

Taibah University
Department of Mathematics

Math 241

Exam I

Thursday, November 2, 2017

Time: 90 minutes

Name: Solution I.D. # _____

Show All Necessary Work

Any Type of Calculator is NOT Allowed in the Exam

	Question #	Points
Part I	1 – 10 (10×2=20)	/20
Part II	1	/20
	2 (a:8 b:5 c:7=20)	/20
	3 (2×10=20)	/20
	4	/20
Total		/100

Part I : M.C.Q

1- The given equation is linear:

a) $2xy - y + z = 0$

b) $x + y^2 - 2z = 1$

c) $5x + 3y - 10z = 22$

d) $x + \sin y - z = 2$

2. The solution of the linear system :

$$\begin{cases} x - 3y = -5 \\ 3x + y = 5 \end{cases} \text{ is:}$$

a) $(x, y) = (-2, 1)$

b) $(x, y) = (2, 1)$

c) $(x, y) = (1, 2)$

d) $(x, y) = (-2, -1)$

3. If A is 2×3 matrix and B is 4×2 matrix and C is 3×4 matrix. The size of BAC is:

a) 2×4

b) 4×4

c) 2×2

d) 3×2

4. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$. The matrix A^2 is equal to:

a) $\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$

5. Let $A = \begin{bmatrix} -5 & 1 \\ 4 & 0 \end{bmatrix}$. The transpose A^T is :

a) $\begin{bmatrix} 25 & 1 \\ 16 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -1 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 4 \\ 1 & -5 \end{bmatrix}$

d) $\begin{bmatrix} -5 & 4 \\ 1 & 0 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 5 & 6 \\ 0 & x & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y & -1 & 0 \\ 1 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$. If $A = B + I$, then

- a) $x=1, y=3$ b) $x=1, y=-2$ **c) $x=1, y=2$** d) $x=-1, y=4$

7. The matrix $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$ is

- a) *Elementary matrix* b) *Triangular* c) *Symmetric* **d) *Not Diagonal***

In questions 8 and 9, assume that A , B and C are matrices of suitable sizes where addition and multiplication are defined. Then

8. **a) $(A+B)^T = B^T + A^T$** b) $AC = I = CA$
 c) $AB = AC \Rightarrow B = C$ d) $AB = BA$

9. a) $(AB)^{-1} = A^{-1}B^{-1}$ b) $(A+B)^2 = A^2 + 2AB + B^2$
c) $I^8 = I$ d) $(AB)^2 = A^2B^2$

10. A non-homogeneous linear System $AX = B$

- a) always consistent **b) has a unique solution if A is invertible**
 c) always inconsistent d) has only the trivial solution

Part II

1. Use Gauss-Jordan-Elimination to solve the following system of linear equations:

$$4x_1 + 2x_2 + x_3 + 2x_4 = 8$$

$$x_1 - x_2 + x_3 + x_4 = 4$$

$$3x_1 + 2x_2 - x_3 + x_4 = 2$$

$$2x_1 - x_2 + x_3 + x_4 = 0$$

$$\begin{bmatrix} 4 & 2 & 1 & 2 & 8 \\ 1 & -1 & 1 & 1 & 4 \\ 3 & 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 2 & 8 \\ 3 & 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} -4R_1 + R_2 \\ -3R_1 + R_3 \\ -2R_1 + R_4 \end{array} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 6 & -3 & -2 & -8 \\ 0 & 5 & -4 & -2 & -10 \\ 0 & 1 & -1 & -1 & -8 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 5 & -4 & -2 & -10 \\ 0 & 6 & -3 & -2 & -8 \end{bmatrix}$$

$$\begin{array}{l} -5R_2 + R_3 \\ -6R_2 + R_4 \end{array} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 3 & 4 & 40 \end{bmatrix} \xrightarrow{-3R_3 + R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & -5 & -50 \end{bmatrix}$$

$$\begin{array}{l} -\frac{1}{5}R_4 \\ -2R_4 + R_2 \\ -3R_4 + R_3 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 & 22 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

\therefore The solution is $x_1 = -4$, $x_2 = 2$, $x_3 = 0$, $x_4 = 10$.

2. Let $A = [2 \ -1 \ 4]$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 7 & -4 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}$. Find, if possible, the following:

a) AB^T

$$B^T = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & -4 \end{bmatrix}$$

$$\begin{matrix} A & B & B^T \\ 1 \times 3 & 2 \times 3 & 3 \times 2 \end{matrix} \left. \vphantom{\begin{matrix} A & B & B^T \\ 1 \times 3 & 2 \times 3 & 3 \times 2 \end{matrix}} \right\} AB^T$$

$$1 \times 2$$

$$AB^T = [2 \ -1 \ 4] \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 3 \times 2 - 1 \times 2 + 4 \times 1 & 2 \times 1 - 1 \times 0 + 4 \times (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -14 \end{bmatrix}$$

b) CA

$$\begin{matrix} C & A \\ 3 \times 2 & 1 \times 3 \end{matrix}$$

Not possible

c) $\text{tr}(CB)$

$$CB = \begin{bmatrix} 7 & -4 \\ 0 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 14 & 23 \\ 3 & 0 & -12 \\ 17 & 10 & -3 \end{bmatrix}$$

$$\text{tr}(CB) = 17 + 0 - 3 = 14$$

3.

a) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

Suppose that A is invertible, i.e. A^{-1} exists.

$$\begin{aligned}(A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1}, \text{ since } A \text{ is symmetric}\end{aligned}$$

$$\therefore (A^{-1})^T = A^{-1}$$

$\Rightarrow A^{-1}$ is symmetric.

b) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and consider the polynomial $p(x) = x^2 - 3x + 2$. Find $p(A)$.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$-3A = -3 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & -6 \end{bmatrix}$$

$$2I = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P(A) = A^2 - 3A + 2I$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ -3 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\therefore P(A) = O.$$

4. Find, if possible, the inverse A^{-1} of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

$$[A | I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right]$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$