SINGLE-DIRECTIONAL PROPERTY OF MULTIVALUED MAPS AND VARIATIONAL SYSTEMS *

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Abstract. In [8], Dontchev and Hager have shown that a monotone set-valued map defined from a Banach space to its dual which satisfies the Aubin property around a point (x, y) of its graph is actually single-valued in a neighbourhood of x. We prove a result which is the counterpart of the above for quasimonotone set-valued maps, based on the concept of single-directional property. As applications, we provide sufficient conditions for this single-valued property to hold for the solution map of general variational systems and quasivariational inequalities. We also investigate the single-directionality property for the normal operator to the sublevel sets of a quasiconvex function.

Key words. Aubin property, Lipschitz-like property, Single-directional property, Metric regularity, Quasimonotone map, Normal operator, Parametric variational systems.

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1. Introduction.

Let Y be a Banach space and $T : Y \to 2^{Y^*}$ a set-valued map. It is an old result in the theory of monotone maps that if T is lower semicontinuous at an interior point y of its domain, then it is single-valued at y [14]. A relatively more recent result, due to Dontchev and Hager [8], states that a monotone set-valued map satisfying the Aubin property around a point (y, y^*) of its graph is actually single-valued in a neighbourhood of y.

When the operator T is not monotone but rather generalized monotone (quasimonotone or pseudomonotone in the Karamardian sense) one cannot hope to obtain results implying single-valuedness from assumptions such as the Aubin property; this will be shown by an example (Remark 3.3). Instead, we show in Section 3 that the Aubin property around a point (y, y^*) of the graph of T with $y^* \neq 0$ implies that T is locally single-directional, i.e, for x near y, $T(x)$ is included in a half-line. The assumption $y^* \neq 0$ is shown to be necessary in general, but is not needed if T is convex valued. In Section 4 we study the single-directional property of a particular kind of map: if $g : Y \to \mathbb{R} \cup \{+\infty\}$ is a quasiconvex function, we are interested in the normal operator whose image at each point is the normal cone to the (adjusted) sublevel set of g corresponding to the value $g(x)$. This operator has very nice properties and is a good candidate to replace the Fenchel subdifferential in several applications when the underlying function is quasiconvex rather than convex, as was shown in [4, 5, 6]. In the special case where the sublevel set is a polyhedron, we give necessary and sufficient conditions for the normal operator to be single-directional (Prop. 4.2) and show the failure of single-directionality in some cases (Prop. 4.4).

The particular case where the set-valued map T is involved in a general variational system is finally considered in Section 5. This important case covers for example the solution set of quasivariational inequalities and of parametric complementarity

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problems. More precisely, let $T : Y \to 2^{Y^*}$ be a quasimonotone and convex-valued set-valued map and f a strictly differentiable function defined from $X \times Y$ to Y^*. We show that if the set-valued map $R : X \to 2^Y$ defined by

$$R(x) = \{y \in Y : 0 \in f(x, y) + T(y)\}.$$

is metrically regular around $(\bar{x}, \bar{y})$, then T is locally single-directional at $(\bar{y})$, provided that f satisfies the ample parametrization condition: $\nabla_x f(\bar{x}, \bar{y})$ is surjective. This is a counterpart, for quasimonotone operators, of a similar result proved very recently in [16] for monotone operators.

2. Definitions and basic properties. In the sequel $Y \neq \{0\}$ will denote a real Banach space, Y^* its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. For $y \in Y$ and $\rho > 0$, we denote by $B(y, \rho)$ and $\overline{B}(y, \rho)$ respectively the open and the closed ball of center y and radius ρ, while for $y, y' \in Y$ we denote by $[y, y']$ the closed segment $\{ty + (1-t)y' : t \in [0, 1]\}$. The segments $]y, y'[$, $]y, y']$, $[y, y'[$ are defined analogously. For any element y^* of X^* we set $\mathbb{R}_+\{y^*\} = \{ty^* \in X^* \ : \ t \geq 0\}$ and $\mathbb{R}_{++}\{y^*\} = \{ty^* \in X^* \ : \ t > 0\}$.

The topological closure, the interior, the boundary and the convex hull of a set $A \subset Y$ will be denoted respectively by $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$ and $\mathrm{conv}(A)$. Given any nonempty subset A of Y and a point $y \in Y$, the distance from y to A will be denoted by $dist(y, A) = \inf\{\|y - y'\| : y' \in A\}$ and $N(A, y)$ stands for the polar cone to A at y, that is,

$$N(A, y) = \big\{y^* \in Y^* \ : \ \langle y^*, u - y\rangle \leq 0, \, \forall\, u \in A\big\}.$$

Given a Banach space $X \neq \{0\}$, the domain and the graph of a set-valued operator $T : Y \to 2^X$ will be denoted, respectively, by $\mathrm{Dom}\, T$ and $\mathrm{Gr}\, T$ while the inverse image of T at x will be $T^{-1}(x) = \{y \in Y \ : \ x \in T(y)\}$.

Recall that a set-valued operator $T : Y \to 2^X$ is said to satisfy the *Aubin property* (also called *Lipschitz-like property*) around $(y, x) \in \mathrm{Gr}\, T$ if there exist neighbourhoods U of y, V of x and a positive real number l such that

$$T(u) \cap V \subset T(u') + l\|u' - u\|\overline{B}_X(0, 1), \quad \forall\, u, u' \in U$$

where $\overline{B}_X(0, 1)$ denotes the closed unit ball of X. Let us observe that, as a direct consequence of the above definition, if T satisfies the Aubin property around a point (y, x) of its graph then T is nonempty-valued in U.

The operator T is said to be *metrically regular around* some point (y, x) of $\mathrm{Gr}\, T$ if there exist neighbourhoods U of y, V of x and a positive real number μ such that

$$d(y', T^{-1}(x')) \leq \mu d(x', T(y')), \quad \forall\, y' \in U \text{ and } x' \in V.$$

The metric regularity of T is known (see Theorem 1.49 in [15]) to be equivalent to the Aubin property of the inverse map T^{-1}.

In the sequel we shall deal with *proper* functions $g : Y \to \mathbb{R} \cup \{+\infty\}$ (i.e. functions for which $\mathrm{dom}\, g = \{y : g(y) < +\infty\}$ is nonempty), and we will consider some generalized convexity assumptions over them. So let us recall that a function $g : Y \to \mathbb{R} \cup \{+\infty\}$ is said to be:

- *quasiconvex* on a subset $C \subset \mathrm{dom}\, g$ if, for any $y, y' \in C$ and any $t \in [0, 1]$,

$$g(ty + (1-t)y') \leq \max\big\{g(y), g(y')\big\},$$

- *semistrictly quasiconvex* on a subset $C \subset \operatorname{dom} g$ if, g is quasiconvex and for any $y, y' \in K$,

$$g(y) < g(y') \Rightarrow g(z) < g(y'), \quad \forall\, z \in [y, y'[.$$

Let us denote, for any $\alpha \in \mathbb{R}$, by $S_\alpha(g)$ and $S_\alpha^<(g)$ the sublevel set and the strict sublevel set associated to g and α :

$$S_\alpha(g) = \{y \in Y \ : \ g(y) \leq \alpha\} \quad \text{ and } \quad S_\alpha^<(g) = \{y \in Y \ : \ g(y) < \alpha\}.$$

Whenever no confusion can occur we will use, for any $y \in \operatorname{dom} g$, the simplified notation $S_{g(y)}$ and $S_{g(y)}^<$ instead of $S_{g(y)}(g)$ and $S_{g(y)}^<(g)$. It is well known that the quasiconvexity of a function g is characterized by the convexity of the sublevel sets (or the convexity of the strict sublevel sets). Analogously, it is easy to check that any lower semicontinuous function g, semistrictly quasiconvex on its domain $\operatorname{dom} g$ satisfies the following property:

$$\forall\, \alpha > \inf_X g, \qquad \operatorname{cl}\big(S_\alpha^<(g)\big) = S_\alpha(g). \tag{2.1}$$

Roughly speaking, this means that a lower semicontinuous semistrictly quasiconvex function g does not have any “flat part” with nonempty interior on $\operatorname{dom} g \setminus \operatorname{argmin}_Y g$.

As shown in [4, 5], an efficient tool to study the properties of quasiconvex functions is the so-called *normal operator* N_g^a defined on $\operatorname{dom} g$ as the normal cone to the adjusted sublevel sets S_g^a, that is:

$$N_g^a(x) = \big\{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\, \forall y \in S_g^a(x)\big\}$$

where $S_g^a(x) = S_{g(x)} \cap \overline{B}\big(S_{g(x)}^<, \rho_x\big)$ $\big($with $\rho_x = dist(x, S_{g(x)}^<)\big)$ if $x \notin \operatorname{argmin} g$, and $S_g^a(x) = S_{g(x)}$ otherwise. Many precious properties of the operator have been proved for quasiconvex functions (see [4], [5]). Let us notice that, since $S_{g(y)}^< \subset S_g^a(y) \subset S_{g(y)}$, one has

$$N(S_{g(y)}, y) \subset N_g^a(y) \subset N(S_{g(y)}^<, y), \quad \forall y \in \operatorname{dom} g. \tag{2.2}$$

Note that in the case of a semistrictly quasiconvex function f, the three cones coincide.

3. Single-directional property of multivalued maps. Let us recall that a set-valued operator $T : Y \to 2^{Y^*}$ is said to be

- *quasimonotone* on $K \subset Y$ if, for all $x, y \in K$,

$$\exists\, x^* \in T(x) \ : \ \langle x^*, y - x\rangle > 0 \Rightarrow \forall\ y^* \in T(y) \ : \ \langle y^*, y - x\rangle \geq 0.$$

- *pseudomonotone* on $K \subset Y$ if, for all $x, y \in K$,

$$\exists\, x^* \in T(x) \ : \ \langle x^*, y - x\rangle > 0 \Rightarrow \forall\ y^* \in T(y) \ : \ \langle y^*, y - x\rangle > 0.$$

Definition 3.1. *A set-valued operator $T : Y \to 2^{Y^*}$ is said to be*

- single-directional *at $y \in \operatorname{Dom} T$ if, $T(y) \subseteq \mathbb{R}_+\{y^*\}$ for some $y^* \in T(y)$.*
- locally single-directional *at $y \in \operatorname{Dom} T$ if there exists a neighbourhood U of y such that for all $y' \in U$, $T(y')$ is single-directional.*

Finally, T is said to be strictly single-directional *(respectively* locally strictly single-directional*) at y if $T(y) \subseteq \mathbb{R}_{++}\{y^*\}$ for some $y^* \neq 0$ (respectively T is strictly single-directional at any point of some neighbourhood of y).*

Dontchev and Hager showed in [8, Proposition 5.1], that a monotone set-valued map which satisfies the Aubin property around a point (x, y) of its graph is actually single-valued in a neighbourhood of x. The following proposition is the counterpart of this result in the non-monotone case.

PROPOSITION 3.2. *Let $T : Y \to 2^{Y^*}$ be a set-valued map satisfying the Aubin property around a point (y, y^*) of its graph. Then*

i) If $y^* \neq 0$ and T is quasimonotone, then T is locally single-directional at y.

ii) If $y^* \neq 0$ and T is pseudomonotone, then T is locally strictly single-directional at y.

Proof. *i)* Let $\varepsilon_1, \varepsilon_2 > 0$ be such that $B(y^*, \varepsilon_1) \subseteq V$ and $B(y, \varepsilon_2) \subseteq U$. Choose $\varepsilon < \min\{\varepsilon_2, \varepsilon_1/l, \|y^*\|/l\}$. For any $x \in B(y, \varepsilon)$ one has

$$y^* \in T(y) \cap V \subseteq T(x) + l\,\|x - y\|\,\overline{B}_{Y^*}(0, 1)$$

thus there exists $x^* \in T(x)$ such that $x^* \in B(y^*, l\varepsilon) \subseteq V$. In addition, $\|x^*\| \geq \|y^*\| - l\varepsilon > 0$, hence $x^* \neq 0$.

Now assume that $T(x) \not\subseteq \mathbb{R}_+\{x^*\}$, i.e., there exists $z^* \in T(x)$ such that $z^* \notin \mathbb{R}_+\{x^*\}$. Then we can find $u \in B_Y(0, 1)$ with $\langle z^*, u\rangle > 0$, $\langle x^*, u\rangle < 0$ (see Lemma 3.3 in [11]). Choose $t \in \,]0, -\langle x^*, u\rangle/l[$ such that $x_t := x + tu \in U$. Then

$$x^* \in T(x) \cap V \subseteq T(x_t) + lt\overline{B}_{Y^*}(0, 1).$$

Thus, there exists $x_t^* \in T(x_t)$ such that $\|x_t^* - x^*\| \leq lt < -\langle x^*, u\rangle$. Thus,

$$\langle x_t^*, u\rangle = \langle x^*, u\rangle + \langle x_t^* - x^*, u\rangle \leq \langle x^*, u\rangle + \|x_t^* - x^*\| < 0.$$

It follows that

$$\langle x_t^*, x_t - x\rangle = t\,\langle x_t^*, u\rangle < 0 \tag{3.1}$$

whereas

$$\langle z^*, x_t - x\rangle = t\,\langle z^*, u\rangle > 0$$

contradicting quasimonotonicity of T.

ii) Arguing as in *i)*, one can find $x^* \in T(x) \cap V$ such that $x^* \neq 0$. We choose $u \in B_Y(0, 1)$ such that $\langle x^*, u\rangle < 0$. Considering again $t \in \,]0, -\langle x^*, u\rangle/l[$ such that $x_t := x + tu \in U$, we get $\langle x_t^*, x - x_t\rangle > 0$ for some $x_t^* \in T(x_t)$. The proof is complete since, by pseudomonotonicity of T, $\langle u^*, x - x_t\rangle > 0$ for any $u^* \in T(x)$, and therefore $0 \notin T(x)$. □

REMARK 3.3. a) *Actually, in the quasimonotone case the proof of Proposition 3.2 shows something more: For x close to y, $T(x) \subseteq \mathbb{R}_+\{x^*\}$ for some $x^* \neq 0$, $x^* \in T(x)$.*

b) *Assuming only quasimonotonicity (or even pseudomonotonicity) of the map T, there is no hope to obtain a general result with single-valuedness. Indeed, the set-valued map $T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $T(x) = \mathbb{R}_{++}$ is pseudomonotone and satisfies the Aubin property (U, V any neighbourhoods, l any nonnegative number), the image does not contain 0 but T is not single-valued.*

c) The following example shows that i) and ii) of Proposition 3.2 can fail if $y^* = 0$. Set $X = \mathbb{R}^2$ and

$$T(x) = \begin{cases} \{x\} & \text{if } x \neq 0 \\ \{0\} \cup \{y : \|y\| = 1\} & \text{if } x = 0. \end{cases}$$

Then T is pseudomonotone. Choosing $U = V = B(0, 1/2)$ and $l = 1$ we see that T satisfies the Aubin property around $(0, 0)$. But it is not single-directional at 0. Note also that T is not lower semi-continuous (nor even lower sign-continuous, see [11]) at 0.

Whenever the map T is convex-valued, then the assumption $y^* \neq 0$ can be dropped. An operator $T : Y \to 2^{Y^*}$ will be said to be *trivial* at y if $T(y) = \{0\}$.

Proposition 3.4. *Let $T : Y \to 2^{Y^*}$ be a quasimonotone set-valued map with convex values and satisfying the Aubin property around a point (y, y^*) of its graph. Then T is locally single-directional at y.*

If moreover T is pseudomonotone, then there exists a neighbourhood U of y such that for each $u \in U$, T is either strictly single-directional or trivial at u.

Proof. As in the proof of *i)* of Prop. 3.2, let $\varepsilon_1, \varepsilon_2 > 0$ be such that $B(y^*, \varepsilon_1) \subseteq V$ and $B(y, \varepsilon_2) \subseteq U$. Choose $\varepsilon \in]\,0, \min\{\varepsilon_2, \varepsilon_1/l\}\,[$. For any $x \in B(y, \varepsilon)$ one has $x \in U$ and

$$y^* \in T(y) \cap V \subseteq T(x) + l\,\|x - y\|\,\overline{B}_{Y^*}(0, 1).$$

Thus there exists $x^* \in T(x)$ such that $x^* \in B(y^*, l\varepsilon) \subseteq V$. If $x^* \neq 0$, according to part *i)* of Prop. 3.2, T is single-directional at x. On the other hand, if $T(x) = \{x^*\}$, T is also trivially single-directional at x. Finally, assume that $x^* = 0$ and that $T(x)$ contains at least one non zero element z^*. Then thanks to the convexity of $T(x)$, there exists $w^* \in [T(x) \setminus \{0\}] \cap V$. Obviously, T satisfies the Aubin property around (x, w^*); since $w^* \neq 0$, again by part *i)* of Prop. 3.2, T is single-directional at x. This finishes the proof in case T is quasimonotone. The pseudomonotone case is similar. □

4. Single-directional property of the normal operator. In this section, we concentrate our attention on a particular multivalued map, namely the normal operator N_g^a associated to a quasiconvex function g, and we give sufficient conditions for this operator to be single-directional. We restrict ourselves to the case where Y is a reflexive Banach space, equipped with a norm such that both Y and Y^* are strictly convex. The strict convexity of Y^* implies that the so-called duality map $J : Y \to 2^{Y^*}$ which is defined by

$$J(y) = \{y^* \in Y^* : \langle y^*, y\rangle = \|y\|^2 = \|y^*\|^2\},$$

is singled-valued.

Let us fix some notation. If $g : Y \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous quasiconvex function and y an element of $\operatorname{dom} g \setminus \arg\min g$, we will denote by $\pi(y)$ the projection of y on the nonempty closed convex $\operatorname{cl}(S^<_{g(y)})$. Since Y is reflexive and strictly convex, this projection exists and is unique. In the sequel we will assume that the sublevel set $S_{g(y)}$ (and/or the closure of the strict sublevel set $\operatorname{cl}(S^<_{g(y)})$) is a polyhedron that is a finite intersection of distinct halfspaces

$$H^-(a_i, b_i) = \{y \in Y \;:\; \langle a_i, y\rangle \leq b_i\}$$

for i in a finite family $I(S_{g(y)})$. The associated hyperplanes will be denoted by $H(a_i, b_i)$.

For any $y \in \operatorname{dom} g$, the -possibly empty- subset $I(y)$ of $I(S_{g(y)})$ stands, roughly speaking, for the set of indices of hyperplanes touching y, that is

$$I(y) = \big\{ i \in I(S_{g(y)}) \ : \ y \in H(a_i, b_i) \big\}. \tag{4.1}$$

Similarly, if $\mathrm{cl}(S^<_{g(y)})$ is a polyhedron (say $\mathrm{cl}(S^<_{g(y)}) = \bigcap_{i \in I(\mathrm{cl}(S^<_{g(y)}))} H^-(a'_i, b'_i)\big)$, then

$$I^<(y) = \big\{ i \in I(\mathrm{cl}(S^<_{g(y)})) \ : \ \pi(y) \in H(a'_i, b'_i) \big\}.$$

Note that for any $y \in \operatorname{dom} g \setminus \arg\min g$, the index set $I^<(y)$ is nonempty.

Let us observe that a very simple example for which the sublevel sets are polyhedra is the case of polyhedral quasiconvex functions.

We will need the following elementary lemma, whose proof is given for the sake of completeness.

Lemma 4.1. *Let $C \subset Y$ be a nonempty convex set and let $y \notin \mathrm{cl}(C)$. If $\pi(y)$ is the projection of y on $\mathrm{cl}(C)$, $\rho_y = \|y - \pi(y)\|$ and*

$$H^-_y = \big\{ u \in Y : \big\langle J(y - \pi(y)), u - y \big\rangle \leq 0 \big\},$$

then

a) $N\big(\overline{B}(C, \rho_y), y\big) = \mathbb{R}^+ \big\{ J(y - \pi(y)) \big\}$;

b) $\overline{B}(C, \rho_y) \subset H^-_y$;

c) For all $x \in C$, $\langle J(y - \pi(y)), x - \pi(y) \rangle \leq 0$.

Proof. Let us check first that for each $y, a \in Y$ one has

$$N\big(\overline{B}(a, \|y - a\|), y\big) = \mathbb{R}_+ J(y - a). \tag{4.2}$$

Indeed,

$$\begin{aligned} y^* \in N\big(\overline{B}(a, \|y - a\|), y\big) &\Leftrightarrow \forall u \in \overline{B}(0, 1), \quad \langle y^*, \, a + \|y - a\| \, u - y \rangle \leq 0 \\ &\Leftrightarrow \langle y^*, y - a \rangle \geq \sup_{u \in \overline{B}(0,1)} \langle y^*, u \rangle \, \|y - a\| = \|y^*\| \, \|y - a\| \\ &\Leftrightarrow y^* \in \mathbb{R}^+ J(y - a). \end{aligned}$$

Using relation (4.2) and the inclusion $\overline{B}\left(\pi(y), \rho_y\right) \subset \overline{B}\left(C, \rho_y\right)$ we deduce:

$$N\left(\overline{B}\left(C, \rho_y\right), y\right) \subset N\left(\overline{B}\left(\pi(y), \rho_y\right), y\right) = \mathbb{R}^+ J(y - \pi(y)).$$

Since Y^* is strictly convex, J is single-valued; thus, *a)* follows. Assertion *b)* is an immediate consequence of *a)*. Finally, let us note that for all $x \in C$, $x + y - \pi(y) \in \overline{B}(C, \rho_y)$. Thus *b*) implies

$$\langle J(y - \pi(y)), (x + y - \pi(y)) - y \rangle \leq 0$$

i.e., assertion *c)*. □

Recall that at any point $y \in \operatorname{dom} g \setminus \arg\min g$, $N^a_g(y)$ is the polar cone at y of the adjusted sublevel set $S^a_g(y) = S_{g(y)} \cap \overline{B}\left(S^<_{g(y)}, \rho_y\right)$ with $\rho_y = dist(y, S^<_{g(y)}) = \|y - \pi(y)\|$.

Proposition 4.2. *Let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function and y be an element of* $\operatorname{dom} g$ *such that $y \notin \mathrm{cl}(S^<_{g(y)})$ and $S_{g(y)}$ is a polyhedron with $I(y)$ being a singleton (say $I(y) = \{i_0\}$). Then the following assertions are equivalent:*

i) N_g^a *is single-directional at* y*;*
ii) $N_g^a(y) = \mathbb{R}^+\big\{J(y-\pi(y))\big\}$*;*
iii) $J(y-\pi(y)) \perp H(a_{i_0}, b_{i_0})$*;*
iv) there exists a neighbourhood V *of* y *such that*

$$\Big[\overline{B}\big(S^{<}_{g(y)}, \rho_y\big) \cap V\Big] \subset \big[S_{g(y)} \cap V\big] .$$

Proof. *i)* $\Rightarrow$ *ii)* Since $S_g^a(y) \subset \overline{B}\big(S^{<}_{g(y)}, \rho_y\big)$, we have $N\big(\overline{B}(S^{<}_{g(y)}, \rho_y), y\big) \subset N_g^a(y)$ and therefore, according to Lemma 4.1 *a)*,

$$0 \neq J(y-\pi(y)) \in N_g^a(y).$$

Assertion *ii)* follows from the fact that N_g^a is single-directional at y.
ii) $\Rightarrow$ *iii)* Since $S^a_{g(y)} \subset S_{g(y)} \subset H^-(a_{i_0}, b_{i_0})$, it follows that

$$a_{i_0} \in N(H^-(a_{i_0}, b_{i_0}), y) \subset N(S_{g(y)}, y) \subset N_g^a(y).$$

Thus, from *ii)* it exists $\lambda > 0$ such that $J\left(y-\pi(y)\right) = \lambda a_{i_0}$, hence $J\left(y-\pi(y)\right) \perp H(a_{i_0}, b_{i_0})$.
iii) $\Rightarrow$ *iv)* Since $I(y) = \{i_0\}$ there exists $\varepsilon > 0$ such that $B(y,\varepsilon) \cap H(a_i, b_i) = \emptyset$ for all $i \neq i_0$. From $y \in H^-(a_i, b_i)$ it follows that $B(y, \varepsilon) \subseteq H^-(a_i, b_i)$ for $i \neq i_0$. Hence,

$$\begin{aligned} S_{g(y)} \bigcap B(y,\varepsilon) &= \textstyle\bigcap_i \left[H^-(a_i, b_i) \cap B(y,\varepsilon)\right] \\ &= H^-(a_{i_0}, b_{i_0}) \textstyle\bigcap \left[\cap_{i\neq i_0} H^-(a_i, b_i) \cap B(y,\varepsilon)\right] \\ &= H^-(a_{i_0}, b_{i_0}) \textstyle\bigcap B(y,\varepsilon). \end{aligned} \tag{4.3}$$

On the other hand, by lemma 4.1 *b)*,

$$\overline{B}(S^{<}_{g(y)}, \rho_y) \subset H^-_y = H^-\big(J(y-\pi(y)), \langle J(y-\pi(y)), y\rangle\big)$$

But from *iii)* it is clear that $H^-_y = H^-(a_{i_0}, b_{i_0})$ and thus, combining with (4.3), we obtain for $V = B(y,\varepsilon)$

$$\Big[\overline{B}(S^{<}_{g(y)}, \rho_y) \cap V\Big] \subset \big[H^-(a_{i_0}, b_{i_0}) \cap V\big] = S_{g(y)} \cap V.$$

iv) $\Rightarrow$ *i)* From the definition of $S_g^a(y)$ and assumption *iv)* we infer

$$S_g^a(y) \cap V = \overline{B}(S^{<}_{g(y)}, \rho_y) \cap S_{g(y)} \cap V = \overline{B}(S^{<}_{g(y)}, \rho_y) \cap V. \tag{4.4}$$

Whenever $C \subseteq Y$ is convex, $x \in C$ and V is a neighbourhood of x, then $N(C\cap V, x) = N(C, x)$. We calculate the normal cone $N_g^a(y)$ by using, successively, this remark, equation (4.4) and Lemma 4.1 *a)*:

$$\begin{aligned} N_g^a(y) &= N(S_g^a(y), y) = N(S_g^a(y) \cap V, y) \\ &= N(\overline{B}(S^{<}_{g(y)}, \rho_y) \cap V, y) \\ &= N(\overline{B}(S^{<}_{g(y)}, \rho_y), y) = \mathbb{R}^+\{J(y-\pi(y))\}. \end{aligned}$$

Thus *i)* follows. $\square$
Remark 4.3. *Let us observe that*

$$I(y) = \emptyset \Longleftrightarrow y \in \operatorname{int}(S_{g(y)}).$$

In this case, if $y \in \mathrm{cl}(S^<_{g(y)})$ *then* $N^a_g(y) = N(S^<_{g(y)}, y)$ *and* N^a_g *is single-directional at* y *if and only if* $\mathrm{card}(I^<(y)) = 1$.

Finally, if $I(y) = \emptyset$ *and* $y \in \mathrm{int}(S_{g(y)}) \backslash \mathrm{cl}(S^<_{g(y)})$ *then it holds* $N^a_g(y) = N(\overline{B}(S^<_{g(y)}, \rho_y), y)$ *and* N^a_g *is single-directional at* y.

PROPOSITION 4.4. *Let* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous quasiconvex function and let* $y \in \mathrm{dom}\, g$ *be such that* $y \notin \mathrm{cl}(S^<_{g(y)})$. *If* $S_{g(y)}$ *and* $\mathrm{cl}(S^<_{g(y)})$ *are polyhedra and* $\mathrm{card}\big(I(y)\big) \neq \mathrm{card}\big(I^<(y)\big)$, *then* N^a_g *is not locally single-directional at* y, *that is, for any neighbourhood* V *of* y,

$$\exists\, z \in V \text{ such that } N^a_g(z) \text{ is not single-directional.}$$

Proof. Without loss of generality we can assume that $\mathrm{card}\big(I(y)\big) = 1$. Indeed, if $\mathrm{card}(I(y)) > 1$ then the normal cone $N(S_{g(y)}, y)$ is not single-directional and therefore, thanks to (2.2), this is also true for $N^a_g(y)$.

So let us now assume that $I(y) = \{i_0\}$. If $J(y - \pi(y)) \not\perp H(a_{i_0}, b_{i_0})$ then the conclusion follows directly from Proposition 4.2.

So suppose now that $J(y - \pi(y)) \perp H(a_{i_0}, b_{i_0})$. Since $I(y) = \{i_0\}$ there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) \cap \mathrm{bd}\big(S_{g(y)}\big) = B(y, \varepsilon) \cap H(a_{i_0}, b_{i_0}). \tag{4.5}$$

and thus, for any $z \in B(y, \varepsilon) \cap \mathrm{bd}\big(S_{g(y)}\big)$, $I(y) = I(z)$. If ε is small enough, then we know that $z \notin \mathrm{cl}(S^<_{g(y)})$, hence $g(z) = g(y)$ and $\mathrm{cl}(S^<_{g(y)}) = \mathrm{cl}(S^<_{g(z)})$.

Let $0 < \varepsilon' \leq \varepsilon$. Suppose that for all $z \in B(y, \varepsilon') \cap \mathrm{bd}\big(S_{g(y)}\big)$, $N^a_g(z)$ is single-directional. According to Proposition 4.2 and since $I(y) = I(z) = \{i_0\}$, we have

$$J(z - \pi(z)) \perp H(a_{i_0}, b_{i_0}), \quad \forall\, z \in B(y, \varepsilon') \cap \mathrm{bd}\big(S_{g(y)}\big).$$

In particular $J(z - \pi(z))$ and $J(y - \pi(y))$ are positive multiples of a_{i_0}, so $J\big(z - \pi(z)\big) = \lambda J\big(y - \pi(y)\big)$ for some $\lambda > 0$. Thus,

$$z - \pi(z) = \lambda\big(y - \pi(y)\big), \tag{4.6}$$

because J is bijective and positively homogeneous [13].

Since y and z are elements of $H(a_{i_0}, b_{i_0})$, it follows that

$$\big\langle J\big(y - \pi(y)\big), y - z\big\rangle = 0. \tag{4.7}$$

Since $\pi(z) \in \mathrm{cl}\big(S^<_{g(y)}\big)$, Lemma 4.1 *c*) implies

$$\langle J(y - \pi(y)), \pi(z) - \pi(y)\rangle \leq 0. \tag{4.8}$$

Likewise,

$$\langle J(z - \pi(z)), \pi(y) - \pi(z)\rangle \leq 0. \tag{4.9}$$

Combining (4.6), (4.8) and (4.9) we deduce

$$\big\langle J\big(y - \pi(y)\big), \pi(y) - \pi(z)\big\rangle = 0$$

which, together with (4.7), implies

$$\left\langle J\big(y-\pi(y)\big), y-\pi(y)\right\rangle = \left\langle J\big(y-\pi(y)\big), z-\pi(z)\right\rangle.$$

Therefore, in (4.6) we obtain $\lambda = 1$, so

$$y-\pi(y) = z-\pi(z), \quad \forall\, z \in B(y,\varepsilon') \cap H(a_{i_0}, b_{i_0}). \tag{4.10}$$

Thus for each $z \in B(y,\varepsilon') \cap H(a_{i_0}, b_{i_0})$, $z + (\pi(y) - y) = \pi(z) \in \operatorname{bd}\big(\operatorname{cl}(S^{<}_{g(y)})\big)$. It follows that the translation $\pi(y)-y+B(y,\varepsilon')\cap H(a_{i_0}, b_{i_0})$ is included in $\operatorname{bd}\big(\operatorname{cl}(S^{<}_{g(y)})\big)$. Obviously $\pi(y) - y + B(y,\varepsilon') = B(\pi(y),\varepsilon')$ and $\pi(y) - y + H(a_{i_0}, b_{i_0}) = H(a_{i_0}, c_{i_0})$ where $c_{i_0} = b_{i_0} + \langle a_{i_0}, \pi(y) - y\rangle$. Hence

$$[B(\pi(y),\varepsilon') \cap H(a_{i_0}, c_{i_0})] \subset \operatorname{bd}\big(\operatorname{cl}(S^{<}_{g(y)})\big)$$

and thus $I^{<}(i_0) = \{i_0\}$ which contradicts $\operatorname{card}\big(I^{<}(y)\big) \neq \operatorname{card}\big(I(y)\big) = 1$. □

5. Links with the metric regularity of solution maps. In this section we will consider the solution map of the following general variational system, i.e. the set-valued map $R : X \to 2^Y$ defined by

$$R(x) = \{y \in Y \ : \ 0 \in f(x,y) + T(y)\} \tag{5.1}$$

where X, Y are Banach spaces, $f : X \times Y \to Y^*$ is a differentiable function and $T : Y \to 2^{Y^*}$ is a set-valued map. We will enlighten some links between the metric regularity of the solution R and the single-directional property of the operator T defining the variational system.

5.1. Solution map of general variational systems. In the following Theorem 5.1, we provide some sufficient conditions under which the metric regularity of the solution map R will imply that the operator T is locally single-directional.

THEOREM 5.1. *Let us suppose that $f : X \times Y \to Y^*$ is strictly differentiable at $(\bar{x}, \bar{y}) \in \operatorname{Gr} R$ and satisfies the ample parametrization condition: $\nabla_x f(\bar{x}, \bar{y})$ is surjective. If the set-valued map $T : Y \to 2^{Y^*}$ satisfies the following hypothesis*

i) T is quasimonotone;

ii) $-f(\bar{x}, \bar{y}) \neq 0$ or T is convex-valued in a neighbourhood of y;

iii) the solution map R is metrically regular around $(\bar{x}, \bar{y})$,

then T is locally single-directional at $\bar{y}$.

The proof of Theorem 5.1 is based on Proposition 3.2, Proposition 3.4 and the following result.

THEOREM 5.2. *Let X, Y and Z be Banach spaces. Suppose that $f : X \times Y \to Z$ is strictly differentiable at $(\bar{x}, \bar{y})$ and satisfying the ample parametrization condition*

$$\nabla_x f(\bar{x}, \bar{y}) \text{ is surjective.}$$

Consider a set-valued map $T : Y \to 2^Z$ such that $(\bar{y}, -f(\bar{x}, \bar{y}))$ is an element of its graph. Then the set-valued map S defined by

$$S(y) = \{x \in X \ : \ 0 \in f(x,y) + T(y)\}$$

satisfies the Aubin property around $(\bar{x}, \bar{y})$ if and only if T satisfies the Aubin property around $(\bar{y}, -f(\bar{x}, \bar{y}))$.

The above theorem, which is a particular case of [2, Corollary 3.5], is an extension to the Banach space setting of the following results: [12, Theorem 1.57 and Corollary 1.59] for the single-valued case in Banach spaces, [10, Theorem 5.6] for T set-valued map with locally closed graph in Asplund spaces and Exercise 3F.14 in the forthcoming book [9] for T set-valued map in the finite dimension case.

Proof. of Theorem 5.1 As mentioned in Section 2, the metric regularity of the set-valued map R around $(\bar{x}, \bar{y})$ is equivalent to the fact that the set-valued map R^{-1} satisfies the Aubin property around $(\bar{y}, \bar{x})$. But, according to Theorem 5.2, this also equivalently express that the map T has the Aubin property around $(\bar{y}, -f(\bar{x}, \bar{y}))$. The conclusion follows from Proposition 3.2 if $-f(\bar{x}, \bar{y}) \neq 0$ and from Proposition 3.4 if T is convex-valued. □

Remark 5.3. *The single-valuedness result of Dontchev and Hager was very recently used by Mordukhovich [16] to show the failure of metric regularity for the solution map R of the general variational systems (5.1) whenever the set-valued map T is supposed to be monotone. The above Theorem 5.1 can also be considered as a non metric regularity result for the general variational systems (5.1) in the quasimonotone case and corresponds therefore to an counterpart of Theorem 5.1 of Mordukhovich [16]. Indeed, by considering the contrapositive of Theorem 5.1, we obtain:*

> *Suppose that $f : X \times Y \to Y^*$ is strictly differentiable at $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, R$ and satisfies the ample parametrization condition: $\nabla_x f(\bar{x}, \bar{y})$ is surjective. If the set-valued map $T : Y \to 2^{Y^*}$ is quasimonotone, convex-valued in a neighbourhood of y and not locally single-directional at $\bar{y}$, then the solution map R is not metrically regular around $(\bar{x}, \bar{y})$.*

5.2. Solution map of quasivariational inequalities with quasiconvex constraints. Like in §4, in this section Y stands for a reflexive Banach space, equipped with a norm such that both Y and Y^* are strictly convex. We will focus our interest on the particular case of the perturbed quasi-variational inequality problem (P_x) where for any y, the constraint set $K(y)$ is the sublevel set of a given quasiconvex function g. Depending of the kind of sublevel set (the large sublevel set $S_{g(y)}$ or the adjusted sublevel set S_g^a), we will consider the following problems:

$$(P_x^{\leq}) \qquad \text{Find } \bar{y} \in Y \text{ such that} \quad \langle f(x, \bar{y}), y - \bar{y} \rangle \geq 0, \quad \forall\, y \in Y \text{ such that } g(y) \leq g(\bar{y})$$

and

$$(P_x^a) \qquad \text{Find } \bar{y} \in Y \text{ such that} \quad \langle f(x, \bar{y}), y - \bar{y} \rangle \geq 0, \quad \forall\, y \in S_g^a(\bar{y})$$

where $f : X \times Y \to Y^*$ is a differentiable function and $g : Y \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous quasiconvex function. The solution maps associated to those problems will be denoted respectively by $R^{\leq}$ and R^a

$$\begin{array}{rccl} R^{\leq}: & X & \to & Y \\ & x & \mapsto & R^{\leq}(x) := \big\{ y \in Y \ : \ y \text{ solution of } (P_x^{\leq}) \big\} \end{array}$$

and

$$\begin{array}{rccl} R^{a}: & X & \to & Y \\ & x & \mapsto & R^{a}(x) := \big\{ y \in Y \ : \ y \text{ solution of } (P_x^{a}) \big\} \end{array}$$

In the forthcoming Theorem 5.4, we show that the solution map $R^{\leq}$ of the above problem $(P_{\bar{x}}^{\leq})$ is not metrically regular at points (x, y) such that, roughly speaking, the sublevel set $S_{g(y)}$ is a polyhedron and y is on an edge of $S_{g(y)}$. Taking into account the specific hypothesis considered, this result, as well as the forthcoming Theorem 5.6, is expressed here similarly as in Remark 5.3 but can also be formulated in the "positive version" of Theorem 5.1.

Theorem 5.4. *Let us suppose that* $f : X \times Y \to Y^*$ *is strictly differentiable at* $(\bar{x}, \bar{y}) \in \operatorname{Gr} R^{\leq}$ *with* $\nabla_x f(\bar{x}, \bar{y})$ *surjective and that the lower semicontinuous semistrictly quasiconvex function* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *is such that* $S_{g(\bar{y})}$ *is a polyhedron and* $\operatorname{card}(I(\bar{y})) > 1$.

Then the solution map $R^{\leq}$ *is not metrically regular around* $(\bar{x}, \bar{y})$.

Proof. Since g is semistrictly quasiconvex, one has $S_{g(u)} = \operatorname{cl}(S^{<}{}_{g(u)})$ for any $u \in \operatorname{dom} g$ and therefore $N(S^{<}_{g(u)}, u) = N^a_g(u)$. Moreover, since $\operatorname{card}(I(\bar{y})) > 1$, N^a_g is not single-directional at $\bar{y}$. Now observe that

$$y \in R^{\leq}(x) \Leftrightarrow -f(x, y) \in N(\operatorname{cl}(S^{<}_{g(y)}), y) = N^a_g(y) \Leftrightarrow y \in R^a(x).$$

Thus the non metric regularity of the solution map $R^{\leq}$ follows from Theorem 5.1 since the normal operator N^a_g is quasimonotone on Y (see [4, Prop. 3.3]). $\square$

Remark 5.5. *The above result obviously holds also in the more general case where* $S_{g(\bar{y})}$ *is not necessarily polyhedral, but there exists a neighbourhood* U *such that* $S_{g(\bar{y})} \cap U$ *can be written as the intersection of* U *with a polyhedron. The same is true, for instance, for Proposition 4.4.*

Now assuming some polyhedral structure of the strict sublevel set, the more general case of quasiconvex functions will be obtained as a direct consequence of [4, Prop. 3.3], Theorem 5.1 and Proposition 4.4.

Theorem 5.6. *Let us suppose that* $f : X \times Y \to Y^*$ *is strictly differentiable at* $(\bar{x}, \bar{y}) \in \operatorname{Gr} R^a$ *with* $\nabla_x f(\bar{x}, \bar{y})$ *surjective, and the lower semicontinuous quasiconvex function* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *is such that the sublevel sets* $S_{g(\bar{y})}$ *and* $\operatorname{cl}(S^{<}{}_{g(\bar{y})})$ *are polyhedra and* $\operatorname{card}\big(I(\bar{y})\big) \neq \operatorname{card}\big(I^{<}(\bar{y})\big)$.

Then the solution map R^a *is not metrically regular around* $(\bar{x}, \bar{y})$.

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