ADJUSTED SUBLEVEL SETS, NORMAL OPERATOR AND QUASICONVEX PROGRAMMING

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Abstract. A new notion of "adjusted sublevel set" of a function is introduced and studied. These sets lie between the sublevel and strict sublevel sets of the function. In contrast to the normal operators to sublevel or strict sublevel sets that were studied in the literature so far, the normal operator to the adjusted sublevel sets is both quasimonotone and, in the case of quasiconvex functions, cone-upper semicontinuous. This makes this new notion appropriate for all kinds of quasiconvex functions and in particular for quasiconvex functions whose graph presents a "flat part". Application is given to quasiconvex optimization through the study of an associated variational inequality problem.

Key words. Quasiconvexity, normal operator, quasiconvex programming, sublevel set.

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1. Introduction. Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function. The aim of this paper is to propose and study a new concept of sublevel set and its associated normal operator to sublevel sets of a quasiconvex function, and then provide an application to quasiconvex optimization. The idea of using normal cones to the sublevel sets $S_{f(x)} = \{y \in X : f(y) \le f(x)\}$ or strict sublevel sets $S^{<}_{f(x)} = \{y \in X : f(y) < f(x)\}$ is due to the fact that, in contrast to convexity that can be described through the convexity of the epigraph, quasiconvexity is related to convexity of the sublevel sets. The idea was exploited by Borde and Crouzeix [5], who mainly established continuity properties, by Aussel and Daniilidis [3] who characterized some classes of quasiconvex functions, and by Eberhard and Crouzeix [8] who studied the integration of these operators as a means to obtain the quasiconvex function.

However, in those papers the most meaningful results were found in the case where, roughly speaking, f admits no "flat parts". If $S_{f(x)} \backslash \overline{S^{<}_{f(x)}} \neq \emptyset$ for some $x \in \operatorname{dom} f$ (i.e., there exists a flat part), then none of the normal operators defined in the literature is able to satisfy at the same time quasimonotonicity and upper semicontinuity in a sense appropriate for cone-valued operators, even if the considered function is lower semicontinuous and quasiconvex (see [5, Example 2.2] and example 2.1 below). This has induced the authors of previous studies on the subject to restrict their attention to the class of quasiconvex functions such that each local minimum is a global minimum (or equivalently $\overline{S^{<}_{\lambda}} = S_\lambda$, $\forall\, \lambda > \inf f$).

In section 2 we propose a concept of "adjusted sublevel set" $S^a(x)$ which allows to deal with all kinds of quasiconvex functions. Based on these adjusted sublevel sets we then define the normal operator. In section 3 we study the properties of the normal operator and in particular some nonemptyness properties, quasimonotonicity and continuity results.

Finally, using this normal operator and our recent study of quasimonotone vari-

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ational inequalities [4] we prove an existence result for the minimization of a quasiconvex function over a convex set.

2. Definitions and basic properties. Let X be a real Banach space, X^* its topological dual and $\langle\cdot,\cdot\rangle$ the duality pairing. The topological closure of a set A will be denoted by $\overline{A}$ for the norm topology and $\overline{A}^*$ for the w* topology whereas $x_i^* \xrightarrow{w^*} x^*$ means that $x_i^* \to x^*$ in the w* topology. As usual, $\mathrm{co}\,A$ will denote the convex hull of A. We denote by $B(x,\varepsilon)$ and $\overline{B}(x,\varepsilon)$ the open ball $\{y \in X : \|y-x\| < \varepsilon\}$ and the closed ball $\{y \in X : \|y-x\| \le \varepsilon\}$. Also, given any nonempty $A \subseteq X$ we denote by $B(A,\varepsilon)$ and $\overline{B}(A,\varepsilon)$ the sets $\{x \in X : dist(x,A) < \varepsilon\}$ and $\{x \in X : dist(x,A) \le \varepsilon\}$ respectively, where $dist(x,A) = \inf\{\|x-y\| : y \in A\}$ is the distance of x from A. Given $x,y \in X$, we set $[x,y] = \{tx+(1-t)y : t \in [0,1]\}$. The domain and the graph of a multivalued operator $T : X \to 2^{X^*}$ will be denoted, respectively, by $\mathrm{dom}\,(T)$ and $\mathrm{gr}\,T$. We will mainly deal with operators whose values are convex cones; in this case, since the values are unbounded, we have to consider a modified definition of upper semicontinuity. We first recall that a convex subset C of a convex cone L in X^* is called a base if $0 \notin \overline{C}^*$ and $L = \bigcup_{t\ge 0} tC$.

Definition 2.1. *An operator $T : X \to 2^{X^*}$ whose values are convex cones is called norm-to-w* cone-usc at $x \in \mathrm{dom}\,(T)$ if there exists a neighborhood U of x and a base $C(u)$ of $T(u)$ for each $u \in U$, such that $u \to C(u)$ is norm-to-w* usc at x.*

It turns out that we may always suppose that, locally, the base $C(u)$ is the intersection of $T(u)$ with a fixed hyperplane. To see this, we first define a conic w*-neighborhood of a cone L in X^* to be a w*-open cone M (i.e., a w*-open set such that $tM \subseteq M$ for all $t > 0$) such that $L \subseteq M \cup \{0\}$.

Proposition 2.2. *Let $T : X \to 2^{X^*}$ be a multivalued operator whose values are convex cones different from $\{0\}$. Given $x \in dom(T)$, the following are equivalent:*

i) T is norm-to-w* cone-usc at x.

ii) $T(x)$ has a base, and for every conic w*-neighborhood M of $T(x)$ there exists a neighborhood U of x such that $T(u) \subseteq M \cup \{0\}$ for all $u \in U$.

iii) There exists a w*-closed hyperplane A of X^* and a neighborhood U of x such that $\forall u \in U, D(u) = T(u) \cap A$ is a base of $T(u)$ and the operator D is norm-to-w* usc at x.

Proof. If *i)* holds and M is a conic w*-neighborhood of $T(x)$, then M is a w*-neighborhood of $C(x)$. Hence there exists a neighborhood U of x such that $C(u) \subseteq M$ for every $u \in U$. Then obviously $T(u) \subseteq M \cup \{0\}$.

Suppose that *ii)* holds. Then $T(x)$ has a base $C(x)$. Since $0 \notin \overline{C(x)}^*$, by convex separation we deduce the existence of some $x_1 \in X$ such that $\langle x^*, x_1\rangle > 0$ for all $x^* \in C(x)$. The set $B = \{x^* \in X^* : \langle x^*, x_1\rangle > 0\}$ is a conic neighborhood of $T(x)$, hence there exists a neighborhood U of x such that for every $u \in U$ one has $T(u) \subseteq B \cup \{0\}$. Set $A = \{x^* \in X^* : \langle x^*, x_1\rangle = 1\}$. Since $T(u) \neq \{0\}$ it follows that $D(u) := T(u) \cap A$ is a base of $T(u)$. To show the semicontinuity of D let us consider a w*-open neighborhood V of $D(x)$. Then $V \cap A$ is w*-open in A. The function $f : B \to A$ defined by $f(x^*) = \frac{x^*}{\langle x^*, x_1\rangle}$ is w*-continuous; thus, the set $\bigcup_{t>0} t\,(V \cap A) = f^{-1}(V \cap A)$ is w*-open. From $D(x) \subseteq (V \cap A)$ we deduce that $\bigcup_{t>0} t\,(V \cap A)$ is a conic w*-neighborhood of $T(x)$. Since *ii)* holds, there exists a neighborhood U_1 of x, $U_1 \subseteq U$, such that $T(u) \subseteq \bigcup_{t>0} t\,(V \cap A) \cup \{0\}$. It follows immediately that $D(u) \subseteq V$, i.e., *iii)* holds.

Finally, *iii)* obviously implies *i)*. □

A definition of upper semicontinuity suitable for cone-valued operators, similar to property *ii)* in the proposition above, was given in [13] (where continuity was taken with respect to the norm topology) and in [5] (where the definition was given in a finite-dimensional setting), the main difference being that in these papers no reference to bases was made.

Given a set $A \subseteq X$, the negative polar cone of A will be denoted by A^-. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. For any $\lambda \in \mathbb{R}$, define $S_\lambda = \{y \in X \ : \ f(y) \leq \lambda\}$, $S_\lambda^< = \{y \in X \ : \ f(y) < \lambda\}$, $S_\lambda^= = \{y \in X : f(y) = \lambda\}$ and, for any $x \in \operatorname{dom} f \setminus \arg\min f$, $\rho_x = dist(x, S^<_{f(x)})$.

Definition 2.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function. To any element $x \in \operatorname{dom} f$ we associate the* adjusted sublevel set *$S_f^a(x)$ defined by*

$$S_f^a(x) = S_{f(x)} \cap \overline{B}\left(S^<_{f(x)}, \rho_x\right)$$

if $x \notin \arg\min f$, and $S_f^a(x) = S_{f(x)}$ otherwise.

Clearly x is always an element of $S_f^a(x)$. If $x \in \operatorname{dom} f \setminus \arg\min f$ is such that $\rho_x = 0$, then $S_f^a(x) = S_{f(x)} \cap \overline{S^<_{f(x)}}$; if moreover f is lower semicontinuous on $\operatorname{dom} f$, then $S_f^a(x) = \overline{S^<_{f(x)}}$.

The convexity of the sublevel sets (resp. strict sublevel sets) characterizes the quasiconvexity of the function. This still holds true for the adjusted sublevel sets.

Proposition 2.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain $\operatorname{dom} f$. Then*

$$f \text{ is quasiconvex} \iff S_f^a(x) \text{ is convex}, \forall x \in \operatorname{dom} f.$$

Proof. Let us suppose that $S_f^a(u)$ is convex for every $u \in \operatorname{dom} f$. We will show that for any $x \in \operatorname{dom} f$, $S_{f(x)}$ is convex. If $x \in \arg\min f$ then $S_{f(x)} = S_f^a(x)$ is convex by assumption. Assume now that $x \notin \arg\min f$ and take $y, z \in S_{f(x)}$.

If both y and z belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then $y, z \in S_f^a(x)$ thus $[y, z] \subseteq S_f^a(x) \subseteq S_{f(x)}$.

If both y and z do not belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then $f(x) = f(y) = f(z)$, $\overline{S^<_{f(z)}} = \overline{S^<_{f(y)}} = \overline{S^<_{f(x)}}$ and ρ_y, ρ_z are positive. If, say, $\rho_y \geq \rho_z$ then $y, z \in \overline{B}\left(\overline{S^<_{f(y)}}, \rho_y\right)$ thus $y, z \in S_f^a(y)$ and $[y, z] \subseteq S_f^a(y) \subseteq S_{f(y)} = S_{f(x)}$.

Finally, suppose that only one of y, z, say z, belongs to $\overline{B}(S^<_{f(x)}, \rho_x)$ while $y \notin \overline{B}(S^<_{f(x)}, \rho_x)$. Then $f(x) = f(y)$, $\overline{S^<_{f(y)}} = \overline{S^<_{f(x)}}$ and $\rho_y > \rho_x$ so we have $z \in \overline{B}\left(S^<_{f(x)}, \rho_x\right) \subseteq \overline{B}\left(\overline{S^<_{f(y)}}, \rho_y\right)$ and we deduce as before that $[y, z] \subseteq S_f^a(y) \subseteq S_{f(y)} = S_{f(x)}$.

The other implication is straightforward. □

An operator T is called:

Quasimonotone, if for every (x, x^*), $(y, y^*) \in \operatorname{gr} T$ the following implication holds:

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0;$$

Cyclically quasimonotone, if for every $(x_i, x_i^*) \in \operatorname{gr} T$, $i = 1, 2, \ldots n$, the following implication holds:

$$\langle x_i^*, x_{i+1} - x_i\rangle > 0, \forall i = 1, 2, \ldots n - 1 \Rightarrow \langle x_n^*, x_{n+1} - x_n\rangle \leq 0$$

where $x_{n+1} = x_1$;

Cyclically monotone, if for every $(x_i, x_i^*) \in \operatorname{gr} T$, $i = 1, 2, \dots n$,

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \le 0.$$

By analogy to convex functions, it is known that a lower semicontinuous function is quasiconvex if and only if its Clarke-Rockafellar subdifferential is quasimonotone [2], [12], or cyclically quasimonotone [6].

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Set

$$N(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \le 0,\ \forall y \in S_{f(x)}\right\}$$
$$N^<(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \le 0,\ \forall y \in S^<_{f(x)}\right\}$$

for every $x \in \operatorname{dom} f$, while we set $N(x) = N^<(x) = \emptyset$ for $x \notin \operatorname{dom} f$. Equivalently, $x^* \in N(x)$ if and only if the following implication holds:

$$\langle x^*, y - x\rangle > 0 \Rightarrow f(y) > f(x);$$

also, $x^* \in N^<(x)$ if and only if

$$\langle x^*, y - x\rangle > 0 \Rightarrow f(y) \ge f(x).$$

These "normal operators" were studied in [5] for functions defined on $\mathbb{R}^n$. They have interesting properties: N is always cyclically quasimonotone. Also, it can be shown that $N^<$ is cone-usc at every point x where f is lower semicontinuous, provided that there exists $\lambda < f(x)$ such that $\operatorname{int} S_\lambda \ne \emptyset$ (see Proposition 2.2 of [5] for an equivalent statement). However, these two operators are essentially adapted to the class of quasiconvex functions such that any local minimum is a global minimum (in particular, semi-strictly quasiconvex functions). In this case, for each $x \in \operatorname{dom} f \setminus \arg\min f$, the sets $S_{f(x)}$ and $S^<_{f(x)}$ have the same closure and $N(x) = N^<(x)$. For quasiconvex functions outside of this class, in general N is not cone-usc (see Example 2.2 in [5]) while $N^<$ is not in general quasimonotone.

EXAMPLE 2.1. *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(a,b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \le 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$

Then f *is quasiconvex. Consider* $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$ *and* $y^* = (2, 1)$. *We see that* $x^* \in N^<(x)$ *and* $y^* \in N^<(y)$ *(since* $|a|+|b| < 1$ *implies* $(1, 2)\cdot(a - 10, b) \le 0$ *and* $(2, 1) \cdot (a, b - 10) \le 0$*) while* $\langle x^*, y - x\rangle > 0$ *and* $\langle y^*, y - x\rangle < 0$. *Hence* $N^<$ *is not quasimonotone.*

In what follows, we will define an operator that has both these properties (cone-usc and quasimonotonicity) and, consequently, is suitable for relating the minimization of a quasiconvex, lsc function f to the variational inequality problem.

DEFINITION 2.5. *To any function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *we associate the set-valued map* $N^a : \operatorname{dom} f \to 2^{X^*}$ *defined for any* $x \in \operatorname{dom} f$ *as the normal cone to the adjusted sublevel set* $S^a_f(x)$ *at* x, *i.e.,*

$$N^a(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \le 0,\ \forall y \in S^a_f(x)\right\}.$$

Note that $S^<_{f(x)} \subseteq S^a_f(x) \subseteq S_{f(x)}$ implies that $N(x) \subseteq N^a(x) \subseteq N^<(x)$ for all $x \in \operatorname{dom} f$.

3. Properties of the normal operator. In this section we investigate properties of the normal operator N^a for quasiconvex functions: equivalent definition, nonemptyness, quasimonotonicity and cone upper semicontinuity are considered.

We will give for quasiconvex functions an equivalent definition of N^a which clearly suggests that this operator corresponds to a refined version of the operator N. Let us first define for any $x \in \operatorname{dom} f$ the *extended normal cone* of f at x as follows. For every $x \in \operatorname{dom} f \setminus \arg\min f$ we set

$$EN(x) = \{x^* \in X^* \ : \ \langle x^*, y\rangle \le \langle x^*, z\rangle, \ \forall\, y \in S^<_{f(x)}, \ \forall\, z \in \overline{B}(x,\rho_x)\},$$

while for $x \in \arg\min f$ we set $EN(x) = \{0\}$. Note that $EN(x)$ is a closed convex cone. In fact, for $x \in \operatorname{dom} f \setminus \arg\min f$ it is the normal cone at x to the set $S^<_{f(x)} + \overline{B}(0,\rho_x)$ or, equivalently, to its closure $\overline{B}(S^<_{f(x)},\rho_x)$. In addition, x^* is an element of $EN(x)$ if and only if for all $y \in S^<_{f(x)}$ and all $v \in \overline{B}(0,1)$ one has $\langle x^*, x-y\rangle \ge -\rho_x\langle x^*, v\rangle$. Consequently, for any $x \in \operatorname{dom} f \setminus \arg\min f$, $EN(x)$ admits the following equivalent definition

$$x^* \in EN(x) \Longleftrightarrow \langle x^*, x-y\rangle \ge \rho_x\|x^*\|, \quad \forall\, y \in S^<_{f(x)}. \tag{3.1}$$

Proposition 3.1. *Let f be quasiconvex. Then for each $x \in \operatorname{dom} f$,*

$$N^a(x) = N(x) + EN(x) = \operatorname{co}\left(N(x) \cup EN(x)\right). \tag{3.2}$$

Before proving Proposition 3.1, let us state the following well-known basic lemma.

Lemma 3.2. *Let A, B be convex subsets of X. If $A \cap \operatorname{int} B \neq \emptyset$ then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.*

Proof. (of Proposition 3.1) If $x \in \arg\min f$, the equality is obvious. Assume that $x \notin \arg\min f$. We consider two cases. If $\rho_x = 0$ then $S^a_f(x) = \overline{S^<_{f(x)}} \cap S_{f(x)}$ thus $S^<_{f(x)} \subseteq S^a_f(x) \subseteq \overline{S^<_{f(x)}}$. It follows that $N^a(x) = N^<(x) = EN(x)$. Since $N(x) \subseteq N^<(x)$, we have $N(x) + EN(x) = EN(x) = N^a(x)$.

Now assume that $\rho_x > 0$. Obviously, $N^a(x)$ is the normal cone to the set $\overline{S_{f(x)} \cap \overline{B}\left(S^<_{f(x)},\rho_x\right)}$ at x. However,

$$S_{f(x)} \cap \operatorname{int}\overline{B}\left(S^<_{f(x)},\rho_x\right) \supseteq S^<_{f(x)} \neq \emptyset \tag{3.3}$$

hence by Lemma 3.2,

$$\overline{S_{f(x)} \cap \overline{B}\left(S^<_{f(x)},\rho_x\right)} = \overline{S_{f(x)}} \cap \overline{B}\left(S^<_{f(x)},\rho_x\right).$$

Therefore, $N^a(x)$ is the normal cone to $\overline{S_{f(x)}} \cap \overline{B}\left(S^<_{f(x)},\rho_x\right)$ at x. From (3.3) and using [1, Th. 4.1.16] we deduce that $N^a(x) = N(x) + EN(x)$. The second equality is obvious. □

Let us set $S^*(0,1) = \{x^* \in X^* : \|x^*\| = 1\}$.

Proposition 3.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function. Then*

i) $EN \cap S^*(0,1)$ is cyclically monotone on any nonempty subset

$$S^=_a = \{x \in X \ : \ f(x) = a\}.$$

ii) N^a is cyclically quasimonotone.

Proof. *i*) Let us consider $x_1, x_2, \ldots, x_n \in S_a^=$. We assume that $x_i \notin \arg\min f$ since otherwise $EN(x_i) \cap S^*(0,1)$ is empty. Set $x_{n+1} = x_1$ and take $x_i^* \in EN(x_i) \cap S^*(0,1)$, $i = 1, 2, \ldots n$. According to (3.1), for any $y \in S^<_{f(x_i)}$, $\langle x_i^*, x_i - y\rangle \geq \rho_{x_i}$. This yields

$$\|x_{i+1} - y\| \geq \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_i^*, x_i - y\rangle \geq \rho_{x_i} + \langle x_i^*, x_{i+1} - x_i\rangle$$

from which, remembering that $f(x_i) = a$, we get

$$\forall i = 1, 2, \ldots n, \quad \rho_{x_{i+1}} = d(x_{i+1}, S^<_{f(x_i)}) \geq \rho_{x_i} + \langle x_i^*, x_{i+1} - x_i\rangle.$$

Adding the inequalities for all i's we obtain

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0,$$

i.e., $EN \cap S^*(0,1)$ is cyclically monotone on $S_a^=$.

ii) If N^a is not cyclically quasimonotone, then there exist $x_i \in \mathrm{dom}(f)$, $x_i^* \in N^a(x_i)$, $i = 1, 2, \ldots n$ such that

$$\langle x_i^*, x_{i+1} - x_i\rangle > 0, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

where $x_{n+1} = x_1$.

Since $N^a(x_i) \subseteq N^<(x_i)$, (3.4) implies that for all $i = 1, 2, \ldots, n$, $f(x_i) \leq f(x_{i+1})$. Consequently, $f(x_1) = f(x_2) = \ldots = f(x_n)$. This means that $S^<_{f(x_i)}$ is the same for all i. We denote this set by A. From (3.4) and $x_i^* \in N^a(x_i)$ it also follows that $x_{i+1} \notin S_{f(x_i)} \cap \overline{B}(A, \rho_{x_i})$. Since $f(x_{i+1}) = f(x_i)$, we have $x_{i+1} \in S_{f(x_i)}$. Hence, $x_{i+1} \notin \overline{B}(A, \rho_{x_i})$ for all $i = 1, 2, \ldots n$. It follows that $\rho_{x_{i+1}} > \rho_{x_i}$ for all $i = 1, 2, \ldots n$. This easily leads to $\rho_{x_{n+1}} > \rho_{x_1}$, a contradiction. □

According to the preceding proposition, the operator N^a is always quasimonotone. Just as the so-called "quasiconvex subdifferential" [7], N^a has the property to characterize the quasiconvexity of the associated function not by its quasimonotonicity, but by its non-emptyness on a dense subset of $\mathrm{dom}(f)$.

PROPOSITION 3.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Suppose that either f is radially continuous, or* $\mathrm{dom}(f)$ *is convex and $int(S_a) \neq \emptyset$, $\forall a > \inf_X f$. Then*

i) If $N^a(x) \setminus \{0\}$ is nonempty on a dense subset of $\mathrm{dom}(f) \setminus \arg\min f$*, then f is quasiconvex.*

ii) If f is quasiconvex, then $N^a(x) \setminus \{0\} \neq \emptyset$, $\forall x \in \mathrm{dom}(f) \setminus \arg\min f$.

iii) f is quasiconvex if and only if $\mathrm{dom}(N^a \setminus \{0\})$ *is dense in* $\mathrm{dom}(f) \setminus \arg\min f$.

Proof. *i*) Looking closely into the proof of Proposition 11 of [7] one can observe that it has been shown that, under the assumptions of the present proposition, the function f is quasiconvex provided that the domain of $N^< \setminus \{0\}$ is dense in $\mathrm{dom} f \setminus \arg\min f$. Since $N^a(x) \setminus \{0\} \subseteq N^<(x) \setminus \{0\}$, the assertion follows.

ii) For every $x \in \mathrm{dom}(f) \setminus \arg\min f$ one has $x \notin S^<_{f(x)}$. It is known that a quasiconvex, lsc and radially continuous function is continuous [7, Prop. 9]. Thus, our assumptions imply that $\mathrm{int}(S^<_{f(x)}) \neq \emptyset$. Hence there exists $x^* \in X^* \setminus \{0\}$ such that

$$\forall\, y \in S^<_{f(x)}, \forall\, z \in \overline{B}(x, \rho_x), \quad \langle x^*, y\rangle \leq \langle x^*, z\rangle.$$

Therefore $x^* \in EN(x)$ and from Proposition 3.1 follows that $N^a(x) \setminus \{0\} \neq \emptyset$. Finally assertion *iii)* resumes the previous ones. □

PROPOSITION 3.5. *Let f be quasiconvex and such that $intS_a \neq \emptyset$ for all $a > \inf f$. If f is lsc at $x \in \mathrm{dom}(f) \setminus \arg\min f$, then N^a is norm-to-w^* cone-usc at x.*

Before proving Proposition 3.5 we establish the following lemma. For any set $U \subseteq X$, $N^<(U)$ denotes as usual the set $\cup_{x \in U} N^<(x)$.

LEMMA 3.6. *Let f be quasiconvex and such that $intS_a \neq \emptyset$ for all $a > \inf f$. If f is lsc at $x \in \mathrm{dom} f \setminus \arg\min f$, then there exists a neighborhood U of x and an element $z \in X \setminus \{0\}$ such that the set $N^<(U) \cap A$, with $A = \{x^* \in X^* : \langle x^*, z\rangle = 1\}$, is a bounded base for the cone $N^<(U)$.*

Proof. Choose $y_0 \in X$ and $\delta > 0$ such that $y_0 \in \mathrm{int}\, S^<_{f(x)-\delta}$. There exists $\varepsilon > 0$ such that

$$\forall z \in B(0,1), f(y_0 + \varepsilon z) < f(x) - \delta.$$

Since f is lsc at x, we can choose $\varepsilon_1 > 0$ such that for every $u \in x + \varepsilon_1 B(0,1)$, $f(u) > f(x) - \delta$. Thus

$$\forall u \in x + \varepsilon_1 B(0,1),\ y_0 + \varepsilon B(0,1) \subseteq S^<_{f(u)}. \tag{3.5}$$

Set $\varepsilon_2 = \min\{\varepsilon/2, \varepsilon_1\}$, $U = x + \varepsilon_2 B(0,1)$. For every $u \in U$, from (3.5) we deduce that $f(y_0 + \varepsilon w) < f(u)$ for all $w \in B(0,1)$ and thus, for every $x^* \in N^<(u)$ we obtain:

$$\forall w \in B(0,1), \langle x^*, y_0 + \varepsilon w - u\rangle \leq 0.$$

It follows that

$$\begin{aligned}\varepsilon \|x^*\| &= \sup_{w \in B(0,1)} \langle x^*, \varepsilon w\rangle \leq \langle x^*, u - y_0\rangle \\ &= \langle x^*, x - y_0\rangle + \langle x^*, u - x\rangle \leq \langle x^*, x - y_0\rangle + \|x^*\| \frac{\varepsilon}{2}\end{aligned}$$

Thus,

$$\forall u \in U, \forall x^* \in N^<(u),\ \langle x^*, x - y_0\rangle \geq (\varepsilon/2)\|x^*\|. \tag{3.6}$$

In particular, $\langle x^*, x - y_0\rangle > 0$ whenever $x^* \in N^<(u) \setminus \{0\}$. Now set $A = \{x^* \in X^* : \langle x^*, x - y_0\rangle = 1\}$. Obviously, for every $u \in U$ and $x^* \in N^<(u) \cap A$, one has $\|x^*\| \leq 2/\varepsilon$, i.e., $N^<(U) \cap A$ is bounded. □

Proof. (of Proposition 3.5) Let U and A be the neighborhood and hyperplane given by Lemma 3.6. Define $C(u) = N^a(u) \cap A$, $u \in U$. Obviously, $C(u)$ is a convex, w*-compact base of $N^a(u)$. We have to show that C is norm-to-w* usc at x. Define $D(u) = (N(u) \cup EN(u)) \cap A$, $u \in U$. We first show that D is norm-to-w* usc. According to [10, Prop. 1.2.23] it is sufficient to show that if $(x_i, x_i^*)_{i \in I}$ is a net in $\mathrm{gr}\, D$ such that $x_i \to x$ in norm and $x_i^* \xrightarrow{w^*} x^*$, then $x^* \in D(x)$. Since obviously $x^* \in A$, we have to show that $x^* \in EN(x) \cup N(x)$. Since $x_i^* \in EN(x_i) \cup N(x_i)$ we may consider, without loss of generality, that either $x_i^* \in N(x_i)$ for all $i \in I$ or $x_i^* \in EN(x_i)$ for all $i \in I$.

Suppose first that $x_i^* \in N(x_i)$. For every $y \in S^<_{f(x)}$, there exists i_0 such that for all $i > i_0$, $f(y) < f(x_i)$. Thus, $\langle x_i^*, x_i - y\rangle \geq 0$. Taking into account that

x_i^* are bounded as they belong to $N^<(U) \cap A$, we obtain at the limit $\langle x^*, x-y\rangle \geq 0$. This means that $x^* \in N^<(x)$. If x is not a local minimum, then $\rho_x = 0$ hence $N^<(x) = EN(x)$ so that $x^* \in EN(x) \cup N(x)$ and we are done. If x is a local minimum, then for i sufficiently large, $f(x_i) \geq f(x)$. Hence, for every $y \in S_{f(x)}$ we have $y \in S_{f(x_i)}$. Consequently, $\langle x_i^*, x_i - y\rangle \geq 0$ thus implying $\langle x^*, x-y\rangle \geq 0$ for all $y \in S_{f(x)}$. It follows that $x^* \in N(x) \subseteq EN(x) \cup N(x)$.

Now suppose that $x_i^* \in EN(x_i)$. Without loss of generality, we may assume that for all i's we have either $f(x_i) > f(x)$ or $f(x_i) \leq f(x)$. If $f(x_i) > f(x)$ holds, then $S_{f(x)} \subseteq S^<_{f(x_i)}$. Thus,

$$\forall y \in S_{f(x)}, \langle x_i^*, x_i - y\rangle \geq 0$$

and at the limit $\langle x^*, x-y\rangle \geq 0$ for all $\forall y \in S_{f(x)}$ which shows that $x^* \in N(x)$. If on the contrary $f(x_i) \leq f(x)$ holds, then $S^<_{f(x_i)} \subseteq S^<_{f(x)}$ thus

$$\liminf \rho_{x_i} = \liminf \operatorname{dist}\left(x_i, S^<_{f(x_i)}\right) \geq \lim \operatorname{dist}\left(x_i, S^<_{f(x)}\right) = \rho_x. \tag{3.7}$$

Now for each $y \in S^<_{f(x)}$ there exists $i_0 \in I$ such that for all $i > i_0$, $f(x_i) > f(y)$. Thus, $y \in S^<_{f(x_i)}$ and

$$\langle x_i^*, x_i - y\rangle \geq \rho_{x_i} \|x_i^*\|.$$

Using (3.7) and lower semicontinuity of $\|\cdot\|$ at x^*, we find

$$\forall y \in S^<_{f(x)}, \langle x^*, x-y\rangle \geq \rho_x \|x^*\|$$

which means that $x^* \in EN(x)$. Thus, in all cases $x^* \in EN(x) \cup N(x)$. This shows that D is norm-to-w* usc at x, as desired.

To show that C is norm-to-w* usc at x, it is again sufficient to show that if $(x_i, x_i^*)_{i\in I}$ is a net in $\operatorname{gr} C$ such that $x_i \to x$ in norm and $x_i^* \overset{w^*}{\rightharpoonup} x^*$, then $x^* \in C(x)$. Note that in view of Proposition 3.1,

$$C(x_i) = \operatorname{co}\left((N(x_i) \cap A) \cup (EN(x_i) \cap A)\right);$$

hence, each x_i^* can be written in the form $x_i^* = \lambda_i y_i^* + (1-\lambda_i) z_i^*$ where $y_i^* \in N(x_i) \cap A$, $z_i^* \in EN(x_i) \cap A$ and $\lambda_i \in [0,1]$. Since y_i^* and z_i^* are bounded (as they belong to $N^<(U) \cap A$), by considering subnets if necessary, we may assume that $y_i^* \overset{w^*}{\rightharpoonup} y^*$, $z_i^* \overset{w^*}{\rightharpoonup} z^*$ and $\lambda_i \to \lambda$. By the norm-to-w* upper semicontinuity of D, we know that $y^*, z^* \in D(x)$; hence, $x^* \in C(x)$ and C is norm-to-w* usc at x. □

4. Quasiconvex Programming. In [4] an existence result for quasimonotone variational inequality has been proved under weak assumptions, in particular without compactness nor hypothesis on inner points. Taking advantage of the good properties of the normal operator N^a, our aim in this section is to obtain an existence result for the minimization of a quasiconvex function over a convex set, through the study of an associated variational inequality.

Given $K \subseteq X$ and an operator $T : K \to 2^{X^*}$ we denote by $S_{str}(T,K)$ the set of *strong solutions* of the Stampacchia variational inequality

$$x_0 \in S_{str}(T,K) \iff x_0 \in K \text{ and } \exists\, x_0^* \in T(x_0) \;:\; \forall\, x \in K,\ \langle x_0^*, x - x_0\rangle \geq 0.$$

Given $K \subseteq X$ we set $K^{\perp} = \{x^* \in X^* \ : \ \forall\, x, y \in K, \ \ \langle x^*, x\rangle = \langle x^*, y\rangle\}$. If we define aff$K$ as the affine hull of K, i.e.

$$\text{lin}K = \{\sum_{i=1}^{n} \lambda_i x_i \ : \ \sum_{i=1}^{n} \lambda_i = 1, \ x_i \in K, \ i = 1, \ldots, n\}$$

and $\overline{\text{aff}}K$ the closure of aff$K$, then it is easy to see that $K^{\perp} = \{0\}$ if and only if $\overline{\text{aff}}K = X$. In optimization problems one can often assume that $K^{\perp} = \{0\}$ with no harm of generality. It is enough to translate K so that $0 \in K$ and then restrict the problem to the subspace $X_1 = \overline{\text{aff}}K$; Then condition $K^{\perp} = \{0\}$ is fulfilled.

Proposition 4.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, radially upper semicontinuous on* dom(f), *and $K \subseteq$ dom(f) be a convex set such that $K^{\perp} = \{0\}$. Assume that either*
i) $x_0 \in S_{str}(N^{<} \setminus \{0\}, K)$, or
ii) $x_0 \in S_{str}(N^{a} \setminus \{0\}, K)$.
Then $\forall\, x \in K$, $f(x_0) \leq f(x)$.

Proof. *i)* By assumption, there exists $x_0^* \in N^{<}(x_0) \setminus \{0\}$ such that $\forall\, x \in K$, $\langle x_0^*, x - x_0\rangle \geq 0$. Since $x_0^* \notin K^{\perp}$, there exists $y \in K$ such that $\langle x_0^*, y - x_0\rangle \neq 0$; thus $\langle x_0^*, y - x_0\rangle > 0$. Fix such a y and for any $x \in K$ and any $t \in]0, 1[$ define $x_t = (1-t)x + ty$. Then

$$\langle x_0^*, x_t - x_0\rangle = (1-t)\langle x_0^*, x - x_0\rangle + t\langle x_0^*, y - x_0\rangle > 0.$$

Since $x_0^* \in N^{<}(x_0)$, this gives $f(x_t) \geq f(x_0)$ and by radial upper semicontinuity, $f(x) \geq f(x_0)$.

ii) This is an immediate consequence of (i) since $N^{a}(x) \subseteq N^{<}(x)$ for all x. □

We will use a very weak kind of continuity for multivalued operators (cf. [9]): Given a convex subset $K \subseteq X$ and an operator $T : K \to 2^{X^*} \setminus \{\emptyset\}$, T is called *upper sign-continuous* on K if for any $x, y \in K$,

$$\forall\, t \in]0, 1[, \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0 \Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0$$

where $x_t = (1-t)x + ty$. If for example the restriction of T to every line segment of K is usc with respect to the w*-topology in X^*, then T is upper sign-continuous.

Let us recall the following existence result for the Stampacchia variational inequality [4].

Proposition 4.2. *Let K be a convex subset of X such that $K \cap \overline{B}(0, n)$ is weakly compact for every $n \in \mathbb{N}$. Let further $T : K \to 2^{X^*} \setminus \{\emptyset\}$ be a quasimonotone operator such that the following coercivity condition holds*

$$\begin{array}{l} \exists\, n \in \mathbb{N}, \ \forall\, x \in K \setminus \overline{B}(0, n), \ \exists\, y \in K \ \textit{with} \ \|y\| < \|x\| \\ \textit{such that} \ \forall\, x^* \in T(x), \langle x^*, x - y\rangle \geq 0. \end{array} \tag{4.1}$$

Suppose moreover that for every $x \in K$ there exist a neighbourhood V_x of x and an upper sign-continuous operator $S_x : V_x \cap K \to 2^{X^*} \setminus \{\emptyset\}$ with convex, w^*-compact values satisfying $S_x(y) \subseteq T(y)$, $\forall\, y \in V_x \cap K$. Then $S_{str}(T, K) \neq \emptyset$.

Note that condition (4.1) is automatically satisfied if K is bounded. We now apply the above results to optimization.

THEOREM 4.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc quasiconvex function, radially continuous on* dom(f). *Assume that for every $\lambda > \inf_X f$, $int(S_\lambda) \neq \emptyset$. Let $K \subseteq$* dom(f) *be convex with $K^\perp = \{0\}$, and such that $K \cap \overline{B}(0,n)$ is weakly compact for every $n \in \mathbb{N}$.*

If condition (4.1) holds with $T = N^a$, then there exists $x_0 \in K$ such that

$$\forall x \in K, \quad f(x) \geq f(x_0).$$

Proof. If $\arg\min f \cap K \neq \emptyset$, we have nothing to prove. Suppose that $\arg\min f \cap K = \emptyset$. According to Proposition 3.3, N^a is quasimonotone. Further, according to Proposition 3.5, it is norm-to-w* cone-usc on K. Thus, all assumptions of Proposition 4.2 hold for the operator $N^a \setminus \{0\}$, so $S_{str}(N^a \setminus \{0\}, K) \neq \emptyset$. Finally, using Proposition 4.1 we infer that f has a global minimum on K. □

COROLLARY 4.4. *Assumptions on f and K as in Theorem 4.3. Assume that there exists $n \in \mathbb{N}$ such that for all $x \in X$, $\|x\| > n$, there exists $y \in X$, $\|y\| < \|x\|$ such that $f(y) < f(x)$. Then there exists $x_0 \in K$ such that*

$$\forall x \in K, \quad f(x) \geq f(x_0).$$

Proof. If $f(y) < f(x)$ then for every $x^* \in N^a(x) \subseteq N^<(x)$, $\langle x^*, y - x\rangle \leq 0$. Hence, coercivity condition (4.1) with $T = N^a$ holds. The corollary follows from Theorem 4.3. □

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