if, for all $x, y \in K$,

$$F(x,y) \not\subseteq -\operatorname{int} C(x) \qquad \text{implies} \qquad G(y,x) \not\subseteq \operatorname{int} C(x);$$

(ii) *maximal G-pseudomonotone* if it is *G-pseudomonotone* and for all $x, y \in K$,

$$G(z,x) \not\subseteq \operatorname{int} C(x) \quad \text{for all } z \in]x,y] \quad \text{implies} \quad F(x,y) \not\subseteq -\operatorname{int} C(x),$$

where $]x,y]$ denotes the line segment joining x and y but not containing x.

PROPOSITION 3.1. *If*

$$\Phi \circ \Phi(F(x,y)) = F(x,y), \qquad \textit{for all } x, y \in K,$$

or equivalently,

$$\Phi(-G(y,x)) = F(x,y), \qquad \textit{for all } x, y \in K,$$

then the dual problem of (3.1) *is* (*GVEP*).

Proof. It is straightforward. ∎

We now establish the following results on the relationship between K^p and K_G^d.

PROPOSITION 3.2. *If F is maximal G-pseudomonotone, then $K^p = K_G^d$.*

Proof. By G-pseudomonotonicity of F, we have $K^p \subseteq K_G^d$.

Let $\bar{x} \in K_G^d$; then $G(y, \bar{x}) \not\subseteq \operatorname{int} C(\bar{x})$ for all $y \in K$. For any $y \in K$, $]\bar{x}, y] \subseteq K$. Therefore, $G(z, \bar{x}) \not\subseteq \operatorname{int} C(\bar{x})$ for all $z \in]\bar{x}, y]$. Since F is maximal G-pseudomonotone, $F(\bar{x}, y) \not\subseteq -\operatorname{int} C(\bar{x})$. Hence $\bar{x} \in K^p$. This completes the proof. ∎

PROPOSITION 3.3. *Let $C: K \to \Pi(Y)$ be a multivalued map such that for each $x \in K$, $C(x)$ is a proper, closed, and convex cone with* $\operatorname{int} C(x) \neq \emptyset$. *Let $F, G \in \mathscr{F}(K)$ such that*

(i) *for all $x, y \in K$, $F(y, y) \subseteq C(x)$,*

(ii) *F is explicitly $\delta(C_x)$-quasiconvex and G-pseudomonotone,*

(iii) *for all $x, y \in K$, $F(x, y) \subseteq \operatorname{int} C(x)$ implies $G(x, y) \subseteq \operatorname{int} C(x)$,*

(iv) *for all $y \in K$, the multivalued map $x \mapsto F(x, y)$ is u-hemicontinuous.*

Then $K^p = K_G^d$.

Proof. It is similar to the proof of Lemma 2.1 in [18]. ∎

Remark 3.1. If $F \equiv G$, then Proposition 3.3 reduces to the Corollary 2.1 in [18].

By using the dual formulation (3.1) of (GVEP), we obtain the following existence result for a solution to (GVEP).

THEOREM 3.1. *Let $F, G \in \mathscr{F}(K)$ and let $C: K \to \Pi(Y)$ be a multivalued map such that for each $x \in K$, $C(x)$ is a proper, closed, and convex cone with* $\operatorname{int} C(x) \neq \emptyset$. *Assume that*

(i) *$W: K \to \Pi(Y)$ is a multivalued map defined as $W(x) = Y \setminus \{\operatorname{int} C(x)\}$ for all $x \in K$, such that $\mathscr{G}(W)$ is closed in $K \times Y$;*

(ii) *F is C_x-quasiconvex and maximal G-pseudomonotone;*

(iii) *$F(y, y) \subseteq C(x)$, for all $x, y \in K$;*

(iv) *for each $y \in K$, $G(\cdot, y)$ is upper semicontinuous with compact values on K;*

(v) *there exist a nonempty compact and convex subset D of K and an element $\tilde{y} \in D$ such that*

$$F(z, \tilde{y}) \subseteq -\operatorname{int} C(z), \qquad \text{for all } z \in K \setminus D.$$

Then there exists $\bar{x} \in K^p$.

Proof. For each $y \in K$, we define two multivalued maps $S, T: K \to \Pi(K)$ by

$$S(y) = \{x \in K : G(y, x) \not\subseteq \operatorname{int} C(x)\}$$

and

$$T(y) = \mathrm{cl}\{x \in K : F(x, y) \not\subseteq -\mathrm{int}\, C(x)\}.$$

By condition (iii), for all $y \in K$, $T(y)$ is nonempty. Since F is C_x-quasiconvex, it is easy to see that T is a KKM-map (see, for example, the proof of Theorem 3.1 in [18]). Then T is a KKM-map with closed values and $T(\tilde{y})$ is contained in compact set D by condition (v) and hence $T(\tilde{y})$ is compact. It follows from the Fan–KKM Lemma 2.1 that there exists $\bar{x} \in D$ such that $\bar{x} \in T(y)$ for all $y \in K$. By conditions (i) and (iv), $S(y)$ is closed in K for all $y \in K$ (see, for example, the proof of Theorem 2.1 in [6]). By G-pseudomonotonicity of f, we have $T(y) \subseteq S(y)$ for all $y \in K$. Therefore, we obtain that $\bar{x} \in S(y)$ for all $y \in K$, that is, $G(y, \bar{x}) \not\subseteq \mathrm{int}\, C(\bar{x})$ for all $y \in K$. By Proposition 3.2, $\bar{x} \in D$ is a solution of (GVEP). ∎

THEOREM 3.2. *Let all the conditions of Proposition* 3.3 *and Theorem* 3.1 *hold except condition* (ii) *of Theorem* 3.1. *If F is C_x-quasiconvex, then there exists* $\bar{x} \in K^p$.

Proof. It follows from Proposition 3.3 and Theorem 3.1. ∎

Remark 3.2. Theorem 3.2 generalizes Theorem 3.1 and Theorem 3.2 in [18] for $C(x) = D(x)$ for all $x \in K$, in the following way that X need not be Hausdorff.

4. ADDITIVE DUALITY

In this section, we consider the case where

$$\Phi(F(x, y)) = -G(y, x) = -(F(y, x) + H(y, x)),$$

for some $H \in \mathscr{F}(K)$ and specialize the results of previous section. In other words, we study the following additive dual problem:

$$\text{Find } \bar{x} \in K \text{ such that } F(y, \bar{x}) + H(y, \bar{x}) \not\subseteq \mathrm{int}\, C(\bar{x}), \qquad \text{for all } y \in K. \tag{DGVEP$_H$}$$

In other words, to define the additive dual problem, we add $H(y, x)$ on the left-hand side of the dual of primal problem, that is, (DGVEP). We shall denote by K_H^d the set of all solutions of (DGVEP$_H$).

PROPOSITION 4.1. *If*

$$H(x, y) + H(y, x) \subseteq -\mathrm{int}\, C(x), \qquad \text{for all } x, y \in K,$$

then the additive dual problem of (*DGVEP$_H$*) *is* (*GVEP*).

Proof. Since $G(y,x) = F(y,x) + H(y,x)$, (DGVEP$_H$) can be written as to find $\bar{x} \in K$ such that

$$G(y,\bar{x}) \not\subseteq \operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K. \tag{4.1}$$

By the rule of additive duality, the additive dual of (4.1) is

$$G(\bar{x},y) + H(y,\bar{x}) \not\subseteq -\operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K,$$

and therefore

$$F(\bar{x},y) + H(\bar{x},y) + H(y,\bar{x}) \not\subseteq -\operatorname{int} C(\bar{x}).$$

Since $H(x,y) + H(y,x) \subseteq -\operatorname{int} C(x)$ for all $x, y \in K$, we have

$$F(\bar{x},y) \not\subseteq -\operatorname{int} C(\bar{x}).$$

This completes the proof. ∎

DEFINITION 4.1. Let $F, H \in \mathscr{F}(K)$ and C be the same as in Definition 3.1. The multivalued map F is called

(i) (H)-*pseudomonotone* if, for all $x, y \in K$,

$$F(x,y) \not\subseteq -\operatorname{int} C(x) \qquad \text{implies} \qquad F(y,x) + H(y,x) \not\subseteq \operatorname{int} C(x);$$

(ii) *maximal* (H)-*pseudomonotone* if it is (H)-*pseudomonotone* and for all $x, y \in K$,

$$F(z,x) + H(z,x) \not\subseteq \operatorname{int} C(x) \qquad \text{for all } z \in \,]x,y] \qquad \text{implies} \qquad F(x,y) \not\subseteq -\operatorname{int} C(x).$$

PROPOSITION 4.2. *If F is maximal (H)-pseudomonotone, then $K^p = K_H^d$.*

Proof. It is similar to the proof of Proposition 3.2. ∎

PROPOSITION 4.3. *Let C be the same as in Proposition* 3.3 *and let $F, H \in \mathscr{F}(K)$ such that*

(i) *for all $x, y \in K$, $F(y,y) \subseteq C(x)$,*

(ii) *F is explicitly $\delta(C_x)$-quasiconvex and (H)-pseudomonotone,*

(iii) *for all $x, y \in K$, $F(x,y) \subseteq \operatorname{int} C(x)$ implies $F(x,y) + H(x,y) \subseteq \operatorname{int} C(x)$,*

(iv) *for all $y \in K$, the multivalued map $x \mapsto F(x,y)$ is u-hemicontinuous.*

Then $K^p = K_H^d$.

Proof. It is similar to the proof of Lemma 2.1 in [18]. ▮

Remark 4.1. Proposition 4.3 extends and generalizes Proposition 4.2 in [17].

In view of Theorem 3.1, we have the following existence result for a solution to (GVEP).

THEOREM 4.1. *Let $F, H \in \mathscr{F}(K)$ and let C be the same as in Theorem* 3.1. *Assume that*

(i) *$W: K \to \Pi(Y)$ is a multivalued map defined as $W(x) = Y \setminus \{\operatorname{int} C(x)\}$ for all $x \in K$, such that $\mathscr{G}(W)$ is closed in $K \times Y$;*

(ii) *F is C_x-quasiconvex and maximal (H)-pseudomonotone;*

(iii) *$F(y, y) \subseteq C(x)$, for all $x, y \in K$;*

(iv) *for each $y \in K$, $(F + H)(\cdot, y)$ is upper semicontinuous with compact values on K;*

(v) *there exist a nonempty compact and convex subset D of K and an element $\tilde{y} \in D$ such that*

$$F(z, \tilde{y}) \subseteq -\operatorname{int} C(z), \qquad \text{for all } z \in K \setminus D.$$

Then there exists $\bar{x} \in K^p$.

Proof. It is similar to the proof of Theorem 3.1. ▮

THEOREM 4.2. *Let all the conditions of Proposition* 4.3 *and Theorem* 4.1 *hold except condition* (ii) *of Theorem* 4.1. *If F is C_x-quasiconvex, then there exists $\bar{x} \in K^p$.*

Proof. It follows from Proposition 4.3 and Theorem 4.1. ▮

5. MULTIPLICATIVE DUALITY

In order to define the multiplicative dual problem, we multiply by $H(y, x)$ for some $H \in \mathscr{F}(K)$ on the left-hand side and by $\operatorname{int} C(x)$ on the right-hand side of the dual problem of primal problem, that is, (DGVEP). In other words, we consider the following multiplicative dual problem:

$$\text{Find } \bar{x} \in K \qquad \text{such that} \qquad F(y, \bar{x}) \times H(y, \bar{x}) \nsubseteq \operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}), \quad \text{for all } y \in K. \qquad (\mathrm{DGVEP}_{m(H)})$$

The set of all solutions of $(\mathrm{DGVEP}_{m(H)})$ will be denoted by $K^d_{m(H)}$.

PROPOSITION 5.1. *If*

$$H(x,y) \times H(y,x) \subseteq \operatorname{int} C(x) \times \operatorname{int} C(x), \qquad \text{for all } x, y \in K,$$

then the multiplicative dual problem of $(DGVEP_{m(H)})$ *is* $(GVEP)$.

Proof. Let $G(y,x) = F(y,x) \times H(y,x)$. Then $(\mathrm{DGVEP}_{m(H)})$ can be written as to find $\bar{x} \in K$ such that

$$G(y,\bar{x}) \nsubseteq \operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K. \tag{5.1}$$

By the rule of multiplicative duality, the multiplicative dual of (5.1) is

$$G(\bar{x},y) \times H(y,\bar{x}) \nsubseteq -\operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K,$$

and therefore,

$$F(\bar{x},y) \times H(\bar{x},y) \times H(y,\bar{x}) \nsubseteq -\operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x}).$$

Since $H(x,y) \times H(y,x) \subseteq \operatorname{int} C(\bar{x}) \times \operatorname{int} C(\bar{x})$ for all $x, y \in K$, we have

$$F(\bar{x},y) \nsubseteq -\operatorname{int} C(\bar{x}).$$

This completes the proof. ∎

DEFINITION 5.1. Let $F, H \in \mathscr{F}(K)$ and C be the same as in Definition 3.1. The multivalued map F is called

(i) *$m(H)$-pseudomonotone* if, for all $x, y \in K$,

$$F(x,y) \nsubseteq -\operatorname{int} C(x) \quad \text{implies } F(y,x) \times H(y,x) \nsubseteq \operatorname{int} C(x) \times \operatorname{int} C(x);$$

(ii) *maximal $m(H)$-pseudomonotone* if it is *$m(H)$-pseudomonotone* and for all $x, y \in K$,

$$F(z,x) \times H(z,x) \nsubseteq \operatorname{int} C(x) \times \operatorname{int} C(x) \qquad \text{for all } z \in]x,y] \quad \text{implies } F(x,y) \nsubseteq -\operatorname{int} C(x).$$

PROPOSITION 5.2. *If F is maximal $m(H)$-pseudomonotone, then* $K^p = K^d_{m(H)}$.

Proof. It is similar to the proof of Proposition 3.2. ∎

PROPOSITION 5.3. *Let C be the same as in Proposition* 3.3 *and let* $F, H \in \mathscr{F}(K)$ *such that*

(i) *for all* $x, y \in K$, $F(y,y) \subseteq C(x)$,

(ii) *F is explicitly $\delta(C_x)$-quasiconvex and $m(H)$-pseudomonotone,*

(iii) *for all $x, y \in K$, $F(x, y) \subseteq \operatorname{int} C(x)$ implies $F(x, y) \times H(x, y) \subseteq \operatorname{int} C(x) \times \operatorname{int} C(x)$,*

(iv) *for all $y \in K$, the multivalued map $x \mapsto F(x, y)$ is u-hemicontinuous.*

Then $K^p = K^d_{m(H)}$.

Proof. It is similar to the proof of Lemma 2.1 in [18]. ∎

Now, we obtain the following existence result for a solution to (GVEP).

THEOREM 5.1. *Let $F, H \in \mathscr{F}(K)$ and C be the same as in Theorem 3.1. Assume that*

(i) *$W: K \to \Pi(Y)$ is a multivalued map defined as $W(x) = Y \setminus \{\operatorname{int} C(x)\}$ for all $x \in K$, such that $\mathscr{G}(W)$ is closed in $K \times Y$;*

(ii) *F is C_x-quasiconvex and maximal $m(H)$-pseudomonotone;*

(iii) *$F(y, y) \subseteq C(x)$, for all $x, y \in K$;*

(iv) *for each $y \in K$, $F(\cdot, y)$ and $H(\cdot, y)$ are upper semicontinuous with compact values on K;*

(v) *there exist a nonempty compact and convex subset D of K and an element $\tilde{y} \in D$ such that*

$$F(z, \tilde{y}) \subseteq -\operatorname{int} C(z), \qquad \text{for all } z \in K \setminus D.$$

Then there exists $\bar{x} \in K^p$.

THEOREM 5.2. *Let all the conditions of Proposition 5.3 and Theorem 5.1 hold except condition (ii) of Theorem 5.1. If F is C_x-quasiconvex, then there exists $\bar{x} \in K^p$.*

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