Minimal Coercivity Conditions and Exceptional Families of Elements in Quasimonotone Variational Inequalities[1]

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Abstract. A coercivity condition is usually assumed in variational inequalities over non-compact domains to guarantee the existence of a solution. We derive *minimal*, i.e., necessary coercivity conditions for pseudomonotone and quasimonotone variational inequalities to have a nonempty, possibly unbounded solution set. Similarly, a minimal coercivity condition is derived for quasimonotone variational inequalities to have a nonempty, bounded solution set, hereby complementing recent studies for the pseudomonotone case. Finally, for quasimonotone complementarity problems previous existence results involving so-called exceptional families of elements are strengthened by considerably weakening assumptions in the literature.

Key Words. Variational inequalities, quasimonotone maps, pseudomonotone maps, coercivity conditions, exceptional families of elements.

1 Introduction

The study of variational inequality problems (VIP) over unbounded domains is usually based on a coercivity condition, which is set in order to guarantee existence of solutions. Such conditions abound in the literature. Some of them are clearly stronger than others. In some cases, a comparison between different coercivity conditions is not obvious; at first look, a condition may seem stronger than another. But in closer examination one may prove that the two conditions are equivalent.

In recent years, there has been some effort to find "minimal" coercivity conditions, i.e., conditions that cannot be weakened. In a sense, these conditions should not only be sufficient, but also necessary for the solution set to be nonempty (provided that some other generalized monotonicity and continuity assumptions hold). In this direction, Crouzeix (Ref. 1) found a minimal coercivity condition for a finite-dimensional, pseudomonotone VIP to have a nonempty, *bounded* set of solutions. In Ref. 2 , Crouzeix' result was generalized to infinite-dimensional spaces. It was also shown that several seemingly different coercivity conditions are actually equivalent to each other.

Other authors focused their efforts on finding minimal coercivity conditions that permit the solution set to be unbounded. To this end, Smith (Ref. 3) in a study of complementarity problems introduced a condition involving a so-called "exceptional sequence of elements" instead of using a coercivity condition. His results and methods were considerably extended by Isac and other authors who instead use the notion of "exceptional family of elements" (Refs. 4 - 7). Another approach is that of Flores-Bazan who even considered the more general (pseudomonotone) equilibrium problems by using recession cones (Refs. 8 - 9).

In this paper, we will consider various coercivity conditions in connection with the nonemptiness and the boundedness of the set of solutions. Following the preliminary Section 2, in Section 3 we will show that for pseudomonotone

VIP two of the frequently encountered coercivity conditions are equivalent to each other as well as to the nonemptiness of the set of solutions. Also, a minimal coercivity condition is given for single-valued, quasimonotone VIP. In both cases very weak assumptions are used. For instance, for the quasi-monotone case we do not use any assumption on the existence of inner points as in Ref. 10 or that the map is densely pseudomonotone as in Ref. 11 . In Section 4, we focus our attention on multivalued quasimonotone VIP and give a minimal coercivity condition for the solution set to be nonempty and bounded. The necessity of the coercivity condition is established under the assumption that the VIP has nontrivial solutions. For reasons we will explain, this assumption cannot be avoided.

Finally, in Section 5 we relate our results to those involving exceptional families of elements rather than a coercivity condition. We show that one can obtain results analogous to those existing in the literature under considerably weaker assumptions: the map needs to be only quasimonotone rather than pseudomonotone and needs to satisfy only a very weak continuity assumption. It may well be discontinuous.

2 Notation and Preliminary Results

Let X be a normed space and X^* be its dual. For every closed convex subset K of X and every $r > 0$ we define

$$K_r = \{x \in K : \|x\| \leq r\}, \qquad K_r^- = \{x \in K : \|x\| < r\}.$$

Let $T : K \rightrightarrows X^*$ be a multivalued map. The variational inequality problem (VIP) and the Minty variational inequality problem (MVIP) are defined as follows:

(VIP) find $x_0 \in K$ such that

$$\forall x \in K, \exists x_0^* \in T(x_0) : \langle x_0^*, x - x_0 \rangle \geq 0$$

and

(MVIP) find $x_0 \in K$ such that

$$\forall x \in K, \forall x^* \in T(x) : \langle x^*, x - x_0 \rangle \geq 0.$$

A solution $x_0 \in K$ of VIP is called strong if there exists $x_0^* \in T(x_0)$ such that $\langle x_0^*, x - x_0 \rangle \geq 0$ holds for all $x \in K$, i.e., if x_0^* in VIP does not depend on x. Let $S(K)$ be the set of solutions of VIP, $S_{str}(K)$ the set of strong solutions of VIP and $S_M(K)$ the set of solutions of MVIP in the set K.

In Ref. 12, another notion of solution was introduced. An element $x_0 \in K$ is called a local solution of MVIP if there exists a neighborhood U of x_0 such that $x_0 \in S_M(K \cap U)$. The set of such local solutions will be denoted by $S_{LM}(K)$. Obviously, $S_M(K) \subseteq S_{LM}(K)$.

A map T is called:

(i) pseudomonotone, if for every $x, y \in K$ and every $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0;$$

(ii) quasimonotone, if for every $x, y \in K$ and every $x^* \in T(x)$, $y^* \in T(y)$, the following implication holds

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

It is well known that properties of T entail some relations between the sets S, S_{str} and S_M. Following Ref. 13, we call T upper sign-continuous if for all $x, y \in K$ the following implication holds, where $x_t = ty + (1-t)\,x$:

$$\left(\forall t \in (0,1)\,,\ \inf_{x^* \in T(x_t)} \langle x^*, y - x\rangle \geq 0\right) \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0.$$

For example, any upper hemicontinuous map is upper sign-continuous. We recall that a map is upper hemicontinuous if its restriction to every line segment of K is upper semicontinuous with respect to the w*-topology on X^*. Also, any positive function in $\mathbb{R}$ is upper sign-continuous. The following proposition is true; see, for instance, Refs. 2, 12 .

Proposition 2.1 (i) If T is pseudomonotone, then $S\,(K) \subseteq S_M\,(K) = S_{LM}\,(K)$.

(ii) If T is upper sign-continuous with nonempty w*-compact values, then $S_{LM}\,(K) \subseteq S\,(K)$.

(iii) If T has w^*-compact and convex values, then $S(K) = S_{str}(K)$.

The following proposition from Ref. 12 will be of use:

Proposition 2.2 Let K be a nonempty, convex, weakly compact subset of X. Let further $T : K \rightrightarrows X^*$ be a quasimonotone map. Then $S_{LM}(K) \neq \emptyset$.

Combining with Proposition 2.1(ii), we get immediately the following existence result that is a simplified version of Proposition 2.2 of Ref. 12.

Proposition 2.3 Let K be a nonempty, convex, weakly compact subset of X. Let further $T : K \rightrightarrows X^*$ be an upper sign-continuous quasimonotone map. If the values of T are nonempty and w*-compact, then $S(K) \neq \emptyset$.

We will make use of Sion's minimax theorem (Ref. 14), in the version given in Ref. 15:

Theorem 2.1 Let M be a convex subset of a linear topological space, N be a convex compact subset of a linear topological space, and $f : M \times N \to \mathbb{R}$ be upper semicontinuous on M and lower semicontinuous on N. Suppose that f is quasiconcave on M and quasiconvex on N. Then

$$\min_N \sup_M f = \sup_M \min_N f.$$

Given a nonempty subset K of X, an element $x^* \in X^*$ is called perpendicular to K if $\langle x^*, y - z \rangle = 0$ for all $y, z \in K$. If we fix $y \in K$, then equivalently $x^* \in X^*$ is perpendicular to K if $\langle x^*, y - z \rangle = 0$ for all $z \in K$.

If K is closed and convex and $T : K \rightrightarrows X^*$ is a multivalued map, set

$$Z = \{x \in K : \exists x^* \in T(x) \text{ such that } x^* \text{ is perpendicular to } K\}.$$

We will call the elements of Z nodes. Note that $Z \subseteq S_{str}(K)$. The elements of the set $S(K) \setminus Z$ are usually called nontrivial solutions of VIP (Ref. 10). One of the main difficulties in dealing with quasimonotone maps stems from the nodes. In fact, if T is any quasimonotone map and x_0 any element in K, then the map T' defined by

$$T'(x) = \begin{cases} T(x_0) \cup \{0\}, & \text{if } x = x_0 \\ T(x), & \text{if } x \neq x_0 \end{cases}$$

is quasimonotone and has x_0 as a solution of VIP. Note that if T satisfies one of the coercivity conditions to be studied below, then so does T'. Thus the structure of the solution set $S(K)$ can be modified by adjoining some nodes to it, without harm to the properties of the map. That is why in many theorems, where we deduce properties of a quasimonotone map from properties of solutions, we will have to exclude nodes.

3 Coercivity Conditions for Nonempty (Possibly Unbounded) Solution Sets

As noted earlier, for pseudomonotone maps, necessary and sufficient coercivity conditions for the set of solutions of VIP and MVIP to be nonempty and bounded have been found in Refs. 1 , 2 . We now establish an analogous result on coercivity conditions that does not entail boundedness of the set of solutions.

We consider the following conditions:

$$\text{(C)} \quad \exists n \in \mathbb{N} : \forall x \in K \backslash K_n, \exists y \in K_n \text{ such that } \forall x^* \in T(x), \langle x^*, x - y \rangle \geq 0$$

and

$$\text{(C')} \qquad \exists n \in \mathbb{N} : \forall x \in K \backslash K_n, \exists y \in K, \|y\| < \|x\|,$$

$$\text{such that } \forall x^* \in T(x), \langle x^*, x - y \rangle \geq 0.$$

Condition (C') is equivalent to the following condition (H2), considered by Luc (Corollary 4.5 in Ref. 11) for single-valued maps: for every sequence $(x_n) \subseteq K$ with $\lim \|x_n\| = +\infty$, there exist $n_0 \in \mathbb{N}$ and $y \in K$ with $\|y\| < \|x_{n_0}\|$ such that $\langle T(x_{n_0}), x_{n_0} - y \rangle \geq 0$. Luc shows that this condition, under suitable assumptions for the map T and the set K, is sufficient for the VIP to have a solution. Condition (C') was also considered by Isac (Definition 4 in Ref. 4) for single-valued maps. In Ref. 4 it is shown that in case of a complementarity problem, (C') implies that T has no exceptional family of elements and thus under some rather restrictive assumptions (T has to be defined on the whole space, be continuous and a k-set field) the complementarity problem has a solution.

Condition (C) is the one considered in Ref. 10 , modified so as to apply for multivalued maps. Obviously, (C) entails (C'). Less obviously, as we will see in the next theorem, the two conditions are actually equivalent under suitable generalized monotonicity and continuity assumptions. In fact, each of them is necessary and sufficient for VIP to have a solution.

Lemma 3.1 Let K be a convex subset of a normed space X, and $T : K \rightrightarrows X^*$ be a map with nonempty values. Let $r > 0$ be given.

(i) If $x_r \in S(K_r)$, T has w*-compact values and there exists $y \in K_r^-$ such that $\forall x^* \in T(x_r)$, $\langle x^*, x_r - y\rangle \geq 0$, then $x_r \in S(K)$.

(ii) If $x_r^* \in T(x_r)$ is such that

$$\langle x_r^*, x - x_r\rangle \geq 0 \tag{1}$$

holds for every $x \in K_r$, and there exists $y \in K_r^-$ such that $\langle x_r^*, x_r - y\rangle \geq 0$, then (1) holds for every $x \in K$, i.e., $x_r \in S_{str}(K)$.

Proof. (i) Define the function $g : K \to \mathbb{R}$ by $g(x) = \sup_{x^* \in T(x_r)} \langle x^*, x - x_r\rangle$.

Note that g is convex and x_r is a global minimum of g on K_r. By our assumption on y, $g(y) \leq g(x_r) = 0$. It follows that $g(y) = g(x_r)$ and y is also a global minimum of g on K_r. But $\|y\| < r$, hence y is also a local minimum of g on K. Since g is convex, y is a global minimum of g on K. Using again $g(y) = g(x_r)$, we infer that x_r is a global minimum of g in K. Hence $\max_{x^* \in T(x_r)} \langle x^*, x - x_r \rangle = \sup_{x^* \in T(x_r)} \langle x^*, x - x_r \rangle \geq 0$ holds for all $x \in K$. This means that $x_r \in S(K)$ as asserted.

(ii) The proof is the same as in part (i) with $g(x) = \langle x_r^*, x - x_r \rangle$. $\square$

In the above lemma, note that whenever $\|x_r\| < r$, we may take $y = x_r$. Hence in this case, if $x_r \in S(K_r)$ (respectively, $x_r \in S_{str}(K_r)$), then $x_r \in S(K)$ (respectively, $x_r \in S_{str}(K)$).

Now we deduce the following existence theorem for quasimonotone maps.

Theorem 3.1 Let X be reflexive, $K \subseteq X$ be nonempty, closed and convex, $T : K \rightrightarrows X^*$ be quasimonotone, upper sign-continuous with nonempty, w*-compact values. If condition (C') holds, then $S(K) \neq \emptyset$.

Proof. Suppose that (C') holds. For all $r > n$ where n is given by condition (C'), K_r is w*-compact and convex, nonempty for r sufficiently large. Hence by Proposition 2.3, $S(K_r) \neq \emptyset$. Choose $x_r \in S(K_r)$. By condition (C'), there exists $y \in K$, $\|y\| < r$ such that $\langle x^*, x_r - y\rangle \geq 0$ for all $x^* \in T(x_r)$. Indeed, if $\|x_r\| = r$, this is exactly what condition (C') says, while in case $\|x_r\| < r$ we may take $y = x_r$. Hence by Lemma 3.1, $x_r \in S(K)$. □

Comparing with the corresponding result in Ref. 11 (Corollary 4.5), we see that there the map was assumed to be densely pseudomonotone, which is stronger than quasimonotone.

As an immediate corollary, we obtain the following theorem, which shows the equivalence of conditions (C) and (C') and also the fact that each is necessary and sufficient for VIP to have a nonempty (possibly unbounded) set of solutions.

Theorem 3.2 Let X be reflexive, $K \subseteq X$ be nonempty, closed and convex,

$T : K \rightrightarrows X^*$ be pseudomonotone, upper sign-continuous with nonempty, w*-compact values. Then the following are equivalent:

(a) condition (C) holds,

(b) condition (C') holds,

(c) $S(K) \neq \emptyset$.

Proof. Obviously, (a) implies (b) and (b) implies (c) by the preceding theorem. Finally, if (c) holds, choose any $x_0 \in S$ and $n > \|x_0\|$. Since T is pseudomonotone, $x_0 \in S_M$. Hence it is obvious that (C) holds with $y = x_0$. □

We should note that in the above theorem implications (a) ⇒ (b) ⇒ (c) need only quasimonotonicity of the map, rather than pseudomonotonicity. Unfortunately, (c) ⇒ (a) or (c) ⇒ (b) do not hold for quasimonotone maps, even if there are no nodes.

Example 3.1 Let $X = \mathbb{R}^2$, $K = \mathbb{R} \times [0, +\infty)$ and T be defined by

$$\forall (a,b) \in K, \qquad T(a,b) = \begin{cases} (0,1) & b \neq 1 \\ \{(t,1), t \in [0,1]\} & b = 1 \end{cases}.$$

Then T is quasimonotone, upper semicontinuous with compact, convex values. Also, $Z = \emptyset$, $S(K) = \mathbb{R} \times \{0\} \neq \emptyset$, but coercivity conditions (C) and (C') do not hold.

However for single-valued quasimonotone maps we do have the implication (c)$\Rightarrow$(a) provided that we exclude the nodes. This result follows easily from the lemma below. Let us recall first that a map $T : K \to X^*$ is said to be hemicontinuous if its restriction to every line segment of K is continuous with respect to the weak topology on X^*.

Lemma 3.2 (See Ref. 10.) Let $T : K \to X^*$ be a quasimonotone hemicontinuous map. If $x_0, x \in K$ are such that $\langle T(x_0), x - x_0 \rangle \geq 0$ holds, then at least one of the following holds:

$$\langle T(x), x - x_0 \rangle \geq 0 \qquad \text{or} \qquad \forall z \in K, \langle T(x_0), z - x_0 \rangle \leq 0.$$

Proposition 3.1 Let X be a normed space, $K \subseteq X$ be closed and convex, and $T : K \to X^*$ be quasimonotone and hemicontinuous. If $S(K) \backslash Z \neq \emptyset$, then condition (C) holds.

Proof. Choose any $x_0 \in S(K) \backslash Z$ and $n \in \mathbb{N}$ such that $\|x_0\| < n$. Then $\langle T(x_0), x - x_0 \rangle \geq 0$ holds for every $x \in X$. Also, there exists at least some $z \in K$ such that $\langle T(x_0), z - x_0 \rangle > 0$ because otherwise one would have $x_0 \in Z$, a contradiction. Thus the previous lemma implies that for every $x \in K$, $\langle T(x), x - x_0 \rangle \geq 0$ holds. In particular, coercivity condition (C) is satisfied with $y = x_0$. □

4 Coercivity Conditions for Nonempty Bounded Solution Sets

Consider the following coercivity condition for T on K:

(C1) $\exists n \in \mathbb{N} : \forall x \in K \backslash K_n, \exists y \in K_n$ such that $\forall x^* \in T(x), \langle x^*, x - y \rangle > 0.$

It is known that, in case X is reflexive, $K \subseteq X$ is nonempty, closed and convex, and T is pseudomonotone, upper sign-continuous with nonempty, convex, w*-compact values, condition (C1) is necessary and sufficient for $S(K)$ to be nonempty and bounded (Ref. 2). Actually, in Ref. 2 T was assumed to be upper hemicontinuous, but the same proof is valid for upper sign-continuous maps. For quasimonotone operators, sufficiency of (C1) under these assumptions is an obvious consequence of Theorem 3.1, because condition (C1) is obviously stronger than condition (C) which implies (C'). Furthermore (C1) implies that $S(K) \subseteq K_n$, hence $S(K)$ is bounded.

If we exclude nodes, we can show that condition (C1) is also necessary, in the following sense.

Theorem 4.1 Let X be reflexive, $K \subseteq X$ be nonempty, closed and convex, and T be a quasimonotone, upper sign-continuous map with nonempty, convex, w*-compact values. If $S(K)$ is bounded and $S(K) \setminus Z$ is nonempty, then (C1) holds.

Proof. Since $S(K)$ is bounded, we can find n_0 such that $S(K) \subseteq K_{n_0 - 1}$. By reflexivity of X, K_n is w*-compact. If (C1) does not hold, then for each $n > n_0$ we may find x_n in $K \backslash K_n$ such that

$$\forall x \in K_n, \exists x_n^* \in T(x_n), \ \ \langle x_n^*, x - x_n \rangle \geq 0.$$

By Sion's minimax theorem (Theorem 2.1), we may suppose that x_n^* does not depend on x, i.e.,

$$\exists x_n^* \in T(x_n), \forall x \in K_n, \ \ \langle x_n^*, x - x_n \rangle \geq 0. \tag{2}$$

Let us show that

$$\forall x \in K_n^-, \langle x_n^*, x - x_n \rangle > 0. \tag{3}$$

From (2) we know that $\langle x_n^*, x - x_n \rangle \geq 0$. Suppose that $\langle x_n^*, x - x_n \rangle = 0$. Since $x_n \notin S(K)$, there exists $y \in K$ such that $\langle x_n^*, y - x_n \rangle < 0$. It follows that for all $t \in (0,1)$, $\langle x_n^*, ty + (1-t)x - x_n \rangle < 0$. However for t sufficiently small $ty + (1-t)x \in K_n$, and this contradicts (2).

Note that our assumptions imply that $S(K) = S_{str}(K)$. Choose $x_0 \in$

$S_{str}(K) \setminus Z$ and $x_0^* \in T(x_0)$ such that

$$\forall x \in K,\ \langle x_0^*, x - x_0 \rangle \geq 0. \tag{4}$$

Then choose $z_n = \lambda x_0 + (1-\lambda)x_n$ with $\lambda \in (0,1)$ and $n-1 < \|z_n\| < n$. This is possible because $\|x_0\| \leq n-1$ while $\|x_n\| > n$. Note that $z_n \notin S(K)$ since $S(K) \subseteq K_{n_0-1}$.

Let us first note that (3) and quasimonotonicity of T imply that

$$\forall x \in K_n^-, \forall x^* \in T(x), \langle x^*, x - x_n \rangle \geq 0. \tag{5}$$

Set A to be the open halfspace $\{x \in X : \langle x_0^*, x - x_0 \rangle > 0\}$. Since $x_0 \notin Z$, $K \cap A \neq \emptyset$. For every $x \in K \cap A$, we can find $x' = tx + (1-t)x_0$, $t \in (0,1]$ such that $x' \in K_n^-$. One can immediately verify that $x' \in A$, thus $K_n^- \cap A$ is a nonempty convex set.

For each $x \in K_n^- \cap A$ we have $\langle x_0^*, x - x_0 \rangle > 0$. By quasimonotonicity,

$$\forall x^* \in T(x),\ \langle x^*, x - x_0 \rangle \geq 0. \tag{6}$$

Combining relations (5) and (6) with the definition of z_n we infer that

$$\forall x \in K_n^- \cap A, \forall x^* \in T(x), \langle x^*, x - z_n \rangle \geq 0. \tag{7}$$

Using upper sign-continuity we infer that

$$\forall x \in K_n^- \cap A, \exists z_n^* \in T(z_n), \langle z_n^*, x - z_n \rangle \geq 0.$$

In virtue of Sion's minimax theorem, we can find $z_n^* \in T(z_n)$ such that

$$\forall x \in K_n^- \cap A, \langle z_n^*, x - z_n \rangle \geq 0. \tag{8}$$

If $x \in K_n^-$, choose any $x_1 \in K_n^- \cap A$. Using (4) and the definition of A, we see that $(x, x_1] \subseteq K_n^- \cap A$. Thus by continuity (8) implies that

$$\forall x \in K_n^-, \langle z_n^*, x - z_n \rangle \geq 0. \tag{9}$$

Finally, for any $x \in K \backslash K_n^-$ we may choose $x' \in (z_n, x) \cap K_n^-$. Since $\langle z_n^*, x' - z_n \rangle \geq 0$ in virtue of (9), it is obvious that $\langle z_n^*, x - z_n \rangle \geq 0$. But this means that $z_n \in S(K)$, a contradiction. $\square$

Remark 4.1 Note that (C1) may not hold if $S(K) \backslash Z$ is empty. Consider for instance in $\mathbb{R}$ the map $T : [0, +\infty) \to \mathbb{R}$ defined by $T(x) = -x$. Then T

is a continuous quasimonotone map. The set $S(K) = \{0\}$ is bounded, but (C1) does not hold.

5 Exceptional Families of Elements

In this section we are going to relate the so-called "exceptional families of elements" for complementarity problems (CP) (Refs. 4 - 7) to "partial solutions" of CP and consequently obtain substantially stronger versions of the corresponding results.

Let $K \subset H$ be a closed, convex cone in a Hilbert space H and $T : K \rightrightarrows H$ be a multivalued map. We consider the complementarity problem:

$$\text{(CP)} \qquad \text{find } x_0 \in K \text{ such that } \exists x_0^* \in T(x_0) : x_0^* \in K^* \text{ and } \langle x_0^*, x_0 \rangle = 0$$

where K^* is the dual cone of K.

It is known that CP is equivalent to finding a strong solution for the associated VIP. We now recall the concept of an exceptional family of elements (Refs. 3 - 6).

Definition 5.1 A family $\{x_r\}_{r>0} \subset K$ is an exceptional family of elements for T with respect to K, if for every $r > 0$ there exist a real number $\mu_r > 0$ and an element $x_r^* \in T(x_r)$ such that the following conditions are satisfied:

(i) $u_r = \mu_r x_r + x_r^* \in K^*$,

(ii) $\langle u_r, x_r \rangle = 0$,

(iii) $\|x_r\| \to \infty$ as $r \to +\infty$.

The following theorem characterizes exceptional families of elements as partial solutions of VIP.

Theorem 5.1 Let K be a closed convex cone in a Hilbert space H and $T : K \rightrightarrows H$ be a map. Suppose that for every $r > 0$ there exists $x_r \in K_r$ and $x_r^* \in T(x_r)$ such that $\langle x_r^*, x - x_r \rangle \geq 0$ holds for all $x \in K_r$, but not for all $x \in K$. Then (x_r) is an exceptional family of elements for T.

Conversely, if there exists an exceptional family of elements (x_r), and x_r^* are the corresponding elements of $T(x_r)$, then for every $x_r \neq 0$, $\langle x_r^*, x - x_r \rangle \geq 0$ holds for all $x \in K_{\|x_r\|}$, but not for all $x \in K$. If in particular T is single-valued, then x_r is a solution of VIP in $K_{\|x_r\|}$, but not in K.

Proof. Suppose that for every $r > 0$ there exist $x_r \in K_r$ and $x_r^* \in T(x_r)$ with the asserted properties. We will show that (x_r) is an exceptional family of elements with $u_r = \mu_r x_r + x_r^*$ where

$$\mu_r = \frac{-\langle x_r^*, x_r\rangle}{r^2}.$$

If we assume that $\|x_r\| < r$, then by taking $y = x_r$ in Lemma 3.1(ii) we would infer that $\langle x_r^*, x - x_r\rangle \geq 0$ for all $x \in K$, a contradiction. Hence $\|x_r\| = r$, from which follows easily that $\lim_{r\to+\infty} \|x_r\| = +\infty$ and $\langle u_r, x_r\rangle = 0$. Finally, $\mu_r > 0$ since otherwise we would have $\langle x_r^*, x_r - 0\rangle \geq 0$ and by taking $y = 0$ in Lemma 3.1(ii) we would infer that $\langle x_r^*, x - x_r\rangle \geq 0$ holds for all $x \in K$, a contradiction.

It remains to show that $u_r \in K^*$. Using the definition of u_r, we have to verify that

$$\forall x \in K, \left\langle x_r^*, x - \frac{x_r \langle x_r, x\rangle}{r^2} \right\rangle \geq 0.$$

For a fixed r set $y = x - \frac{x_r\langle x_r, x\rangle}{r^2}$ and $z_t = y + tx_r$, $t > \frac{\langle x_r, x\rangle}{r^2}$. Then $z_t \in K$

and $\frac{z_t}{\|z_t\|}r \in K_r$. Hence

$$\left\langle x_r^*, \frac{z_t}{\|z_t\|}r - x_r \right\rangle \geq 0 \Rightarrow$$

$$\left\langle x_r^*, y + \left(t - \frac{\|z_t\|}{r}\right) x_r \right\rangle \geq 0. \tag{10}$$

Note that $\langle y, x_r \rangle = 0$ implying $\|z_t\| = \sqrt{\|y\|^2 + t^2 r^2}$. By taking the limit as $t \to +\infty$ in (10) we deduce that $\langle x_r^*, y \rangle \geq 0$ as desired.

To show the converse, suppose that (x_r) is an exceptional family of elements and take any $x_r \neq 0$ and the corresponding x_r^*. Then there exists $\mu_r > 0$ such that $u_r \in K^*$ and $\langle u_r, x_r \rangle = 0$ where $u_r = \mu_r x_r + x_r^*$. Note that

$$\langle x_r^*, 2x_r - x_r \rangle = \langle u_r - \mu_r x_r, x_r \rangle = -\mu_r \langle x_r, x_r \rangle < 0.$$

This means that $\langle x_r^*, x - x_r \rangle \geq 0$ does not hold for $x = 2x_r \in K$. If $x \in K_{\|x_r\|}$, then

$$\langle x_r^*, x - x_r \rangle = \langle u_r - \mu_r x_r, x - x_r \rangle =$$

$$\langle u_r, x \rangle + \mu_r \left(\langle x_r, x_r \rangle - \langle x_r, x \rangle \right) \geq \mu_r \left(\|x_r\|^2 - \|x_r\| \|x\| \right) \geq 0.$$

Hence $\langle x_r^*, x - x_r \rangle \geq 0$ holds for all $x \in K_{\|x_r\|}$. □

The above theorem shows: if the assumptions on T imply that VIP has a strong solution in every K_r, then we have an alternative of the type "either the CP has a solution or there exists an exceptional family of elements". For instance we have:

Corollary 5.1 Let K be a closed convex cone in a Hilbert space H and $T : K \rightrightarrows H$ be a map. Suppose that T has nonempty, w*-compact and convex values. If T is quasimonotone and upper sign-continuous, then either the CP has a solution or there exists an exceptional family of elements for T.

Proof. Our assumptions on T imply that VIP has a strong solution in every K_r; see Proposition 2.1(iii) and Proposition 2.3. Thus, if the CP has no solution, i.e., if the VIP has no strong solution in K, then by Theorem 5.1 there exists an exceptional family of elements. $\square$

The corollary considerably generalizes some of the results on exceptional families of elements (Refs. 4 - 6):

(a) The map T is assumed to be only quasimonotone instead of pseudomonotone,

(b) T does not have to be defined on the whole space H but only on K,

(c) T does not have to be a completely continuous or completely upper semicontinuous field. The latter is a very restrictive assumption. For instance constant maps $T(x) = c$ are not completely continuous fields in an infinite-dimensional space, since $h(x) = x - T(x)$ does not map bounded sets to relatively compact subsets.

The only restriction with respect to Corollary 5.1 of Ref. 5 is that we assume $T(x)$ to be convex. Note that there it is assumed to be contractible.

We finish this section with a remark. Corollary 5.1 does not entail that the existence of an exceptional family of elements implies $S_{str}(K) = \emptyset$. If T is pseudomonotone, then this implication is true; see Ref. 5 , Theorem 5.1. However if T is only quasimonotone, then $S_{str}(K) \neq \emptyset$ is possible. For instance, set $H = \mathbb{R}$, $K = [0, +\infty)$ and $T(x) = -x$. Then $S_{str}(K) = \{0\}$

while $(0, +\infty)$ is an exceptional family of elements. In fact, the existence of an exceptional family of elements implies only that $S_M(K) = \emptyset$.

Proposition 5.1 Let K be a closed convex cone and $T : K \rightrightarrows X^*$ be a multivalued map. If there exists an exceptional family of elements for T with respect to K, then $S_M(K) = \emptyset$.

Proof. Let $\{x_r\}_{r>0}$ be an exceptional family of elements and x_r^* be the corresponding elements in $T(x_r)$. Suppose that $S_M(K) \neq \emptyset$ and choose $x_0 \in S_M(K)$. Then consider any x_r in the exceptional family of elements such that $\|x_r\| > \|x_0\|$. By Theorem 5.1, $\langle x_r^*, x - x_r \rangle \geq 0$ holds for all $x \in K_{\|x_r\|}$, but not for all $x \in K$. Also, we have $\langle x^*, x_r - x_0 \rangle \geq 0$ for all $x^* \in T(x_r)$ because $x_0 \in S_M(K)$. By Lemma 3.1(ii) $\langle x_r^*, x - x_r \rangle \geq 0$ holds for every $x \in K$, a contradiction. $\square$

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