TECHNICAL NOTE

On Quasimonotone Variational Inequalities[1]

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Abstract The purpose of this paper is to prove the existence of solutions of the Stampacchia variational inequality for a quasimonotone multivalued operator without any assumption on the existence of inner points. Moreover the operator is not supposed to be bounded valued. The result strengthens a variety of other results in the literature.

Key Words: Variational inequalities, quasimonotone operators, generalized monotonicity, existence results.

1 Introduction and Definitions

Given a Banach space X with topological dual X^*, a subset K of X and a multivalued operator $T : K \to 2^{X^*}$, the *Stampacchia variational inequality* problem is to find $x \in K$ such that

$$\forall\, y \in K, \exists\, x^* \in T\,(x) : \langle x^*, y - x\rangle \geq 0. \tag{1}$$

Existence of solutions of (1) under a generalized monotonicity assumption for T has been intensively investigated in recent years. In most cases, T was assumed to be pseudomonotone (in the sense of Karamardian), see e.g. Refs. 1-2 . Extension of these results to the broader class of quasimonotone operators has also been established, but only at the cost of restrictive assumptions. For instance, in Ref. 3 , K was assumed to contain “inner points”; in addition, in case T is multivalued, its values were assumed to be compact in the norm topology (Ref. 4); in Ref. 5 , T was assumed to be “densely pseudomonotone” which is more restrictive than quasimonotone, etc.

The purpose of this note is to show existence of solutions of (1) for quasimonotone operators with no additional assumptions apart from those used for pseudomonotone operators (i.e., a kind of continuity along lines and w*-compactness and convexity of the values). In fact, even the latter assumptions will be stated in a very weak form.

We recall that an operator T is called *quasimonotone* (Ref. 6) if for all (x, x^*), (y, y^*) in the graph grT,

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

The operator T is called *properly quasimonotone* (Ref. 7) if for all $x_1, \ldots, x_n \in \operatorname{dom} T$, and all $x \in co\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\forall x^* \in T(x_i) : \langle x^*, x_i - x\rangle \geq 0.$$

Finally, T is called *pseudomonotone* (in the Karamardian sense (Ref. 6)) if for all (x, x^*), $(y, y^*) \in grT$,

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

Pseudomonotone operators are properly quasimonotone, and properly quasimonotone operators are quasimonotone. We denote by $S(T,K)$ the set of solutions of the Stampacchia variational inequality

$$x \in S(T,K) \Longleftrightarrow x \in K \text{ and } \forall\, y \in K,\ \exists\, x^* \in T(x):\ \ \langle x^*, y-x\rangle \geq 0$$

and by $S_{str}(T,K)$ the set of "strong" solutions of the same inequality:

$$x \in S_{str}(T,K) \Longleftrightarrow x \in K \text{ and } \exists\, x^* \in T(x):\ \forall\, y \in K,\ \langle x^*, y-x\rangle \geq 0.$$

Also, we denote by $M(T,K)$ the set of solutions of the *Minty variational inequality:*

$$x \in M(T,K) \Longleftrightarrow x \in K \text{ and } \forall y \in K, \forall y^* \in T\,(y):\ \langle y^*, y-x\rangle \geq 0.$$

Finally, we call $x \in K$ a *local solution* of the Minty variational inequality if there exists a neighborhood U of x such that $x \in M(T, K\cap U)$. We denote by $LM(T,K)$ the set of these local solutions. Clearly, $M\,(T,K) \subseteq LM(T,K)$. In the following lemma we will clarify the relations between those different sets of solutions. Before this, let us recall (Ref. 8) the definition of a very weak kind of continuity: Given a convex subset $K \subseteq X$ and an operator

$T : K \to 2^{X^*}$ with nonempty values, T is called *upper sign-continuous* on K if for any $x, y \in K$, the following implication holds:

$$(\forall\, t \in]0,1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x\rangle \geq 0) \Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0$$

where $x_t = (1-t)x + ty$. If for example T is upper hemicontinuous (i.e., the restriction of T to every line segment of K is usc with respect to the w*-topology in X^*), then T is upper sign-continuous. Any strictly positive real function is upper sign-continuous.

2 Existence Result

It is known that a solution of the Minty variational inequality is also a strong solution of the Stampacchia variational inequality, provided that T is upper hemicontinuous with convex, w*-compact values. Using essentially the same argument, we show that the same is true under weaker assumptions.

Lemma 2.1 Let K be a nonempty convex subset of the Banach space X and $T : K \to 2^{X^*}$ be an operator.

(i) If T is pseudomonotone, then $LM\,(T,K) = M(T,K)$.

(ii) If for every $x \in K$ there exists a convex neighborhood V_x of x and an upper sign-continuous operator $S_x : V_x \cap K \to 2^{X^*}$ with nonempty, w^*-compact values satisfying $S_x(y) \subseteq T(y)$, $\forall\, y \in V_x \cap K$, then $LM(T,K) \subseteq S(T,K)$.

(iii) If additionally to the assumptions of (ii), the operators S_x are convex valued, then $LM(T,K) \subseteq S(T,K) = S_{str}(T,K)$.

Proof :

(i) Let x be an element of $LM(T,K)$. Then there exists a neighborhood U of x such that $x \in M(T, K \cap U)$. For any $y \in K$, there exists $z = x + t(y-x)$, $t \in]0,1[$, such that $z \in K \cap U$. Then for any $z^* \in T(z)$, $\langle z^*, y-z\rangle = \frac{1-t}{t}\langle z^*, z-x\rangle \geq 0$. By pseudomonotonicity, $\langle y^*, y-x\rangle \geq 0$, for all $y^* \in T(y)$. Therefore x is an element of $M(T,K)$.

(ii) Let x be an element of $LM(T,K)$. Thus there exists a neighborhood U of x such that $x \in M(S_x, K \cap V_x \cap U)$. Let $y \in K \cap V_x$. Since $K \cap V_x$ is convex, there exists $\tilde{y} \in]x, y]$ for which $[x, \tilde{y}] \subset (K \cap V_x \cap U)$ and thus

$$\inf_{u \in]x, \tilde{y}]} \inf_{u^* \in S_x(u)} \langle u^*, u - x \rangle \geq 0.$$

By upper sign-continuity of S_x,

$$\sup_{x^* \in S_x(x)} \langle x^*, y - x \rangle \geq 0.$$

But $S_x(x)$ is w^*-compact and we deduce that

$$\inf_{y \in V_x \cap K} \max_{x^* \in S_x(x)} \langle x^*, y - x \rangle \geq 0 \tag{2}$$

which means that for all $y \in V_x \cap K$, there exists $x^* \in S_x(x) \subseteq T(x)$ such that $\langle x^*, y - x \rangle \geq 0$. Therefore x is an element of $S(T, K)$ since, using the convexity of K one can easily prove that the above relation holds for any $y \in K$.

(iii) This is a consequence of the Sion's minimax theorem applied to relation (2). □

If in particular T itself is upper sign-continuous and has nonempty, convex and w*-compact values, then we can take in the lemma $V_x = K$, $S_x = T$.

However, the lemma in its present form (as well as the forthcoming Theorem 2.1) permits application to operators whose values are unbounded, such as cone-valued operators.

We now establish an alternative, valid for every quasimonotone operator:

Proposition 2.1 Let K be a nonempty, convex subset of the Banach space X and $T : K \to 2^{X^*}$ be quasimonotone. Then one of the following assertions holds:

(i) T is properly quasimonotone

(ii) $LM(T, K) \neq \emptyset$.

If in addition K is weakly compact, then $LM\,(T, K) \neq \emptyset$ in both cases.

Proof : Suppose that T is not properly quasimonotone. Then there exist $x_1, \ldots, x_n \in K$, $x_i^* \in T(x_i)$, $i = 1, \ldots, n$ and $x \in \text{co}\{x_1, \ldots, x_n\}$ such that $\langle x_i^*, x - x_i \rangle > 0$, $i = 1, \ldots, n$. By continuity of the functionals x_i^*, there exists a neighborhood U of x such that for any $y \in K \cap U$ one has

$$\langle x_i^*, y - x_i \rangle > 0.$$

By quasimonotonicity, for all $y^* \in T(y)$, $\langle y^*, y - x_i \rangle \geq 0$. Since $x \in \text{co}\{x_1, \ldots, x_n\}$, it follows easily that

$$\forall y^* \in T(y), \langle y^*, y - x \rangle \geq 0. \tag{3}$$

Thus $x \in LM(T, K)$ since the previous inequality holds for every $y \in K \cap U$.

It remains to show that $LM(T, K) \neq \emptyset$ whenever K is weakly compact and T is properly quasimonotone. But under such assumptions, it is known (Ref. 7) that $M(T, K) \neq \emptyset$; since $M(T, K) \subseteq LM(T, K)$, it follows that $LM(T, K) \neq \emptyset$. □

Combination of the lemma with Proposition 2.1 leads to a result of existence of solutions for the Stampacchia variational inequality without any assumption on the existence of inner points.

Theorem 2.1 Let K be a nonempty convex subset of X. Let further $T : K \to 2^{X^*}$ be a quasimonotone operator such that the following coercivity condition holds:

$$\exists\, \rho > 0,\ \forall\, x \in K \setminus \overline{B}(0,\rho),\ \exists\, y \in K \text{ with } \|y\| < \|x\| \\ \text{such that } \forall\, x^* \in T(x), \langle x^*, x - y\rangle \geq 0. \tag{4}$$

Suppose that there exists $\rho' > \rho$ such that $K \cap \overline{B}(0,\rho')$ is nonempty weakly compact. Suppose moreover that for every $x \in K$ there exist a convex neighborhood V_x of x and an upper sign-continuous operator $S_x : V_x \cap K \to 2^{X^*}$ with nonempty, convex, w^*-compact values satisfying $S_x(y) \subseteq T(y)$, $\forall\, y \in V_x \cap K$. Then $S_{str}(T,K) \neq \emptyset$.

Proof : The set $K_{\rho'} := K \cap \overline{B}(0,\rho')$ is nonempty, convex and weakly compact. According to Proposition 2.1, $LM(T, K_{\rho'}) \neq \emptyset$. By Lemma 2.1 the set $S_{str}(T, K_{\rho'})$ is also nonempty. Choose $x_0 \in S_{str}(T, K_{\rho'})$. Then

$$\exists\, x_0^* \in T(x)\ :\ \forall\, y \in K_{\rho'}, \quad \langle x_0^*, y - x_0\rangle \geq 0. \tag{5}$$

According to (4), there exists $y_0 \in B(0,\rho') \cap K$ such that

$$\forall\, x^* \in T(x_0), \quad \langle x^*, x_0 - y_0\rangle \geq 0. \tag{6}$$

(If $\|x_0\| < \rho'$ we can take $y_0 = x_0$). From (5) and (6) it follows that

$$\langle x_0^*, y_0 - x_0 \rangle = 0. \tag{7}$$

Now for every $y \in K$ there exists $t \in [0, 1[$ such that $(1-t)y + ty_0 \in K_{\rho'}$; hence,

$$\langle x_0^*, (1-t)y + ty_0 - x_0 \rangle \geq 0. \tag{8}$$

If follows immediately from (7) and (8) that $\langle x_0^*, y - x_0 \rangle \geq 0$, i.e. $x_0 \in S_{str}(T, K)$. □

Note that in Theorem 2.1 the condition on the compactness of $K \cap \overline{B}(0, \rho')$ is automatically satisfied if K is weakly compact, or X is reflexive and K is closed; the coercivity condition is also automatically satisfied if K is bounded. Finally, the condition on the existence of S_x is satisfied if T itself is upper sign-continuous with nonempty, convex, w*-compact values. Thus Theorem 2.1 generalizes corresponding results for pseudomonotone operators (Ref. 1), quasimonotone operators where K is assumed to contain "inner points" (Ref. 3), densely pseudomonotone operators (Ref. 5) etc.

Finally, let us compare the results of this paper with Theorem 5.1 of Ref. 7 ; there, it is established (using no continuity assumption) that for every properly quasimonotone operator T defined on a weakly compact convex subset K, $M(T,K) \neq \emptyset$ holds. Starting from this, one usually deduces that $S(T,K) \neq \emptyset$ by adjoining some suitable assumptions (for instance that T is upper hemicontinuous with convex, w*- compact values). If the operator T is quasimonotone, but not properly quasimonotone, then $M(T,K)$ may be empty. However, according to Proposition 2.1, $LM\,(T,K) \neq \emptyset$. This last property is again sufficient for proving that $S(T,K) \neq \emptyset$ under the same (or even weaker) additional assumptions, as shown by Theorem 2.1.

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